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THE EXPRESSIVENESS OF LOOPING TERMS IN THE
SEMANTIC PROGRAMMING

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ABSTRACT. We consider the language of Δ_0 -formulas with list terms interpreted over hereditarily finite list superstructures. We study the complexity of reasoning in extensions of the language of Δ_0 -formulas with non-standard list terms, which represent bounded list search, bounded iteration, and bounded recursion. We prove a number of results on the complexity of model checking and satisfiability for these formulas. In particular, we show that the set of Δ_0 -formulas with bounded recursive terms true in a given list superstructure $HW(\mathcal{M})$ is non-elementary (it contains the class $k\text{ExpTime}$, for all $k \geq 1$). For Δ_0 -formulas with restrictions on the usage of iterative and recursive terms, we show lower complexity.

Keywords: semantic programming, list structures, bounded quantification, reasoning complexity.

1. INTRODUCTION

In [1, 2] a paradigm of the Semantic Programming has been proposed, which paved the way for a new generation of declarative programming languages. The approach of the Semantic Programming allows to abstract away from details of implementation and to focus on the desired properties of a software system under development. It also includes imperative features, which allow to specify the order of computations, when it is necessary. Semantic Programming rests on the computability theory put in terms of Σ -definability in hereditarily finite list superstructures (see, e.g., [3] for recent results). The concept of *list* appears to be general

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enough to represent common datatypes of programming languages. A program in the Semantic Programming is a formula over a signature, which includes basic list functions such as concatenation, adding an element to a list, taking head or tail of a list, as well as predicates, which can be used to refer to elements and initial segments of lists. Computation is implemented in two ways. The first one is based on testing whether a formula is true in an appropriate list superstructure and is conceptually close to the idea of Model Checking in the field of Software Verification. The second way is via deciding the entailment of a formula from an appropriate theory, which axiomatizes properties of list superstructures. The latter approach is close to the idea of Logic Programming and Deductive Verification. The language of the Semantic Programming is powerful enough to formulate statements about syntactic and semantic properties of programs, thus providing a unified framework for program specification, validation, and verification. The approach has numerous applications in model-driven software engineering and in particular, for the development of AI tools. One of the recent use cases is the application of the Semantic Programming to document processing in the area of Business Process Management [9].

As a trade-off between the expressiveness and computational efficiency a number of sublanguages of the Semantic Programming have been proposed. Of the most important ones is the language of Δ_0 -formulas, in which only bounded quantification over elements and initial segments of lists can be used. In [4], this language has been extended with conditional list terms, which implement the “if-then-else” primitive of programming languages. It has been noted in [10] that there are at least two sources of the computational complexity of model checking for Δ_0 -formulas and their extensions. The first one is the complexity of computing list terms in a given list superstructure and the second one is the form of the quantifier prefix. It has been shown that for any polynomially computable structure, there exists a polynomially computable representation of its hereditarily finite list superstructure with the above mentioned basic list functions. Thus, the basic list terms make no contribution to the complexity of model checking (provided it is polynomial or superpolynomial). The same has been shown for conditional terms. The authors have noted the natural relationship with the truth problem for Quantified Boolean Formulas, which provided complexity bounds on the model checking for Δ_0 -formulas with conditional terms and general or restricted quantifier prefix.

In [7, 8], the language of Δ_0 -formulas has been further extended to address primitives, which implement looping in programming languages. The authors have introduced non-standard list terms, which correspond to bounded list search, bounded list recursion, and bounded iteration. In this paper, we study the complexity of the language of Δ_0 -formulas extended with these non-standard terms, with the goal to describe their contribution to the complexity of the basic reasoning tasks. Naturally, the first task we consider is model checking, i.e., given a formula φ and a list superstructure $HW(\mathcal{M})$, decide whether $HW(\mathcal{M}) \models \varphi$. The second one is satisfiability, i.e., for a formula φ decide whether it is true in some list superstructure. The complexity of this task is obviously important for program validation, since it indicates how complex it is to identify modelling errors, which might cause inconsistency of the constructed program. To make the contribution of the non-standard terms vivid, we omit two features of the language of Δ_0 -formulas, which on their own may cause an increased complexity of the reasoning tasks. First of all,

we assume that the language contains constants, basic list functions, but *no other functions*. Second, the range of any variable under a (bounded) quantifier must be given by a *ground* list term. These restrictions are implicitly present in [10]. Our results evidence that the language with the non-standard list terms, which implement bounded recursion or iteration, is more expressive than the language of Δ_0 -formulas under the mentioned restrictions. In particular, the complexity of reasoning is increased, which is due to the ability of non-standard terms to succinctly represent lists of large size.

2. PRELIMINARIES

We assume the reader is familiar with basics of the complexity theory. We introduce below the complexity classes mentioned in this paper; all the necessary details can be found in [5, 6].

For a finite alphabet Σ , let Σ^* be the set of all words over Σ and for a subset $A \subseteq \Sigma^*$, let $f : A \rightarrow \Sigma^*$ be a function. f is said to be P-computable/NP-computable if there is a deterministic/non-deterministic Turing Machine (TM) T , respectively, and a polynomial p such that for any $x \in A$ the value of $f(x)$ can be computed by T in at most $p(|x|)$ steps, where $|x|$ is the length of the word x . The function f is PSpace-computable if for any $x \in A$ the value of $f(x)$ can be computed by a deterministic TM T using at most $p(|x|)$ cells of the tape of T , where p is a polynomial.

For a given $n \geq 0$, let $\mathbf{1exp}(n)$ be the notation for 2^n and for $k \geq 1$, let $(k+1)\mathbf{exp}(n) = 2^{\mathbf{kexp}(n)}$.

For $k \geq 1$, the function f is called $\mathbf{kExpTime}/\mathbf{NkExpTime}$ -computable, respectively, if for any word $x \in A$, the value of $f(x)$ can be computed by a deterministic/non-deterministic TM, respectively, in at most $\mathbf{kexp}(p(|x|))$ steps, where p is a polynomial.

We slightly abuse the terminology and define $C \in \{\mathbf{P}/\mathbf{NP}/\mathbf{PSpace}/\mathbf{kExpTime}/\mathbf{NkExpTime}\}$ as the class of subsets $A \subseteq \Sigma^*$ such that their characteristic function is C -computable. A subset $A \subseteq \Sigma^*$ is called C -hard if any set from C is m -reducible to A by a P-computable function. A subset A is called C -complete if it is C -hard and is contained in C .

A structure \mathcal{M} is P-computable if so are the functions of \mathcal{M} , the characteristic functions of predicates, and the domain of \mathcal{M} .

In this paper, we define the language of Δ_0 -formulas as a first-order language with sorts “urelement” and “list”, in which only bounded quantification of the following form is allowed:

- a restriction onto the list elements $\forall x \in t$ and $\exists x \in t$;
- a restriction onto the initial segments of lists $\forall x \sqsubseteq t$ and $\exists x \sqsubseteq t$.

where t is a variable-free list term (i.e., it does not contain variables). A list term is defined inductively via constant lists, variables of sort “list”, and list functions given below. A constant *list* (which can be nested) is built over constants of sort “urelement” (called “urelements”, for short) and a constant *nil* of sort “list”, which represents the empty list. The *list functions* are:

- (1) *head* – the last element of a non-empty list and *nil*, otherwise;
- (2) *tail* – the list without the last element, for a non-empty list, and *nil*, otherwise;

- (3) *cons* – the list obtained by adding a new last element to a list;
- (4) *conc* – concatenation of two lists;

We assume that the language contains constants, list functions, but no other functions. The predicates \in, \sqsubseteq are allowed to appear in Δ_0 -formulas without any restrictions, i.e., they can be used in bounded quantifiers and atomic formulas.

Δ_0 -formulas are interpreted over hereditarily finite *list superstructures* $HW(\mathcal{M})$, where \mathcal{M} is a structure. Urelements are interpreted as distinct elements of the domain of \mathcal{M} and lists are interpreted as lists over urelements and the distinguished “empty list” *nil*. In particular, the following equations hold in every $HW(\mathcal{M})$ (the free variables below are assumed to be universally quantified):

$$\begin{aligned} \neg \exists x \ x \in \text{nil} \\ \text{cons}(x, y) = \text{cons}(x', y') \rightarrow x = x' \wedge y = y' \\ \text{tail}(\text{cons}(x, y)) = x, \quad \text{head}(\text{cons}(x, y)) = y \\ \text{tail}(\text{nil}) = \text{nil}, \quad \text{head}(\text{nil}) = \text{nil} \\ \text{conc}(\text{nil}, x) = \text{conc}(x, \text{nil}) = x \\ \text{cons}(\text{conc}(x, y), z) = \text{conc}(x, \text{cons}(y, z)) \\ \text{conc}(\text{conc}(x, y), z) = \text{conc}(x, \text{conc}(y, z)) \end{aligned}$$

It was shown in [10] that for any P-computable structure \mathcal{M} , there exists a P-computable representation of its superstructure of finite lists $HW(\mathcal{M})$, in which the value of any list term is given by a P-computable function. In this paper, we omit subtleties related to the representation of models and we simply assume that any $HW(\mathcal{M})$ mentioned in the paper is P-computable and so is the value of any (standard) list term in $HW(\mathcal{M})$. Since we assumed that no functions except constants and list functions are in the language of Δ_0 -formulas, it follows that the set of variable-free Δ_0 -formulas true in a given structure $HW(\mathcal{M})$ is P-computable. In turn, the set of variable-free Δ_0 -formulas satisfiable in some structure $HW(\mathcal{M})$ is NP-complete, which is due to the correspondence with the satisfiability problem for propositional boolean formulas. In particular, the upper bound is shown as follows. Given a variable-free Δ_0 -formula φ , every list term in φ is replaced with its value, a constant list; by Lemma 2 in [10], this transformation can be done by a P-computable function. Next, every ground atom $s \propto t$, where s, t are constant lists and $\propto \in \{\in, \sqsubseteq\}$, is evaluated as *true/false* and every equality $s = t$ is replaced with *true* or *false* if s, t are equal or non-equal, respectively. The resulting formula can again be obtained as a value of a P-computable function. Finally, every ground atom $P(s_1, \dots, s_k)$ is replaced with a boolean variable $x_{P(s_1, \dots, s_k)}$, which gives a propositional boolean formula, which is satisfiable iff so is the initial Δ_0 -formula φ .

3. LOOPING TERMS

We consider extensions of the language of Δ_0 -formulas with *bounded search terms* (or *b-search terms*, for short), *recursive terms*, and *iterative terms* of sort “list”. The corresponding language extensions are denoted as $\Delta_0+\text{bSearch}$, $\Delta_0+\text{Rec}$, $\Delta_0+\text{Iteration}$. By default, any formula or a list term in the language of Δ_0 -formulas is a formula/list term in these language extensions. *Non-standard list terms* are defined as follows.

If $t(\bar{v})$ and $\theta(\bar{v}, x)$ is a Δ_0 +bSearch list term and formula, respectively, then the expression $bSearch_{\in}(\theta, t)(\bar{v})$ and $bSearch_{\sqsubseteq}(\theta, t)(\bar{v})$ is a *b-search term*. It is equal to the first element/initial segment a of $t(\bar{v})$, respectively, such that $\theta(\bar{v}, a)$ holds and it is equal to $t(\bar{v})$, otherwise (i.e., if there is no such a).

If $f(\bar{v}), h(\bar{v}, y)$ are Δ_0 +Iteration list terms and i is a natural number given in either unary or binary representation, then the expression $\langle i \rangle Iter[f, h](\bar{v})$ is an *iterative term* and its value is given by $g^i(\bar{v})$ with the following definition:

- $g^0(\bar{v}) = f(\bar{v})$
- $g^{j+1}(\bar{v}) = h(\bar{v}, g^j(\bar{v}))$

If $f(\bar{v}), h(\bar{v}, y, z)$, and $t(\bar{v})$ are Δ_0 +Rec list terms then the expression $Rec[f, h, t](\bar{v})$ is a *recursive term* and its value is given by $g(\bar{v}, t)$ with the following definition:

- $g(\bar{v}, nil) = f(\bar{v})$
- $g(\bar{v}, cons(\alpha, b)) = h(\bar{v}, g(\bar{v}, \alpha), b)$, for any lists α, b such that $cons(\alpha, b) \sqsubseteq t$.

Let s be a Δ_0 +Rec term. s is called *explicit* if in every term $Rec[f, h, t](\bar{v})$, which occurs in s , the term t is variable-free. The term s is called *flat* if in every term $Rec[f, h, t](\bar{v})$, which occurs in s , f and h are Δ_0 -terms. A Δ_0 +Rec formula is *flat* if so is every term in it. The notion of flat Δ_0 +Iteration term or formula is defined identically.

Intuitively, the terms, which are not flat, may implement nested looping, which is an additional source of computational complexity. In the paper, we will show however that the complexity of reasoning is increased if Δ_0 -formulas are extended only with flat iterative and recursive terms.

Terms $s(\bar{v})$ and $t(\bar{v})$ are called *equivalent* if $HW(\mathcal{M}) \models s(\bar{a}) = t(\bar{a})$, for any structure $HW(\mathcal{M})$ and any substitution for \bar{v} with a vector of values \bar{a} .

The *rank* of a Δ_0 +Rec term s (notation: $\mathbf{rank}(s)$) is defined as follows. If s is a Δ_0 -term then $\mathbf{rank}(s) = 0$. If $s = Rec[f, h, t](\bar{v})$ then $\mathbf{rank}(s)$ is the maximum rank of the terms f, h, t increased by 1. If s is not a recursive term then $\mathbf{rank}(s)$ is the maximum rank of the recursive terms in s . The rank of other non-standard terms is defined similarly wrt the maximum rank of the list terms in their parameters. The rank of a formula φ (notation: $\mathbf{rank}(\varphi)$) equals to the maximum rank of the terms in φ .

For a list s , the *length* of s , denoted as $\mathbf{len}(s)$, is the number of elements in s , i.e., for $s = \langle t_1, \dots, t_n \rangle$ (where every $t_i, i = 1, \dots, n$, is a urelement or a list), we have $\mathbf{len}(s) = n$. For a urelement or a list term t , the *size* of t (denoted by $|t|$) is the length of the string, which represents t . The *size of a formula* (we use the same notation $|\varphi|$) is defined identically.

4. EXPRESSIVENESS OF FORMULAS WITH LOOPING TERMS

We begin with an observation that bounded search terms add no expressiveness to Δ_0 -formulas in terms of the computational complexity of model checking.

Theorem 1 (Complexity of Model Checking for Δ_0 +bSearch Formulas). *The set of Δ_0 +bSearch formulas true in a given structure $HW(\mathcal{M})$ is PSpace-complete.*

Proof. Hardness follows from Theorem 3 in [10], where it is proved that the set of Δ_0 -formulas true in a given structure $HW(\mathcal{M})$ is PSpace-complete. The upper complexity bound is shown as follows.

Let φ be a Δ_0 +bSearch formula and $HW(\mathcal{M})$ a structure. First, we consider the case when every b-search term in φ is variable-free and use induction on the

rank of φ to prove the claim of the theorem. We simultaneously show by induction that there is a **PSpace**-computable function, which for a variable-free b-search term s computes the value of s as a list \underline{s} in $HW(\mathcal{M})$ such that the size of \underline{s} is bounded by $|s|$.

For $\mathbf{rank}(\varphi) = \mathbf{rank}(s) = 0$ the claims above readily follow from Theorem 3 and Lemma 2 in [10]. For $k = \mathbf{rank}(\varphi) \geq 1$, take an arbitrary (variable-free) term $s = bSearch_\alpha[\theta, t]$, $\alpha \in \{\in, \sqsubseteq\}$ of rank k in φ . Then t is variable-free, θ has a single free variable, and the ranks of t and θ are less than k . Then by the induction assumption \underline{t} is given by a **PSpace**-computable function and the size of \underline{t} is bounded by $|t|$. By applying the induction assumption again, we conclude that there is a **PSpace**-computable function, which gives the first element/initial segment a of \underline{t} , for which $HW(\mathcal{M}) \models \theta(a)$. Clearly, the size of a is bounded by $|s|$.

Now let φ' be a Δ_0 -formula obtained from φ by replacing every b-search term s with \underline{s} . By the observation above, φ' can be obtained by a **PSpace**-computable function, it has size bounded by $|\varphi|$, and it holds $HW(\mathcal{M}) \models \varphi$ iff $HW(\mathcal{M}) \models \varphi'$. Since φ' is a Δ_0 -formula, we conclude that the claim of the theorem holds for formulas, in which non-standard terms are variable-free.

For the general case, note that if there is a quantifier $\exists x \alpha t$, with $\exists \in \{\exists, \forall\}$, $\alpha \in \{\in, \sqsubseteq\}$, in a Δ_0 +bSearch formula φ , then by the definition of Δ_0 -formulas, the list term t is variable-free. Let φ' be a formula obtained from φ by replacing every quantifier of the form $\exists x \alpha t$ with $\exists x \alpha \underline{t}$, for $\alpha \in \{\in, \sqsubseteq\}$. By Lemma 2 in [10] and the above shown, φ' can be obtained by a **PSpace**-computable function, it has size bounded by $|\varphi|$, and it holds $HW(\mathcal{M}) \models \varphi$ iff $HW(\mathcal{M}) \models \varphi'$. Then $HW(\mathcal{M}) \models \varphi'$ can be decided by a **PSpace**-computable function. It is given by the standard procedure of bounded quantifier elimination, which stores the selected value for each quantified variable. After all quantifiers are eliminated, a formula ψ from φ' is obtained, in which every variable is substituted with the corresponding selected value. The formula ψ is variable-free, thus, by the above shown, $HW(\mathcal{M}) \models \psi$ can be decided by a **PSpace**-computable function, from which the claim of the theorem follows. \square

In the rest of the paper we focus on the expressiveness of recursive and iterative terms and provide the corresponding complexity results.

Let L be an extension of the language of Δ_0 -formulas.

For $k \geq 0$, we say that a *k-list is expressible in L* if there exists a variable-free L -term t such that for any structure $HW(\mathcal{M})$, the interpretation of t in $HW(\mathcal{M})$ is a list of length k .

Let \times be a map, which for non-empty lists s_1, s_2 gives a list $\times[s_1, s_2]$, which consists of $conc(\alpha_1, \alpha_2)$, for all $\alpha_i \in s_i$, $i = 1, 2$. Now let \circ^k be a map defined as follows: for a non-empty list s , it holds $s^1 = s$ and for $k \geq 2$, we have $s^k = \times[s^{k-1}, s]$.

We say that \times is *expressible in L* if there is a L -term $t(x_1, x_2)$ such that in any structure $HW(\mathcal{M})$ for any non-empty lists s_1, s_2 , the term $t(s_1, s_2)$ is interpreted as $\times[s_1, s_2]$. Similarly, for $k \geq 1$, \circ^k is said to be expressible in L if there is a L -term $t(x)$ such that in any structure $HW(\mathcal{M})$ for any non-empty list s , the term $t(s)$ is interpreted as s^k . Whenever we want to specify the L -term t , we say that t represents \times (\cdot^k , respectively), or \times (\cdot^k , respectively) is expressible by t . We omit a direct reference to the language L , whenever it is clear from the context.

Lemma 1 (Succinctness of Recursive Terms). *For $k \geq 1$, $n \geq 0$, a $\mathbf{kexp}(n)$ -list, \times , and $\circ^{\mathbf{kexp}(n)}$ is expressible by a recursive term of size linear in k and n .*

Proof. Let s_0 denote the list $\langle nil \rangle$ (i.e., the list, which consists of the single element being the empty list) and for all $n \geq 0$, let $s_{n+1} = cons(s_n, nil)$. Given $n \geq 0$, we define by induction on $k \geq 1$ a variable-free recursive term ϵ_k as follows.

For $k = 1$, we let ϵ_1 be the term

$$Rec[\langle nil \rangle, conc(g(\alpha), g(\alpha)), s_n].$$

For $k \geq 2$, we define

$$\epsilon_k = Rec[\langle nil \rangle, conc(g(\alpha), g(\alpha)), \epsilon_{k-1}].$$

It easy to verify by induction that the interpretation of ϵ_k in any structure $HW(\mathcal{M})$ is a list, which consists of $\mathbf{kexp}(n)$ -many elements (being empty lists). Clearly, the size of ϵ_k is linear in k and n .

Now consider a recursive term, which for any lists x, y gives a list consisting of $conc(x, b)$, for all $b \in y$. It is defined as

$$Rec[nil, conc(g(\alpha), conc(x, b)), y](x, y)$$

We denote this term by $multiply_element(x, y)$. Now a term which represents \times is defined as

$$Rec[nil, conc(g(\alpha), multiply_element(b, x_2)), x_1](x_1, x_2)$$

Denote it by $multiply(x_1, x_2)$.

Finally, a recursive term which represents $\circ^{\mathbf{kexp}(n)}$ is given by

$$Rec[x, multiply(g(\alpha), x), tail(\epsilon_k)](x)$$

and it is of size linear in k and n . \square

Lemma 2 (Succinctness of Iterative terms). *For any $n \geq 0$, a $\mathbf{1exp}(n)$ -list and $\mathbf{2exp}(n)$ -list is expressible by an iterative term of size linear in n , in which the number of iterations is given in the unary and binary representation, respectively.*

Proof. The proof is identical to Lemma 1, the variable-free term

$$\langle n \rangle Iter[\langle nil \rangle, conc(g(\alpha), g(\alpha))]$$

is the required one. It represents a $\mathbf{1exp}(n)$ -/ $\mathbf{2exp}(n)$ -list, respectively, if n is given in unary or binary (since the binary representation is exponentially more succinct than the unary one) and its size is linear in n . \square

Lemma 3 (Unfolding Lemma). *For any flat Δ_0 +Iteration term $t(\bar{v})$, there exists an equivalent Δ_0 -term $t_0(\bar{v})$ such that $|t_0| \leq \mathbf{1exp}(p(|t|))$ or $|t_0| \leq \mathbf{2exp}(p(|t|))$, for a polynomial p , if the number of iterations is given in unary or binary, respectively.*

For any explicit flat Δ_0 +Rec term $t(\bar{v})$ of rank bounded by $k \geq 1$, there is an equivalent Δ_0 -term $t_0(\bar{v})$, with $|t_0| \leq \mathbf{kexp}(p(|t|))$, for a polynomial p .

Proof. Let t be a flat Δ_0 +Iteration term. If t is a Δ_0 -term, there is nothing to prove, therefore, we assume there is an iterative term $s = \langle i \rangle Iter[f, h](\bar{v})$ in t . We use induction on i to show that the size of a Δ_0 -term equivalent to s is bounded by $|f| \cdot |h|^i$. The case $i = 0$ is trivial. For $i \geq 1$, consider the term $g^i(\bar{v})$ in the definition of s . It is given as a combination of a definition for $g^{i-1}(\bar{v})$ with list functions, where $g^{i-1}(\bar{v})$ is equivalent to $\langle i-1 \rangle Iter[f, h](\bar{v})$. The

number of occurrences of $g^{i-1}(\bar{v})$ in $g^i(\bar{v})$ is at most $|h|$, thus, by applying the induction assumption, the size of a Δ_0 -term equivalent to $g^i(\bar{v})$ is bounded by $|h| \cdot |f| \cdot |h|^{i-1}$. It follows that the size of the Δ_0 -term equivalent to s is bounded by $|s|^{p(|s|)}$ or $|s|^{\mathbf{1exp}(p(|s|))}$, respectively, for a polynomial p , if the number i is given in unary/binary. Hence, it is bounded by $\mathbf{1exp}(p(|s|))$ or $\mathbf{2exp}(p(|s|))$, respectively, for an appropriate polynomial p . Since the number of iterative terms in t is bounded by t and the choice of s was arbitrary, we conclude that the claim of the lemma holds for t .

Now let t be an explicit flat $\Delta_0 + \text{Rec}$ term. First, we show that in case $t = \text{Rec}[f, h, l_0](\bar{v})$, where l_0 is a constant list, there is a Δ_0 -term equivalent to t of size bounded by $\mathbf{1exp}(p(|t|))$, for a polynomial p .

Consider the terms $g(\bar{v}, \alpha)$ in the definition of t and denote $t_\alpha = \text{Rec}[f, h, \alpha](\bar{v})$, for $\alpha = \text{nil}$ or $\alpha \sqsubseteq l_0$. We use induction on the length of l_0 to show that for any list α such that $\alpha = \text{nil}$ or $\alpha \sqsubseteq l_0$, there is a Δ_0 -term t_α^0 equivalent to t_α such that $|t_\alpha^0|$ is bounded by $|f| \cdot |h|^{\mathbf{1en}(\alpha)}$. The case $l_0 = \text{nil}$ is trivial. For $l_0 \neq \text{nil}$, observe that for all lists α, b the term $g(\bar{v}, \text{cons}(\alpha, b))$ in the definition of t is given as a combination of $g(\bar{v}, \text{cons}(\alpha))$ with list functions, where $g(\bar{v}, \text{cons}(\alpha))$ is equivalent to t_α . Thus, by applying the induction assumption, the size of a Δ_0 -term equivalent to $g(\bar{v}, \text{cons}(\alpha, b))$ is bounded by $|h| \cdot |f| \cdot |h|^{\mathbf{1en}(\alpha)}$. It follows that for all $\alpha = \text{nil}$ or $\alpha \sqsubseteq l_0$, the size of the Δ_0 -term equivalent to t_α is bounded by $|t_\alpha|^{p(|t_\alpha|)}$, for a polynomial p . Hence, the size of a Δ_0 -term equivalent to t is bounded by $\mathbf{1exp}(p(|t|))$, for an appropriate polynomial p .

Now let t be an arbitrary explicit flat $\Delta_0 + \text{Rec}$ term. We use induction on the rank bound $k \geq 1$ for t to show the claim of the lemma. If t is a Δ_0 -term then there is nothing to prove. Assume there is a recursive term $s = \text{Rec}[f, h, l](\bar{v})$ in t , then f, h are Δ_0 -terms (since t is flat). If $k = 1$ then $\text{rank}(s) = 1$ and hence, l is a variable-free Δ_0 -term, since t is explicit. By Lemma 2 in [10], l is equivalent to a constant list l_0 of size bounded by a polynomial in the size of l (and hence, in the size of s). Then by the above shown, there is a Δ_0 -term equivalent to s , for which the claim of the lemma holds.

For $k \geq 2$, observe that the term l is given as a combination of recursive terms of rank less than k with list functions. Hence, by the induction assumption, the size of a constant list term l_0 equivalent to l is bounded by $(k-1)\mathbf{exp}(p(|s|))$, for a polynomial p . Then by the above shown we conclude that there is a Δ_0 -term equivalent to s of size bounded by $k\mathbf{exp}(p(|t|))$, for a polynomial p . Since the choice of s in the term t was arbitrary and the number of non-standard terms in t is bounded by $|t|$, we obtain the required statement. \square

Lemma 4 (Hardness of Model Checking). *Let L be an extension of the language of Δ_0 -formulas such that \times is expressible in L and for all $n \geq 0$ and some $k \geq 1$, $\circ^{k\mathbf{exp}(n)}$ is expressible by a L -term of size polynomial in k and n . Then the set of L -formulas true in a given structure $\text{HW}(\mathcal{M})$ is $Nk\text{ExpTime-hard}$.*

Proof. The lemma is proved by a reduction of the inequality problem for regular-like expressions without the Kleene star, but with the exponentiation operation. Regular-like expressions of this kind are defined over a finite alphabet Σ by using the operation of union \cup , concatenation \cdot , and exponentiation $\cdot^{k\mathbf{exp}(n)}$, where $k \geq 1$ and $n \geq 0$. For a regular-like expression E , the language $L(E)$ is given inductively as a subset of all strings over Σ by the following definition:

- $L(a) = \{a\}$, for $a \in \Sigma$
- $L(E_1 \cup E_2) = L(E_1) \cup L(E_2)$
- $L(E_1 \cdot E_2) = \{s_1 \cdot s_2 \mid s_i \in L(E_i), i = 1, 2\}$
- $L(E^{\mathbf{kexp}(n)}) = \underbrace{L(E) \cdot \dots \cdot L(E)}_{\mathbf{kexp}(n) \text{ times}}$

The size of a regular-expression is the length of the string, which represents it. The inequality problem for regular-like expressions is the set of pairs $\langle E_1, E_2 \rangle$ such that $L(E_1) \neq L(E_2)$. It is shown in [11] that the inequality problem is $\mathbf{NExpTime}$ -hard for regular-like expressions, in which exponentiation is restricted to $\mathbf{1exp}(n)$, for $n \geq 0$. The proof employs a direct reduction from the halting problem for non-deterministic Turing machines making at most $\mathbf{1exp}(n)$ -many steps on an input of size $n \geq 0$. For the reduction, the subsequent configurations of the TM are encoded by regular-like expressions over an alphabet Σ of the TM, which represent the tape content and the state of the TM. In particular, each configuration is represented as a word of length $2 \cdot \mathbf{1exp}(n) + 1$. Expressions of the form $(\sigma)^{f(n)}$ are used to refer to (parts of) configurations of TM, where $\sigma \subseteq \Sigma \cup \{\lambda\}$ and $f(n) = c_0 \cdot \mathbf{1exp}(n) + nc_1 + c_2$, with $c_0 \geq 0$ and c_1, c_2 being integers such that $|c_i| \leq n$, for $i = 0, 1, 2$.

If instead, each configuration is represented by a word of length $2 \cdot \mathbf{1exp}(n) + n + 2$ (which can be made without loss of generality), then every regular-like expression of the form $(\sigma)^{f(n)}$, with $f(n)$ as above, can be replaced by $(\sigma)^{g(n)}$, where $g(n) = c_0 \cdot \mathbf{1exp}(n) + nd_1 + d_2$ is a function, with $0 \leq c_0, d_1, d_2 \leq n$. Then the proof works for exponentiation $\mathbf{kexp}(n)$ with any $k \geq 1$, $n \geq 0$ and gives $\mathbf{NkExpTime}$ -hardness of the inequality problem for regular-like expressions with the operation $\cdot^{\mathbf{kexp}(n)}$. We reduce this problem to checking the truth of L -formulas.

Let $t_\times(x)$ be a L -term, which represents \times , and $t_{\mathbf{kexp}(n)}(x_1, x_2)$ a L -term of size polynomial in k and n , which represents $\circ^{\mathbf{kexp}(n)}$. For a regular-like expression E we inductively define the L -term $list(E)$, which encodes the language $L(E)$ as:

- $list(\{a\}) = \langle\langle a \rangle\rangle$, for $a \in \Sigma$;
- $list(E_1 \cup E_2) = conc(list(E_1), list(E_2))$;
- $list(E_1 \cdot E_2) = t_\times(list(E_1), list(E_2))$;
- $list(E^{\mathbf{kexp}(n)}) = t_{\mathbf{kexp}(n)}(list(E))$.

Clearly, the size of $list(E)$ is linear in the size of the expression E .

Now it suffices to note that for any structure $HW(\mathcal{M})$ and any regular-like expressions E_1, E_2 , it holds $L(E_1) \neq L(E_2)$ iff

$$HW(\mathcal{M}) \models \exists x \in list(E_1) (x \notin list(E_2)) \vee \exists x \in list(E_2) (x \notin list(E_1))$$

Indeed, if there is such x then it has the form $\langle a_1, \dots, a_k \rangle$, where $k \geq 1$, $a_i \in \Sigma$, and then the word $a_1 \dots a_k$ witnesses the difference between $L(E_1)$ and $L(E_2)$. The opposite direction is straightforward. \square

Theorem 2 (Complexity of Model Checking for $\Delta_0 + \text{Rec}$). *The set of $\Delta_0 + \text{Rec}$ formulas true in a given structure $HW(\mathcal{M})$ contains the class $\mathbf{kExpTime}$, for every $k \geq 1$, and hence, it is non-elementary. It follows that there is a $\Delta_0 + \text{Rec}$ formula φ , which is not equivalent to a Δ_0 -formula ψ of size polynomial in $|\varphi|$.*

Proof. By Lemma 1, \times is expressible in $\Delta_0 + \text{Rec}$ and for all $k \geq 1$ and $n \geq 0$, $\circ^{\mathbf{kexp}(n)}$ is expressible by a $\Delta_0 + \text{Rec}$ term of size linear in k and n . Then by Lemma 4, for any $k \geq 1$, there exists a $\mathbf{NkExpTime}$ -hard set of $\Delta_0 + \text{Rec}$ formulas true in a

given $HW(\mathcal{M})$. For all $k \geq 1$, it holds $k\text{ExpTime} \subseteq Nk\text{ExpTime}$ and hence, the set of $\Delta_0+\text{Rec}$ formulas true in $HW(\mathcal{M})$ is non-elementary.

It was proved in [10] that the set of Δ_0 -formulas true in a given structure $HW(\mathcal{M})$ is PSpace -complete. Assume that for any $\Delta_0+\text{Rec}$ formula φ , there is an equivalent Δ_0 -formula ψ of size polynomial in the size of φ . Then $\cup_{k \geq 1} k\text{ExpTime} \subseteq \text{PSpace}$, which is a contradiction, since already 2ExpTime is not contained in PSpace . \square

Theorem 3 (Complexity of Model Checking for Flat $\Delta_0+\text{Iteration}$). *The set of flat $\Delta_0+\text{Iteration}$ formulas true in a given structure $HW(\mathcal{M})$ is PSpace -hard and it is in ExpTime or 2ExpTime if the number of iterations is given in unary or binary, respectively.*

Proof. The lower complexity bound follows from Theorem 3 in [10], where it is shown that the set Δ_0 -formulas true in a given structure $HW(\mathcal{M})$ is PSpace -complete. The upper bound is shown as follows. We consider the case, when the number of iterations in formulas is given in unary, the proof for the binary case is identical.

Let $HW(\mathcal{M})$ be a structure and φ a flat $\Delta_0+\text{Iteration}$ formula of the form

$$\partial_1 x_1 \alpha_1 t_1 \dots \partial_n x_n \alpha_n t_n \psi(x_1, \dots, x_n)$$

where $n \geq 0$, $\partial_i \in \{\exists, \forall\}$, and $\alpha_i \in \{\in, \sqsubseteq\}$, for all $i = 1, \dots, n$. Let T denote the total size of the terms t_1, \dots, t_n . We will show by induction on the complexity of φ that satisfiability of φ can be decided by at most $1\text{exp}(p(|T|))$ -many tests for satisfiability of formulas ψ' obtained from ψ by substitutions of x_1, \dots, x_n with vectors of constant lists, each of size bounded by $1\text{exp}(p(T))$, where p is a polynomial. We refer to this claim further as (*).

Then it follows from the proof of Lemma 3 that satisfiability of φ can be decided by at most $1\text{exp}(p(T))$ -many tests for satisfiability of Δ_0 -formulas, each of which is either ψ' as above (if ψ does not contain non-standard terms) or obtained from ψ' as a value of a ExpTime -computable function and has size bounded by $1\text{exp}(r(|\varphi|))$, for some polynomials p, r . As $T \leq |\varphi|$ and the set of variable-free Δ_0 -formulas true in a given structure $HW(\mathcal{M})$ is P -computable, we obtain the statement of the theorem.

We now show that claim (*) holds. If φ is quantifier-free, there is nothing to prove. Now assume φ has the form $\partial x \alpha t \theta(x)$, where $\partial \in \{\exists, \forall\}$ and $\alpha \in \{\in, \sqsubseteq\}$. It is equivalent to the formula $\varphi' = \partial x \alpha t_0 \theta(x)$, where t_0 is a constant term equivalent to t . It follows from the proof of Lemma 3 that t_0 can be obtained as a value of a ExpTime -computable function. In particular, the number and the size of lists $a \alpha t_0$ is bounded by $1\text{exp}(p(T))$, where p is a polynomial. Then satisfiability of φ can be decided by $1\text{exp}(p(T))$ -many tests of satisfiability of $\theta(a)$, one for each $a \alpha t_0$, and thus, by applying the induction assumption to the formulas $\theta(a)$, we obtain the required claim. \square

We now turn to the complexity results on satisfiability. Note that Theorem 2 provides a lower bound on the complexity of testing satisfiability of $\Delta_0+\text{Rec}$, while Theorem 3 does not provide any lower bound (other than PSpace , which is known already for Δ_0 -formulas). However, it is possible to obtain tight complexity results by using the reduction, which we describe next.

Lemma 5 (Hardness of Satisfiability). *Let L be an extension of the language of Δ_0 -formulas such that for all $n \geq 0$ and some $k \geq 1$, a $\mathbf{kexp}(n)$ -list is expressible in L by a L -term of size polynomial in k and n . Then the set of satisfiable L -formulas is $\mathbf{NkExpTime}$ -hard.*

Proof. The lemma is proved by a reduction from the (bounded) domino tiling problem [12]. A *domino system* is a triple $\mathcal{D} = (T, V, H, \mathit{init})$, where $T = \{1, \dots, p\}$, for $p \geq 1$, is a finite set of tiles, $H, V \subseteq T \times T$ are horizontal and vertical tile matching relations, and $\mathit{init} = \langle t_1, \dots, t_s \rangle$ is an initial tiling condition, where $t_i \in T$, for $1 \leq i \leq s$, and $s \geq 0$. A *tiling* of size $m \times m$ for a domino system \mathcal{D} is a mapping $t : \{1, \dots, m\} \times \{1, \dots, m\} \rightarrow T$ such that $\langle t(y-1, x), t(y, x) \rangle \in V$, for $1 < y \leq m$, $1 \leq x \leq m$, $\langle t(y, x-1), t(y, x) \rangle \in H$, for $1 \leq y \leq m$, $1 < x \leq m$, and $t(1, x) = t_x$, for $1 \leq x \leq s$. The size of a domino system is measured as s plus the sum of the cardinalities of V, H , and T . It is known that the set of domino systems, which admit a tiling of size $\mathbf{kexp}(n) \times \mathbf{kexp}(n)$, where $k \geq 1$, $n \geq 0$, is $\mathbf{NkExpTime}$ -complete.

Let \mathcal{D} be a domino system, $k \geq 1$, $n \geq 0$, and let ϵ be a L -term of size polynomial in k and n , which represents a $\mathbf{kexp}(n)$ -list. We define a set of L -formulas \mathcal{T} with quantification over elements and initial segments of ϵ , which encode the tiling problem for \mathcal{D} and a grid of dimension $\mathbf{kexp}(n) \times \mathbf{kexp}(n)$ to be “tiled”. We assume without loss of generality that $s \leq \mathbf{kexp}(n)$.

The theory \mathcal{T} is defined over a signature Σ , which contains a binary predicate T_i , for every tile $i \in \{1, \dots, p\}$. In particular, it includes predicates t_1, \dots, t_s corresponding to the tiles in the initial condition. In our encoding of the tiling problem, we represent an element of a grid of an exponential size by a pair of lists being initial segments of ϵ (there are $\mathbf{kexp}(n)$ -many of them), which corresponds to the “coordinate” of the grid element.

First of all, the theory \mathcal{T} contains axioms

$$(1) \quad \forall x, y \sqsubseteq \epsilon \bigvee_{i \in T} T_i(y, x)$$

and

$$(2) \quad \forall x, y \sqsubseteq \epsilon \neg (T_i(y, x) \wedge T_j(y, x))$$

for all distinct $i, j \in T$.

These axioms state that every element of the grid is “occupied” by exactly one tile.

The next axiom encodes the initial tiling condition $\langle t_1, \dots, t_s \rangle$ and we assume that it is present in \mathcal{T} if $s \geq 1$:

$$(3) \quad t_1(\epsilon, \epsilon) \wedge t_2(\epsilon, \mathit{tail}(\epsilon)) \wedge \dots \wedge t_s(\epsilon, \underbrace{\mathit{tail}(\mathit{tail}(\dots \mathit{tail}(\epsilon) \dots))}_{s-1 \text{ times}})$$

The following axioms represent the vertical matching condition on tiling:

$$(4) \quad \forall x, y_1, y_2 \sqsubseteq \epsilon \neg (y_1 = \mathit{tail}(y_2) \wedge T_i(y_1, x) \wedge T_j(y_2, x))$$

for all $i, j \in T$ such that $\langle j, i \rangle \notin V$.

Finally, the next axioms represent the horizontal matching condition:

$$(5) \quad \forall x_1, x_2, y \sqsubseteq \epsilon \quad \neg(x_1 = \text{tail}(x_2) \wedge T_i(y, x_1) \wedge T_j(y, x_2))$$

for all $i, j \in T$ such that $\langle j, i \rangle \notin H$.

The definition of the theory \mathcal{T} is complete.

It is easy to see that the size of \mathcal{T} is polynomially bounded by the size of the domino system D . We claim that D admits a tiling of size $\mathbf{kexp}(n) \times \mathbf{kexp}(n)$ iff there is a structure \mathcal{M} such that $HW(\mathcal{M}) \models \mathcal{T}$.

For a list s and $1 \leq z \leq \mathbf{len}(s)$, let $\text{seg}^z(s)$ denote the initial segment of s , which consists of $(\mathbf{len}(s) - z + 1)$ -many elements.

(\Leftarrow): Given a model $HW(\mathcal{M})$ of \mathcal{T} , define a mapping $t : \{1, \dots, \mathbf{kexp}(n)\} \times \{1, \dots, \mathbf{kexp}(n)\} \rightarrow T$ by setting $t(y, x) = k$ iff $HW(\mathcal{M}) \models T_k(\text{seg}^y(\epsilon), \text{seg}^x(\epsilon))$. By axioms 1,2, the mapping t is well defined. By axiom 3, it respects the initial tiling condition and by axioms 4,5 it satisfies the vertical and horizontal matching conditions. Thus, the mapping t is a tiling.

(\Rightarrow): Given a tiling t , consider a structure $HW(\mathcal{M})$ of signature Σ , in which ϵ is interpreted as some list s (of length $\mathbf{kexp}(n)$) and the binary predicates are interpreted as follows: for any tile $k \in T$ and lists l_1, l_2 , it holds $HW(\mathcal{M}) \models T_k(l_1, l_2)$ iff $l_1 = \text{seg}^y(s)$ and $l_2 = \text{seg}^x(s)$, for some $1 \leq x, y \leq \mathbf{kexp}(n)$ and $t(y, x) = k$. Since t is a map, the structure $HW(\mathcal{M})$ defined this way is a model of axioms 1,2. As t respects the initial condition and matching conditions, $HW(\mathcal{M})$ is a model of axioms 3-5 and hence, it is a model of \mathcal{T} . \square

Now we are in the position to formulate the complexity results on satisfiability of flat Δ_0 -formulas extended with iterative or recursive terms.

Theorem 4 (Complexity of Satisfiability for Flat Δ_0 +Iteration). *The set of satisfiable flat Δ_0 +Iteration formulas is $NExpTime$ -complete if the number of iterations is given in unary and it is $N2ExpTime$ -complete if the number of iterations is given in binary.*

Proof. Hardness follows from Lemma 5 and the construction from the proof of Lemma 2, where it is shown that for any $n \geq 0$ and $k = 1$ or $k = 2$, respectively, a $\mathbf{kexp}(n)$ -list is expressible by a flat iterative term, in which the number of iterations is given in unary/binary. The upper complexity bound is shown by a repetition of the proof of Theorem 3 and by using the fact that the set of satisfiable variable-free Δ_0 -formulas is in NP. \square

Theorem 5 (Complexity of Satisfiability for Flat Δ_0 +Rec). *The set of satisfiable flat Δ_0 +Rec formulas, which contain at most $k \geq 1$ recursive terms, is $NkExpTime$ -complete.*

Proof. Hardness follows from Lemma 5 and the construction from the proof of Lemma 1, which shows that for any $k \geq 1$ and $n \geq 0$, a $\mathbf{kexp}(n)$ -list is expressible by a flat term, which contains k recursive terms. The proof for the upper bound is similar to the proof of Theorem 3.

Let φ be a flat Δ_0 +Rec formula of the form

$$\partial_1 x_1 \propto_1 t_1 \dots \partial_n x_n \propto_n t_n \quad \psi(x_1, \dots, x_n)$$

where $n \geq 0$, $\vartheta_i \in \{\exists, \forall\}$, and $\alpha_i \in \{\in, \sqsubseteq\}$, for all $i = 1, \dots, n$. Let T denote the total size of the terms t_1, \dots, t_n . We will show by induction on the complexity of φ that satisfiability of φ can be decided by at most $\text{mexp}(p(|T|))$ -many tests for satisfiability of formulas ψ' obtained from ψ by substitutions of x_1, \dots, x_n with a vector of constant lists, each of size bounded by $\text{mexp}(p(T))$, where p is a polynomial and $m = \max(\text{rank}(t_1), \dots, \text{rank}(t_n))$. We refer to this claim further as $(*)$. It yields the statement of the theorem due to the following observation.

Since m is bounded by the number of recursive terms in the quantifier prefix of φ , the number of recursive terms in ψ' (and hence, their rank) is less or equal than $k - m$. Then it follows from the proof of Lemma 3 that satisfiability of φ can be decided by at most $\text{mexp}(p(T))$ -many tests for satisfiability of Δ_0 -formulas, each of which is either ψ' as above (if ψ does not contain non-standard terms) or obtained from ψ' as a value of a kExpTime -computable function and has size bounded by $\text{kexp}(r(|\varphi|))$, for some polynomials p, r . Since $m \leq k$, $T \leq |\varphi|$, and the set of satisfiable variable-free Δ_0 -formulas is in NP , we obtain the statement of the theorem.

Let us now show that $(*)$ holds. If φ is quantifier-free, there is nothing to prove. Now assume that φ has the form $\vartheta x \alpha t \theta(x)$, where $\vartheta \in \{\exists, \forall\}$ and $\alpha \in \{\in, \sqsubseteq\}$. It is equivalent to the formula $\varphi' = \vartheta x \alpha t_0 \theta(x)$, where t_0 is a constant list term equivalent to t . It follows from the proof of Lemma 3 that t_0 can be obtained as a value of a mExpTime -computable function, where m is the maximal rank of terms t_i , $i = 1, \dots, n$, in the quantifier prefix of φ . In particular, the number and the size of every list $a \alpha t_0$ is bounded by $\text{mexp}(p(T))$, for a polynomial p . Then satisfiability of φ can be decided by $\text{mexp}(p(T))$ -many tests of satisfiability of $\theta(a)$, one for each $a \alpha t_0$, and thus, by applying the induction assumption to the formulas $\theta(a)$, we obtain the claim. \square

5. CONCLUSIONS

We have shown that looping terms can succinctly represent exponentially long lists and can express Cartesian concatenation of lists, which may be the source of the increased computational complexity. For the latter operation, nested iteration over lists is required. If the number of iterations is bounded by some number k , then it is possible to implement Cartesian concatenation via iterative terms only for lists of k -bounded length. Thus, there remains a certain gap in understanding the expressiveness of iterative terms. On one hand, they can succinctly represent exponentially long lists (even when terms are flat and the number of iterations is given in unary), on the other hand, they allow for expressing Cartesian concatenation of lists only of polynomially bounded length. We leave it open whether the lower bound on the complexity of model checking for flat Δ_0 -formulas with iterative terms matches the upper bound shown in this paper. We have proved tight complexity bounds for satisfiability, which hint to the natural connection with model checking in terms of complexity. If model checking is complete in some complexity class (e.g., ExpTime), then typically satisfiability is complete in the non-deterministic variant of this class (i.e., NExpTime), and vice versa. In the paper we considered extensions of the language of Δ_0 -formulas with bounded search, iterative, and recursive terms as separate languages. In further research, we plan to study the interplay between non-standard terms, when they are used in formulas simultaneously, and to identify

syntactic restrictions on the form of terms and formulas which guarantee tractable reasoning.

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