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ON THE DE RHAM COMPLEX ON A SCALE OF ANISOTROPIC WEIGHTED HÖLDER SPACES

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Abstract. We obtain a solvabilty criterion for the operator equations induced by de Rham differentials on a scale of anisotropic weighted Hölder spaces on the strip $\mathbb{R}^n \times [0,T]$, $n \geq 1$, where the weight controls the behavior of elements at the infinity point with respect to the space variables. Besides, we give a description of the closures in these space of the set of infinitely differentiable functions on the strip $\mathbb{R}^n \times [0,T]$ that are compactly supported with respect to the space variables. The results are applied to study the properties of the famous Leray-Helmholtz projection from the theory of the Navier-Stokes equations on the scale of these weighted spaces for $n \geq 2$.

Keywords: weighted Hölder spaces, de Rham complex.

1. Introduction

The experience of many decades of studying integro-differential equations related to models of contemporary natural science shows that the choice of suitable function spaces for data and solutions plays often a key role. It appears that the normed spaces of functions with finite smoothness are not always fit for this purpose because the integral operators solving the investigated problems are not continuous. Now the Banach spaces most popular among the researchers, are the Lebesgue spaces, the Sobolev spaces the Hölder spaces, and their various generalizations (see, for instance, [1], [2], [3], [4], [5]). Though the Hölder spaces have evident advantages

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(their elements are continuous and whence they can be easily restricted to subsets of lesser dimension, the compositions of mappings are easily defined in these spaces and there is a natural product operation on the corresponding functions), they also have some considerable disadvantages (these spaces are not separable and not reflexive and their elements cannot be approximated by smooth functions in the Hölder topology, see, for instance, [3], [6]).

The present paper is devoted to the investigation of the differentials of the de Rham complex over $\mathbb{R}^n$ on a scale of anisotropic weighted Hölder spaces over the layer $\mathbb{R}^n \times [0,T]$ for $n \geq 1$ and with a weight controlling the asymptotic behavior at infinity with respect to the space variables, i.e., in the situation where the variable $t \in [0,T]$ occurs in the coefficients of differential forms on $\mathbb{R}^n$ as a parameter. This is partially motivated by the mathematical theory of the Navier-Stokes equations, used for description of the dynamics of viscous fluids where the decreasing of solutions at the infinity with respect to variables $x \in \mathbb{R}^n$ is justified by physical considerations, and the investigation of the operator equations involving the de Rham differentials (for instance, the operators ∇, rot and div) is necessary to describe properties of the famous Leray-Helmholtz projection and/or to exclude the pressure of the fluid from the consideration, see, for example, [7, Ch. 1, §2], [8].

Though the de Rham cohomologies are well investigated on the scale of Hölder spaces on smooth Riemannian manifolds, see, for instance, [9, §§10.2-10.4], the introduction of a weight and an additional parameter (the time) can essentially change the picture, see, for instance, the paper [10], where the Laplace operator and the heat operator were considered on weighted Hölder spaces on a Riemannian manifold with conical singularity.

Recently, in [11], we described necessary and sufficient solvability conditions for operator equations generated by the de Rham complex over $\mathbb{R}^n$ on weighted Hölder spaces both isotropic in $\mathbb{R}^n$ and anisotropic ("parabolic") in the layer $\mathbb{R}^n \times [0,T]$; the proofs were based on the technique elaborated in [12] and [13] for weighted Sobolev spaces. Unfortunately, the approach of [11] is appropriate for data from a closed subspace consisting of the elements from the image of the Laplace operator, that is not natural studying the cohomologies of the de Rham complex. This weak point was eliminated in [14] for isotropic weighted Hölder spaces on $\mathbb{R}^n$. Besides, a criterion for the approximability of weighted Hölder functions $\mathbb{R}^n$ by functions with compact support was obtained in [14] (cf. also [15, §1.3] for the approximability of Hölder functions on open subsets and compacts), which makes it possible to solve the corresponding operator equations on separable subspaces of the weighted Hölder spaces.

In the present paper, we extend the results of [14] to a scale of anisotropic weighted Hölder spaces on the layer $\mathbb{R}^n \times [0,T]$.

2. Anisotropic weighted Hölder spaces

Suppose that $\mathcal{X}$ is an unbounded domain in $\mathbb{R}^n$, $n \geq 1$, with piecewise smooth boundary. The weighted Hölder and Sobolev spaces are well known and widely used for solving boundary value problems for elliptic equations and systems, see, for instance, [11], [14] for $\mathcal{X} = \mathbb{R}^n$, [16] for polyhedral domains, [10] for Riemannian manifolds with conic singularities. In order to define the weighted Hölder spaces, we set

$$w(x) = \sqrt{1+|x|^2}, \ w(x,y) = \max\{w(x), w(y)\}, \ x, y \in \mathbb{R}^n.$$

For $s \in \mathbb{Z}_+$ and $\delta \in \mathbb{R}$ denote by $C^{s,0}_\delta(\overline{\mathcal{X}})$ the space of s times continuously differentiable functions on the closure $\overline{\mathcal{X}}$ of the domain $\mathcal{X}$ having the finite the norm

$$\|u\|_{C^{s,0}_\delta(\overline{\mathcal{X}})} = \sum_{|\alpha|\leq s} \sup_{x\in\overline{\mathcal{X}}} w^{\delta+|\alpha|}(x)|\partial^\alpha u(x)|.$$

Let V be the empty set if $\overline{\mathcal{X}}$ does not contain the origin; otherwise let $V \subset \overline{\mathcal{X}}$ be a nonempty bounded neighborhood of the origin in the relative topology of $\overline{\mathcal{X}}$. For $0 < \lambda \leq 1$, we set

$$\langle u\rangle_{\lambda,\delta,\overline{\mathcal{X}}} = \sup_{\substack{x,y\in\overline{\mathcal{X}}\setminus V, x\neq y\\ |x-y|\leq|x|/2}} w^{\delta+\lambda}(x,y)\frac{|u(x)-u(y)|}{|x-y|^\lambda}.$$

Denote by $C^{0,\lambda}_\delta(\overline{\mathcal{X}})$ the set of continuous functions on $\overline{\mathcal{X}}$ with the finite norm

$$\|u\|_{C^{0,\lambda}_\delta} = \|u\|_{C^{0,\lambda}(\overline{V})} + \|u\|_{C^{0,0}_\delta(\overline{\mathcal{X}})} + \langle u\rangle_{\lambda,\delta,\overline{\mathcal{X}}},$$

where $\|\cdot\|_{C^{0,\lambda}(\overline{V})} = \|\cdot\|_{C^{0,0}(\overline{V})}| + \langle\cdot\rangle_{\lambda,\overline{V}}$ is the norm of the usual Hölder space $C^{0,\lambda}(\overline{V})$ on the compact set $\overline{V}$, see, for instance, [2], [3], [5]. Finally, for $s \in \mathbb{Z}_+$, let $C^{s,\lambda}_\delta(\overline{\mathcal{X}})$ stand for the space of all s times continuously differentiable functions on $\overline{\mathcal{X}}$ with the finite norm

$$\|u\|_{C^{s,\lambda}_\delta(\overline{\mathcal{X}})} = \sum_{|\alpha|\leq s} \|\partial^\alpha u\|_{C^{0,\lambda}_{\delta+|\alpha|}(\overline{\mathcal{X}})}.$$

The normed spaces $C^{s,\lambda}_\delta(\overline{\mathcal{X}})$, obviously, form a scale of Banach spaces parametrized by the indices $s \in \mathbb{Z}_+$, $\lambda \in [0,1]$ and $\delta \in \mathbb{R}$. This construction becomes natural if after a suitable compactification of the domain $\mathcal{X}$ is considered as a closed Riemannian manifold with a singular point corresponding to infinity (cf. the pioneer paper on the weighted spaces of this kind [17] for parabolic cylindrical domains in $\mathbb{R}^n \times (0,+\infty)$ or more general constructions on Riemannian manifolds with conic singularities in [10]). The first summand in the norm corresponds to a coordinate chart of the nonsingular part of the manifold, and the second and third summands correspond to a coordinate chart of the point at the infinity. We do not need to glue the summands with the use of a partition of unity in $\overline{\mathcal{X}}$, because there are global coordinates in $\mathbb{R}^n$.

If $\mathcal{X} = \mathbb{R}^n$ then we set $V = B_2$, where B_R is the ball in $\mathbb{R}^n$ of radius $R > 0$ centered at the origin. We use the notation $C^{s,\lambda}_\delta$ for $C^{s,\lambda}_\delta(\mathbb{R}^n)$. The space $C^{s,\lambda}_\delta$ is continuously embedded into the Fréchet space $C^{s,\lambda}_{\mathrm{loc}}(\mathbb{R}^n)$. By definition, each (partial) derivative ∂^α maps $C^{s,\lambda}_\delta$ continuously into $C^{s-|\alpha|,\lambda}_{\delta+|\alpha|}$ if $|\alpha| \leq s$. Therefore, the following embedding theorem is expectable:

Theorem 1 (see [11], [14])**.** *Assume that $s, s' \in \mathbb{Z}_+$, $\delta, \delta' \in \mathbb{R}_+$ и $\lambda, \lambda' \in [0,1]$. If $\delta \leq \delta'$, $s \leq s'$ and $s+\lambda \leq s'+\lambda'$ then the space $C^{s',\lambda'}_{\delta'}$ is embedded continuously into the space $C^{s,\lambda}_\delta$. The embedding is compact if $\delta < \delta'$ and $s+\lambda < s'+\lambda'$.*

Moreover, in [14], a description was obtained for the closure of the space $\mathcal{D}(\mathbb{R}^n)$ of infinitely differentiable functions with compact support in $\mathbb{R}^n$ in the Banach space $C^{s,\lambda}_\delta$. Namely, let $0 < \lambda < 1$ and let $\mathfrak{C}^{s,\lambda}_\delta$ stand for the subset in $C^{s,\lambda}_\delta$, with elements satisfying the following properties: for any $\varepsilon > 0$ there is such number $\gamma_\varepsilon > 0$ that

$$\sum_{|\alpha|\le s} \langle \partial^\alpha u\rangle_{\lambda,\delta+|\alpha|} < \varepsilon \text{ if } \frac{|x-y|}{w(x,y)} < \gamma_\varepsilon; \tag{1}$$

and, in addition,

$$\lim_{R\to\infty} \|u\|_{C^{s,\lambda}_\delta(\mathbb{R}^n\setminus B_R)} = 0. \tag{2}$$

Theorem 2 (see [14]). *Let $0 < \lambda < 1$, $s, s' \in \mathbb{Z}_+$ and $\delta' > \delta$. If $0 \le \lambda' \le 1$, $s \le s'$ and $s + \lambda < s' + \lambda'$ then the closures of the sets $C^{s',\lambda'}_{\delta'}$ and $\mathcal{D}(\mathbb{R}^n)$ in $C^{s,\lambda}_\delta$ coincide with $\mathfrak{C}^{s,\lambda}_\delta$.*

By analogy with the usual anisotropic Hölder spaces (see, for instance, [18]) weighted anisotropic Hölder spaces were introduced in [11] for a cylindrical domain $\mathcal{X} \times (0,T)$; cf. [10] for the case where the base $\mathcal{X}$ is a Riemannian manifold with conical singularity. The peculiarity of this space consists in the fact that though the parabolic dilation principle is fulfilled as regards the smoothness of the elements, it is violated with respect to the weight indices. This was done to ensure under some additional conditions the continuity of both elliptic and parabolic potentials on the scale of spaces, see [11, §3,§4].

Namely, we set

$$\|v\|_{C^{0,0}[0,T]} = \sup_{t\in[0,T]} |v(t)|, \quad \langle v\rangle_{\mu,[0,T]} = \sup_{\substack{t\neq\tau\\ t,\tau\in[0,T]}} \frac{|v(t)-v(\tau)|}{|t-\tau|^\mu},$$

$$\|v\|_{C^{0,\mu}[0,T]} = \|v\|_{C^{0,0}[0,T]} + \langle v\rangle_{\mu,[0,T]}$$

for functions defined on the segment $[0,T]$. For $\mu \in [0,1]$, $s \in \mathbb{Z}_+$ let $C^{s,\mu}([0,T])$ be the usual Banach space of functions satisfying the Hölder condition on the segment $[0,T]$ endowed with the norm

$$\|v\|_{C^{s,\mu}[0,T]} = \sum_{j=0}^{s} \left\| \frac{d^j v}{dt^j} \right\|_{C^{0,\mu}[0,T]}.$$

Now, as usual, if $\mathcal{B}$ is a Banach space we denote by $C^{s,0}([0,T],\mathcal{B})$ the Banach space of mappings $U : [0,T] \to \mathcal{B}$ with the finite norm

$$\|U\|_{C^{s,0}([0,T],\mathcal{B})} = \sum_{j=0}^{s} \sup_{t\in[0,T]} \|\partial_t^j U(\cdot,t)\|_{\mathcal{B}}, \ s \in \mathbb{Z}_+.$$

We also set

$$\langle U\rangle_{\mu,[0,T],\mathcal{B}} = \sup_{\substack{t\neq\tau\\ t,\tau\in[0,T]}} \frac{\|U(\cdot,t)-U(\cdot,\tau)\|_{\mathcal{B}}}{|t-\tau|^\mu}.$$

Let $C^{s,\mu}([0,T],\mathcal{B})$ stand for the space of functions from $C^{s,0}([0,T],\mathcal{B})$, $s \in \mathbb{Z}_+$, with the finite norm

$$\|U\|_{C^{s,\mu}([0,T],\mathcal{B})} = \sum_{j=0}^{s} \left(\sup_{t\in[0,T]} \|\partial_t^j U(\cdot,t)\|_{\mathcal{B}} + \langle \partial_t^j U(\cdot,t)\rangle_{\mu,[0,T],\mathcal{B}} \right).$$

Let $C^{0,0,0,0}_{\delta,T}(\overline{\mathcal{X}}) = C^{0,0}([0,T], C^{0,0}_\delta(\overline{\mathcal{X}}))$ stand for the space of continuous function on the "layer" $\overline{\mathcal{X}} \times [0,T]$ with the finite norm

$$\|u\|_{C^{0,0,0,0}_{\delta,T}(\overline{\mathcal{X}})} = \sup_{(x,t)\in\overline{\mathcal{X}}\times[0,T]} |u(x,t)| w^\delta(x).$$

For $s \in \mathbb{N}$, denote by

$$C^{2s,s,0,0}_{\delta,T}(\overline{\mathcal{X}}) = \bigcap_{j=0}^{s} C^{j,0}([0,T], C^{2(s-j),0}_{\delta}(\overline{\mathcal{X}}))$$

the space of continuous functions on $\overline{\mathcal{X}} \times [0,T]$ having continuous partial derivatives $\partial^\alpha_x \partial^j_t u$ for $|\alpha| + 2j \leq 2s$, $s \in \mathbb{Z}_+$, with the finite norm

$$\|u\|_{C^{2s,s,0,0}_{\delta,T}(\overline{\mathcal{X}})} = \sum_{|\alpha|+2j\leq 2s} \|\partial^j_t \partial^\alpha_x u\|_{C^{0,0,0,0}_{\delta+|\alpha|,T}(\overline{\mathcal{X}})}.$$

Next, for $\lambda \in (0,1]$, $\mu \in [0,1]$, let

$$C^{0,0,\lambda,\mu}_{\delta,T}(\overline{\mathcal{X}}) = C^{0,0}([0,T], C^{0,\lambda}_{\delta}(\overline{\mathcal{X}})) \cap C^{0,\mu}([0,T], C^{0,0}_{\delta}(\overline{\mathcal{X}}))$$

be the space of continuous functions on $\overline{\mathcal{X}} \times [0,T]$ with the finite norm

$$\|u\|_{C^{0,0,\lambda,\mu}_{\delta,T}(\overline{\mathcal{X}})} = \sup_{t\in[0,T]} \|u(\cdot,t)\|_{C^{0,0}_{\delta}(\overline{\mathcal{X}})} + \langle u \rangle_{\lambda,\mu,\delta,\overline{\mathcal{X}},T}$$

where

$$\langle u \rangle_{\lambda,\mu,\delta,\overline{\mathcal{X}},T} = \begin{cases} \sup\limits_{t\in[0,T]} \langle u(\cdot,t)\rangle_{\lambda,\delta,\overline{\mathcal{X}}}, & \mu = 0, \\ \sup\limits_{t\in[0,T]} \langle u(\cdot,t)\rangle_{\lambda,\delta,\overline{\mathcal{X}}} + \sup\limits_{\substack{t\neq\tau \\ t,\tau\in[0,T]}} \frac{\|u(\cdot,t)-u(\cdot,\tau)\|_{C^{0,0}_{\delta}(\overline{\mathcal{X}})}}{|t-\tau|^{\mu}}, & \mu \in (0,1]. \end{cases}$$

Similarly, for $\lambda \in (0,1]$, $\mu \in [0,1]$ denote by

$$C^{2s,s,\lambda,\mu}_{\delta,T}(\overline{\mathcal{X}}) = \bigcap_{j=0}^{s} \Big(C^{j,0}([0,T], C^{2(s-j),\lambda}_{\delta}(\overline{\mathcal{X}})) \cap C^{j,\mu}([0,T], C^{2(s-j),0}_{\delta}(\overline{\mathcal{X}})) \Big)$$

the space of continuous functions on $\overline{\mathcal{X}} \times [0,T]$ having continuous partial derivatives $\partial^\alpha_x \partial^j_t u$ for $|\alpha| + 2j \leq 2s$, $s \in \mathbb{Z}_+$, with the finite norm

$$\|u\|_{C^{2s,s,\lambda,\mu}_{\delta,T}(\overline{\mathcal{X}})} = \sum_{|\alpha|+2j\leq 2s} \|\partial^j_t \partial^\alpha_x u\|_{C^{0,0,\lambda,\mu}_{\delta+|\alpha|,T}(\overline{\mathcal{X}})}.$$

Finally, for $s \in \mathbb{Z}_+$, $k \in \mathbb{N}$, $\lambda, \mu \in [0,1]$, let $C^{2s+k,s,\lambda,\mu}_{\delta,T}(\overline{\mathcal{X}})$ be the space of continuous functions on $\overline{\mathcal{X}} \times [0,T]$ having continuous partial derivatives $\partial^{\alpha+\beta}_x \partial^j_t u$ for $|\alpha| + 2j \leq 2s$ and $|\beta| \leq k$, with the finite norm

$$\|u\|_{C^{2s+k,s,\lambda,\mu}_{\delta,T}(\overline{\mathcal{X}})} = \sum_{|\beta|\leq k} \|\partial^\beta_x u\|_{C^{2s,s,\lambda,\mu}_{\delta+|\beta|,T}(\overline{\mathcal{X}})}.$$

It is convenient to identify $C^{2s,s,\lambda,\mu}_{\delta,T}(\overline{\mathcal{X}})$ with $C^{2s+k,s,\lambda,\mu}_{\delta,T}(\overline{\mathcal{X}})$ for $k = 0$. Clearly, $C^{2s+k,s,\lambda,\mu}_{\delta,T}(\overline{\mathcal{X}})$ are Banach spaces.

If $\mathcal{X} = \mathbb{R}^n$ then we will write simply $C^{2s+k,s,\lambda,\mu}_{\delta,T}(\mathbb{R}^n) = C^{2s+k,s,\lambda,\mu}_{\delta,T}$. The natural cases for (Petrovskii) parabolic operators of order $2m$ correspond to $\mu = 0$ or $\mu = \frac{\lambda}{2m}$. In this paper, we will concentrate on the simpler case of $\mu = 0$; this case is more convenient for investigating the de Rham differentials.

The following embedding theorems are quite expectable:

Theorem 3 (see [11]). *Let $s, s' \in \mathbb{Z}_+$, $\delta, \delta' \in \mathbb{R}_+$, $k \in \mathbb{Z}_+$ and $\lambda, \lambda' \in [0,1]$. If $\delta \leq \delta'$, $s \leq s'$, $s + \lambda \leq s' + \lambda'$ then the space $C^{2s'+k,s',\lambda',\frac{\lambda'}{2}}_{\delta',T}$ is embedded continuously into $C^{2s+k,s,\lambda,\frac{\lambda}{2}}_{\delta,T}$. The embedding is compact if $s + \lambda < s' + \lambda'$ and $\delta < \delta'$.*

Theorem 4. *Let $s, s' \in \mathbb{Z}_+$, $\delta, \delta' \in \mathbb{R}_+$, $k \in \mathbb{Z}_+$ and $\lambda, \lambda' \in [0,1]$. If $\delta' \geq \delta$, $s' \geq s$, $s'+\lambda' \geq s+\lambda$ then the space $C^{2s'+k,s',\lambda',0}_{\delta',T}$ is embedded continuously into $C^{2s+k,s,\lambda,0}_{\delta,T}$. $s' > s$, $s'+\lambda' > s+\lambda$ The embedding is compact if and $\delta' > \delta$.*

Proof. The assertion on the continuous embedding follows immediately from the definition of the space and Theorem 1. The compactness of the embedding follows from Theorem 3. Indeed, fix indices δ, δ', $\delta' > \delta$. By the Lagrange theorem on finite differences, the space $C^{2(s+1)+k,s+1,\lambda',0}_{\delta',T}$ is embedded continuously into the space $C^{2s+k,s,1,\frac{1}{2}}_{\delta',T}$, and this space is embedded compactly into $C^{2s+k,s,\lambda,\frac{\lambda}{2}}_{\delta,T}$ if $0 \leq \lambda < 1$. Since $C^{2s+k,s,\lambda,\frac{\lambda}{2}}_{\delta,T}$ is embedded continuously into $C^{2s+k,s,\lambda,0}_{\delta',T}$, the space $C^{2(s+1)+k,s+1,\lambda',0}_{\delta',T}$ is embedded compactly into $C^{2s+k,s,\lambda,0}_{\delta',T}$ if $0 \leq \lambda < 1$. This precisely corresponds to the assumption that $s+1+\lambda' > s+\lambda$ for $\lambda' \in [0,1]$.

As the space $C^{2s'+k,s',\lambda',0}_{\delta',T}$ is embedded continuously into $C^{2(s+1)+k,s+1,\lambda',0}_{\delta',T}$ for $s' > s$, then we conclude that $C^{2s'+k,s',\lambda',0}_{\delta',T}$ embeds compactly into $C^{2s+k,s,\lambda,0}_{\delta',T}$ if $0 \leq \lambda < 1$.

Finally, if $s'-s \geq 2$, $\lambda = 1$, then the space $C^{2s'+k,s',\lambda',0}_{\delta',T}$ is embedded continuously into $C^{2(s+2)+k,s+2,\lambda',0}_{\delta',T}$, the space $C^{2(s+2)+k,s+2,\lambda',0}_{\delta',T}$ is embedded compactly into $C^{2(s+1)+k,s+1,\frac{1}{2},0}_{\delta,T}$, and $C^{2(s+1)+k,s+1,\frac{1}{2},0}_{\delta,T}$ is embedded continuously into $C^{2s+k,s,1,0}_{\delta,T}$. The theorem is proved. □

We are ready to formulate and to prove an analog of Theorem 2 for the scale of anisotropic weighted Hölder spaces $C^{2s+k,s,\lambda,0}_{\delta}$. Denote by $C^\infty([0,T], \mathcal{D}(\mathbb{R}^n))$ the space of infinitely differentiable mappings $U : [0,T] \to \mathcal{D}(\mathbb{R}^n)$. Let $0 < \lambda < 1$ and let $\mathfrak{C}^{2s+k,s,\lambda,0}_{\delta,T}$ for the subset in $C^{2s+k,s,\lambda,0}_{\delta}$ with elements satisfying the following properties: for any $\varepsilon > 0$ there is such a $\gamma_\varepsilon > 0$ that

$$(3) \qquad \sum_{|\beta|\leq k} \sum_{|\alpha|+2j\leq 2s} \langle \partial_x^\beta \partial_x^\alpha \partial_t^j u \rangle_{\lambda,\mu,\delta+|\alpha+\beta|,T} < \varepsilon \text{ if } \frac{|x-y|}{w(x,y)} < \gamma_\varepsilon;$$

and, moreover,

$$(4) \qquad \lim_{R\to\infty} \|u\|_{C^{2s+k,s,\lambda,0}_{\delta,T}(\mathbb{R}^n \setminus B_R)} = 0.$$

Theorem 5. *Let $0 < \lambda < 1$, $k, s, s' \in \mathbb{Z}_+$ and $\delta' > \delta$. If $0 \leq \lambda' \leq 1$, $s \leq s'$, and $s+\lambda < s'+\lambda'$, then the closures of the sets $C^{2s'+k,s',\lambda',0}_{\delta',T}$ and $C^\infty([0,T], \mathcal{D}(\mathbb{R}^n))$ in the space $C^{2s+k,s,\lambda,0}_{\delta,T}$ coincide with $\mathfrak{C}^{2s+k,s,\lambda,0}_{\delta,T}$.*

Proof. First, we note that the continuous embedding $C^{2s'+k,s',\lambda',0}_{\delta',T} \hookrightarrow C^{2s+k,s,\lambda,0}_{\delta,T}$ holds by Theorem 4. If $s' = s$ and $\lambda' > \lambda$ then, for any multi-index $(\alpha, j) \in \mathbb{Z}^{n+1}_+$, satisfying $|\alpha| + 2j \leq 2s$, any $\beta \in \mathbb{Z}^n_+$, satisfying $|\beta| \leq k$, and any $u \in C^{2s'+k,s',\lambda',0}_{\delta'}$, $\delta' \geq \delta$, we have

$$\langle \partial_x^{\alpha+\beta} \partial_t^j u \rangle_{\lambda,0,\delta+|\alpha+\beta|,T} \leq \langle \partial_x^{\alpha+\beta} \partial_t^j u \rangle_{\lambda',0,\delta+|\alpha+\beta|,T} \Big(\frac{|x-y|}{w(x,y)} \Big)^{\lambda'-\lambda}.$$

If $s' > s$ then, for all the multi-indices mentioned above and any $u \in C^{2s'+k,s',\lambda',0}_{\delta'}$, $\delta' \geq \delta$, we have

$$\langle \partial_x^{\alpha+\beta} \partial_t^j \rangle_{\lambda,0,\delta+|\alpha+\beta|,T} \leq \sum_{i=1}^n \|\partial_i \partial_x^{\alpha+\beta} u\|_{C^{0,0}_{\delta+|\alpha+\beta|+1,T}} \Big(\frac{|x-y|}{w(x,y)} \Big)^{1-\lambda}.$$

This shows that the elements of the space $C^{2s'+k,s',\lambda',0}_{\delta',T}$ satisfy (3) if $s'+\lambda' > s+\lambda$. Moreover, if $\delta' > \delta$ and $u \in C^{s',\lambda'}_{\delta'}$ then, for all admissible multi-indices (α, j) and β, we have:

$$\sup_{t\in[0,T]} \sup_{|x|\geq R} |\partial_x^{\alpha+\beta} \partial_t^j u(x,t)| w^{\delta+|\alpha+\beta|}(x) \leq \|u\|_{C^{2s+k,s,0,0}_{\delta'}} w^{\delta-\delta'}(R),$$

$$\sup_{t\in[0,T]} \langle \partial_x^{\alpha+\beta} \partial_t^j u \rangle_{\lambda,\delta+|\alpha+\beta|,\mathbb{R}^n\setminus B_R} \leq \sup_{t\in[0,T]} \langle \partial_x^{\alpha+\beta} \partial_t^j u \rangle_{\lambda',\delta'+|\alpha+\beta|} \Big(\frac{|x-y|}{w(x,y)} \Big)^{\lambda'-\lambda} w^{\delta-\delta'}(R).$$

This shows that the elements of the space $C^{2s'+k,s',\lambda',0}_{\delta',T}$ satisfy (4) if $\mu = 0$, $s'+\lambda' > s+\lambda$ and $\delta' > \delta$, i.e. in this case $C^{2s'+k,s',\lambda',0}_{\delta',T} \subset \mathfrak{C}^{2s+k,s,\lambda,0}_{\delta,T}$. Clearly, $C^\infty([0,T], \mathcal{D}(\mathbb{R}^n)) \subset C^{2s'+k,s',\lambda',0}_{\delta',T}$ for any $s' \in \mathbb{Z}_+$, $0 \leq \lambda' \leq 1$, $\delta' \in \mathbb{R}$, and hence $C^\infty([0,T], \mathcal{D}(\mathbb{R}^n)) \subset \mathfrak{C}^{2s+k,s,\lambda,0}_{\delta,T}$.

Let us check that $\mathfrak{C}^{2s+k,s,\lambda,0}_{\delta,T}$ is a closed subspace in $C^{2s+k,s,\lambda,0}_{\delta,T}$. Indeed, the set is obviously linear. Choose a sequence $\{u_\nu\} \subset \mathfrak{C}^{2s+k,s,\lambda,0}_{\delta,T}$ converging to a limit u in $C^{2s+k,s,\lambda,0}_{\delta,T}$. Then, for any multi-index $(\alpha, j) \in \mathbb{Z}_+^{n+1}$ with $|\alpha| + 2j \leq 2s$ and any $\beta \in \mathbb{Z}_+^n$ with $|\beta| \leq k$, we have

$$\langle \partial_x^{\alpha+\beta} \partial_t^j u \rangle_{\lambda,0,\delta+|\alpha+\beta|,T} \leq \langle \partial_x^{\alpha+\beta} \partial_t^j (u - u_\nu) \rangle_{\lambda,\delta+|\alpha|} + \langle \partial_x^{\alpha+\beta} \partial_t^j u_\nu \rangle_{\lambda,0,\delta+|\alpha+\beta|,T}. \tag{5}$$

By the definition of limit in $C^{2s+k,s,\lambda,0}_{\delta,T}$, for every $\varepsilon > 0$ there is a number $N_\varepsilon \in \mathbb{N}$ such that for any $\nu \geq N_\varepsilon$ we have

$$\sum_{|\alpha|\leq s} \langle \partial_x^{\alpha+\beta} \partial_t^j (u - u_\nu) \rangle_{\lambda,0\delta+|\alpha|,T} \leq \|u - u_\nu\|_{C^{2s+k,s,\lambda,0}_{\delta,T}} < \frac{\varepsilon}{2}. \tag{6}$$

Since $u_{N_\varepsilon} \in \mathfrak{C}^{2s+k,s,\lambda,0}_{\delta,T}$ then, using (3), (5) and (6), we conclude that u satisfies (3), too. Besides, for any $\varepsilon > 0$ there is a number $N_\varepsilon \in \mathbb{N}$ such that for all $\nu \geq N_\varepsilon$ and all $R > 0$ we have

$$\|u - u_\nu\|_{C^{2s+k,s,\lambda,0}_{\delta,T}(\mathbb{R}^n\setminus B_R)} \leq \|u - u_\nu\|_{C^{2s+k,s,\lambda,0}_{\delta,T}} < \frac{\varepsilon}{2}. \tag{7}$$

Moreover, by the triangle inequality, for any $\nu \in \mathbb{N}$,

$$\|u\|_{C^{2s+k,s,\lambda,0}_{\delta,T}(\mathbb{R}^n\setminus B_R)} \leq \|u - u_\nu\|_{C^{2s+k,s,\lambda,0}_{\delta,T}(\mathbb{R}^n\setminus B_R)} + \|u_\nu\|_{C^{2s+k,s,\lambda,0}_{\delta,T}(\mathbb{R}^n\setminus B_R)}. \tag{8}$$

Since $u_{N_\varepsilon} \in \mathfrak{C}^{2s+k,s,\lambda,0}_{\delta,T}$ then, using (4), (7) and (8), we see that $u \in \mathfrak{C}^{2s+k,s,\lambda,0}_{\delta,T}$. Thus, we have proved that the closures of the sets $C^{2s'+k,s',\lambda',0}_{\delta',T}$ and $C^\infty([0,T], \mathcal{D}(\mathbb{R}^n))$ lie in $\mathfrak{C}^{2s+k,s,\lambda,0}_{\delta,T}$. It remains to prove that $\mathfrak{C}^{2s+k,s,\lambda,0}_{\delta,T}$ lie in the closure of the space $C^\infty([0,T], \mathcal{D}(\mathbb{R}^n))$.

For bounded domains in $\mathbb{R}^n$ the construction of the corresponding approximating sequences was given in [15, §1.3]. It was shown in [14] that, using a suitable compactification of $\mathbb{R}^n$, one can reduce the problem of approximating Hölder functions from the isotropic weighted space to the similar problem on relatively compact submanifold with a conic singular point in $\mathbb{R}^{n+1}$ with the induced metric. In the case of anisotropic spaces, the arguments become more complicated but the general scheme is the same. More precisely, we immerse $\mathbb{R}^n$ into a compact closed hypersurface $\mathcal{S} = \mathcal{S}_{-} \cup \mathcal{S}_0 \cup \mathcal{S}_{+}$ with conical singularity in the coordinates $(z^0, z^1, \dots, z^n) \in \mathbb{R} \times \mathbb{R}^n$, where $\mathcal{S}_{-} = \{|z| = 1, -1 \le z_0 \le 0\}$ is the lower semisphere, $\mathcal{S}_{+} = \{1 - z_0 = |(z^1, \dots, z^n)|, 1/3 \le z_0 \le 1\}$ is a cone, and the surface $\mathcal{S}_0 = \{z_0 = \rho(z^1, \dots, z^n), 2/3 \le |(z^1, \dots, z^n)| \le 1\}$ is chosen in such a way that the hypersurface $\mathcal{S} \setminus (1, 0 \dots, 0)$ is infinitely smooth. If we denote $\mathbb{R}^n \cup \{\infty\}$ by $\hat{\mathbb{R}}^n$ then the immersion $\iota : \hat{\mathbb{R}}^n \times [0, T] \to \mathbb{R}^{n+2}$ is given as follows:

$$\iota(x,t) = \begin{cases} \Big(-\sqrt{1-|x|^2}, x, t\Big), & \text{if} \quad |x| \le 1, \\ \Big(\rho\left(\frac{x}{|x|^2}\right), \frac{x}{|x|^2}, t\Big), & \text{if} \quad 1 < |x| < 3/2, \\ \Big(1 - \frac{1}{|x|}, \frac{x}{|x|^2}, t\Big), & \text{if} \quad |x| \ge 3/2, \\ (1, 0, \dots, 0, t), & \text{if} \quad x = \infty, \end{cases}$$

with a smooth invertible function ρ, and the inverse mapping is given by

$$\iota^{-1}(z,t) = \begin{cases} (z^1, \dots, z^n, t), & \text{if} \quad -1 \le z_0 \le 0, \\ \Big(\frac{(z^1, \dots, z^n)}{(\rho^{-1}(z_0))^2}, t\Big) & \text{if} \quad 0 < z_0 \le 1/3, \\ \Big(\frac{(z^1, \dots, z^n)}{(1-z^0)^2}, t\Big) & \text{if} \quad 1/3 \le z^0 < 1, \\ (\infty, t) & \text{if} \quad z = (1, 0, \dots, 0). \end{cases}$$

The mapping ι is obviously continuous on $\hat{\mathbb{R}}^n \times [0, T]$, smooth and nonsingular on $\Big(\mathcal{S} \setminus (1, 0 \dots, 0)\Big) \times [0, T]$. The function $d((x,t),(y,\tau)) = |\iota(x,t) - \iota(y,\tau)|$ is a metric on $\hat{\mathbb{R}}^n \times [0, T] \cong \mathcal{S} \times [0, T]$.

Let $u \in \mathfrak{C}^{2s+k,s,\lambda,0}_{\delta,T}$. Using the standard regularization (see, for instance, [19]), we construct a family $\{u_r\}_{r>0} \subset C^\infty([0,T], \mathcal{D}(\mathbb{R}^n))$ convergent to u in the space $C^{2s+k,s,\lambda,0}_{\delta,T}$ if $r \to +0$. With this purpose, we fix a C^∞-smooth function ψ with the support in $B_1 \times [0,1] \subset \mathbb{R}^n \times [0,1]$, normalized by the condition

$$\int_0^1 \int_{\mathbb{R}^n} \psi(x,t) dx\, dt = 1. \tag{9}$$

According to the famous Bochner's Lemma, for each $0 < r < 2/3$ there is a smooth function $\eta_r(y)$ on $[-1, 1]$ with the support on the segment $[-1, 1-r]$ and such that $0 \le \eta_r \le 1$, $\eta_r(y) = 1$ on $[-1, 1-2r]$ and

$$\Big|\frac{d^k \eta_r}{dy^k}(y)\Big| \le c_k \Big(\frac{1}{r}\Big)^k \tag{10}$$

for all $k \in \mathbb{N}$ with a constant c_k independent on y. Set

$$u_r(x,t) = r^{-n-1} \eta_r(\iota_0(x,t)) \int_0^T \int_{\mathbb{R}^n} u(y,\tau)\, \psi\Big(\frac{(y-x, \tau-t)}{r}\Big) dy d\tau,$$

where ι_0 is the component of the mapping ι indexed by 0.

Since η_r belongs to $\mathcal{D}(\mathbb{R}^n)$ then the Leibniz rule implies that $u_r \in C^\infty([0,T], \mathcal{D}(\mathbb{R}^n))$. On the other hand, after an obvious change of variables,

$$u_r(x,t) = \eta_r(\iota_0(x,t)) \int_0^1 \int_{|y| \le 1} u(ry + x, r\tau + t) \psi(y,\tau) dy d\tau.$$

Property (9) means that

$$u_r(x,t) - u(x,t) = \tag{11}$$

$$\eta_r(\iota_0(x,t)) \int\limits_0^1 \int\limits_{|y|\leq 1} \Big(u(ry+x, r\tau+t) - u(x,t)\Big)\psi(y,\tau)dyd\tau + (1-\eta_r(\iota_0(x,t)))u(x,t).$$

Now, it follows from the properties of the function η_r and from (1) that for all the admissible multi-indices $(\alpha, j) \in \mathbb{Z}_+^{n+1}$ and $\beta \in \mathbb{Z}_+^n$, and $|x| \leq 1$, $0 < r < \frac{2}{3}$ we have

$$\partial_x^{\alpha+\beta}\partial_t^j(u_r(x,t) - u(x,t)) = \tag{12}$$

$$\int_0^1 \int_{|y|\leq 1} \partial_x^{\alpha+\beta}\partial_t^j \Big(u(ry+x, r\tau+t) - u(x,t)\Big)\psi(y,\tau)dyd\tau,$$

$$|\partial_x^{\alpha+\beta}\partial_t^j(u(ry+x, r\tau+t) - u(x,t))| \leq \tag{13}$$

$$|\partial_x^{\alpha+\beta}\partial_t^j(u(ry+x, r\tau+t) - u(x, r\tau+t))| + |\partial_x^{\alpha+\beta}\partial_t^j(u(x, r\tau+t) - u(x,t))| \leq$$

$$|ry|^\lambda \sup_{t\in[0,T]} \langle \partial^{\alpha+\beta} u(\cdot,t)\rangle_{\lambda,\overline{B}_2} + \sup_{|x|\leq 1} |\partial_x^{\alpha+\beta}\partial_t^j(u(x, r\tau+t) - u(x,t))|.$$

Therefore, formulas (9), (12), (13) and the properties of the standard regularization (see, for instance, [19]) imply that

$$\lim_{r\to 0+} \|u_r - u\|_{C^{2s+k,s,0,0}(\overline{B}_2)} = 0. \tag{14}$$

After straightforward calculations, we obtain for any $\gamma \in \mathbb{Z}_+^n$ that

$$|\partial_x^\gamma \iota_0(x,t)| \leq c_\gamma / |x|^{|\gamma|+1} \tag{15}$$

with a constant c_γ independent on x. Since that function $\eta_r(\iota_0(x,t))$ equals identically to 1 if $|x| \leq 1/(2r)$, relations (10), (15) imply that the following equality and inequality are fulfilled for $\gamma \neq 0$:

$$\sup_{\substack{|x|\geq 2\\ t\in[0,T]}} |\partial_x^\gamma \eta_r(\iota_0(x,t))| = \sup_{\substack{|x|\geq (2r)^{-1}\\ t\in[0,T]}} |\partial_x^\gamma \eta_r(\iota_0(x,t))| \leq \tilde{c}_\gamma |x|^{-|\gamma|} \tag{16}$$

with a constant $\tilde{c}_\gamma$ independent on r. Thus, as ι_0 does not depend on t, formulas (3) and (16) imply that, for all admissible multi-indices $(\alpha, j) \in \mathbb{Z}_+^{n+1}$ and $\beta \in \mathbb{Z}_+^n$, we have

$$\sup_{\substack{|x|\geq 2\\ t\in[0,T]}} w^{\delta+|\alpha+\beta|}(x) \Big|\partial_x^{\alpha+\beta}\partial_t^j \Big(\eta_r(\iota_0(x,t))\,(u(ry+x, r\tau+t) - u(x,t)\Big)\Big| \leq \tag{17}$$

$$c_{\alpha,\beta,j} |ry|^\lambda \sup_{t\in[0,T]} \langle \partial^{\alpha+\beta} u\rangle_{\lambda,\delta+|\alpha+\beta|} +$$

$$c_{\alpha,\beta,j} \sup_{\substack{|x|\geq 2\\ t\in[0,T]}} w^{\delta+|\alpha+\beta|}(x) |\partial_x^{\alpha+\beta}\partial_t^j(u(x, r\tau+t) - u(x,t))|,$$

$$\sup_{\substack{|x|\geq 2\\ t\in[0,T]}} w^{\delta+|\alpha+\beta|}(x) \Big|\partial_x^{\alpha+\beta}\partial_t^j \Big((1-\eta_r(\iota_0(x,t)))u(x,t)\Big)\Big| \leq \tag{18}$$

$$c_{\alpha,\beta} \|\partial_x^{\alpha+\beta}\partial_t^j u\|_{C^{0,0,0,0}_{\delta+|\alpha+\beta|,T}(\mathbb{R}^n \setminus B_{\frac{1}{2r}})}$$

with a constant $c_{\alpha,\beta,j}$ independent on r, $0 < r < 2/3$. Now, using the Leibniz rule, formula (4) with $\mu = 0$, and formulas (9), (14) – (18), we conclude that

$$\lim_{r\to 0+} \|u_r - u\|_{C^{2s+k,s,0,0}_\delta} = 0. \tag{19}$$

Next, it is not difficult to prove the inequality:

$$\frac{|x|}{2} \le |y| \le \frac{3}{2}|x|,\ w(x) \le w(x,y) \le \frac{\sqrt{13}}{2} w(x),\ \text{ if } |x-y| \le \frac{|x|}{2}. \tag{20}$$

Since, for any functions f, g and points p, q from their domain, we have

$$|(fg)(p) - (fg)(q)| \le |f(p) - f(q)||g(p)| + |g(p) - g(q)||f(q)|,$$

it follows that, for all $\gamma > 0$ and $|\alpha| \le s$, the Leibniz rule, (11), and (20) give us the following inequality with some constants $c^{\beta}_{\tilde\beta}$:

$$\sup_{t\in[0,T]} \langle \partial_t^j \partial^{\alpha+\beta}(u_r - u)(\cdot,t)\rangle_{\lambda,\delta+|\alpha|} \le \tag{21}$$

$$\sup_{t\in[0,T]} \sum_{\beta+\tilde\beta=\alpha} c^{\beta}_{\tilde\beta}\Big(\langle \partial^{\tilde\beta}\eta_r(\iota_0(\cdot,t))\rangle_{\lambda,|\tilde\beta|}\, \|\partial^{\beta}(u_r-u)(\cdot,t)\|_{C^{0,0}_{\delta+|\beta|}} +$$

$$\sup_{t\in[0,T]}\left(\|\partial^{\tilde\beta}(1-\eta_r(\iota_0(\cdot,t))\|_{C^{0,0}_{\delta+|\tilde\beta|}(\mathbb{R}^n\setminus B_{\frac{1}{2r}})}\|\partial^{\beta}u(\cdot,t)\|_{C^{0,\lambda}_{\delta+|\beta|}(\mathbb{R}^n\setminus B_{\frac{1}{2r}})}\right) +$$

$$\sup_{t\in[0,T]}\left(\langle\partial^{\tilde\beta}(1-\eta_r(\iota_0(\cdot,t)))\rangle_{\lambda,|\tilde\beta|,\mathbb{R}^n\setminus B_{1/2r}}\, \|\partial^{\beta}u(\cdot,t)\|_{C^{0,0}_{\delta+|\beta|}(\mathbb{R}^n\setminus B_{1/2r})}\right) +$$

$$\frac{13}{2}\sup_{t\in[0,T]}\sum_{\beta+\tilde\beta=\alpha} c^{\beta}_{\tilde\beta}\|\partial^{\tilde\beta}(\eta_r(\iota_0(\cdot,t)))\|_{C^{0,0}_{\delta+|\tilde\beta|}}\times$$

$$\Big(\frac{2}{13}\sup_{\substack{x,y\not\in\overline{B}_2\\ \frac{|x-y|}{w(x,y)}\le\gamma}}\Big(\int_0^1\int_{|z|\le 1}\frac{w^{\delta+\lambda+|\beta|}(x,y)|\partial^{\beta}u(x,t)-\partial^{\beta}u(y,t)|}{|x-y|^{\lambda}}\psi(z,\tau)dzd\tau +$$

$$\int_0^1\int_{|z|\le1}\frac{w^{\tilde\delta}(x+rz,y+rz)|\partial^{\beta}u(rz+x,t+\tau r)-\partial^{\beta}u(rz+y,t+\tau r)|}{|x+rz-rz-y|^{\lambda}}\frac{\psi(z,\tau)}{2}dzd\tau\Big)+$$

$$\sup_{\substack{x,y\not\in\overline{B}_2\\ |x-y|\ge\gamma}}\int_0^1\int_{|z|\le1}\frac{w^{\tilde\delta}(x+rz,x)|\partial^{\beta}u(rz+x,t+\tau r)-\partial^{\beta}u(x,t+\tau r)|}{|rz|^{\lambda}}\frac{\psi(z,\tau)}{(\gamma/r)^{\lambda}}dzd\tau+$$

$$\sup_{\substack{x,y\not\in\overline{B}_2\\ |x-y|\ge\gamma}}\int_0^1\int_{|\tau|\le1}\frac{w^{\tilde\delta}(y,y+r\tau)|\partial^{\beta}u(rz+y,t+\tau r)-\partial^{\beta}u(y,t)|}{|rz|^{\lambda}}\frac{\psi(z,\tau)}{(\gamma/r)^{\lambda}}dzd\tau\Big),$$

where $\tilde\delta = \delta + \lambda + |\beta|$.

On the other hand, it follows from Theorem 4 and (16) that for any $\tilde\beta \in \mathbb{Z}^n_+$ there is a constant $c_{\tilde\beta}$ independent on r and such that

$$\sup_{t\in[0,T]}\|\partial^{\tilde\beta}(\eta_r(\iota_0(\cdot,t)))\|_{C^{0,\lambda}_{|\tilde\beta|}} \le \sup_{t\in[0,T]}\|\partial^{\tilde\beta}(\eta_r(\iota_0(\cdot,t)))\|_{C^{1,0}_{|\tilde\beta|}} \le c_{\tilde\beta},$$

and hence, in accordance with (4) with $\mu = 0$,

$$\lim_{r\to0+}\sup_{t\in[0,T]}\left(\|\eta_r(\iota_0(\cdot,t)))\|_{C^{s,\lambda}_0}\|u(\cdot,t)\|_{C^{s,\lambda}_{\delta}(\mathbb{R}^n\setminus B_{\frac{1}{2r}})}\right) = 0. \tag{22}$$

Fix a number $\varepsilon > 0$ and set $\gamma = \gamma_{\varepsilon/16}$ in (21), where γ_ε is the number from (3). Then (3), (9), (19) and (22) imply that there is a number $R_\varepsilon > 0$ such that for all $r \in (0, R_\varepsilon)$ and all $\alpha \in \mathbb{Z}^n_+$, satisfying $|\alpha| \leq s$, we have

$$\sup_{t\in[0,T]} \langle (\partial^\alpha u_r - \partial^\alpha u)(\cdot,t)\rangle_{\lambda,\delta} \leq \frac{3\varepsilon}{4}. \tag{23}$$

Similarly, (3), (9) and (12) imply that for any $\varepsilon > 0$ and any r from the interval $(0, (\varepsilon/16)^{1/\lambda}\gamma_{\varepsilon/16})$ and all $\alpha \in \mathbb{Z}^n_+$, satisfying $|\alpha| \leq s$, we have

$$\sup_{t\in[0,T]} \langle \partial^\alpha (u_r - u)(\cdot,t)\rangle_{\lambda,\overline{B}_2} \leq \tag{24}$$

$$\sup_{t\in[0,T]} \sup_{\substack{x,y\in\overline{B}_2 \\ |x-y|\leq\gamma_{\frac{\varepsilon}{16}}}} \int_0^1 \int_{|z|\leq 1} \frac{|\partial^\alpha u(rz+x,t+r\tau) - \partial^\alpha u(rz+y,t+r\tau)|}{|x-y|^\lambda}\psi(z,\tau)dzd\tau +$$

$$\sup_{t\in[0,T]} \sup_{\substack{x,y\in\overline{B}_2 \\ |x-y|\leq\gamma_{\frac{\varepsilon}{16}}}} \int_0^1 \int_{|z|\leq 1} \frac{|\partial^\alpha u(x,t) - \partial^\alpha u(y,t)|}{|x-y|^\lambda}\psi(z,\tau)dzd\tau +$$

$$\Big(\frac{r}{\gamma_{\frac{\varepsilon}{16}}}\Big)^\lambda \sup_{\substack{t\in[0,T] \\ x,y\in\overline{B}_2 \\ |x-y|\geq\gamma_{\frac{\varepsilon}{16}}}} \int_0^1 \int_{|z|\leq 1} \frac{|\partial^\alpha u(rz+x,t+r\tau) - \partial^\alpha u(x,t+r\tau)|}{|rz|^\lambda}\psi(z,\tau)dzd\tau +$$

$$\sup_{t\in[0,T]} \Big(\frac{r}{\gamma_{\frac{\varepsilon}{16}}}\Big)^\lambda \sup_{\substack{x,y\in\overline{B}_2 \\ |x-y|\geq\gamma_{\frac{\varepsilon}{16}}}} \int_0^1 \int_{|z|\leq 1} \frac{|\partial^\alpha u(rz+y,t) - \partial^\alpha u(y,t)|}{|rz|^\lambda}\psi(z,\tau)dzd\tau \leq \frac{\varepsilon}{4}.$$

Finally, (19), (23) and (24) imply that

$$\lim_{r\to 0+} \|u_r - u\|_{C^{2s+k,s,\lambda,0}_{\delta,T}} = 0.$$

Theorem 5 is proved. □

Example 1. *For each $(x_0,t_0) \in \mathbb{R}^n \times [0,T]$ the function $u_{(x_0,t_0)}(x) = \Big(|x-x_0|^\lambda + |t-t_0|^{\frac{\lambda}{2}}\Big)w^{-\delta-\lambda}(x)$ belongs to $C^{0,0,\lambda,0}_{\delta,T}$ and even to $C^{0,0,\lambda,\frac{\lambda}{2}}_{\delta,T}$, $0 \leq \lambda \leq 1$. It is clear that $\|u_{(x_0,t_0)} - u_{(x_1,t_0)}\|_{C^{0,0,\lambda,0}_\delta} \geq 1$, if $x_0 \neq x_1$ and $0 < \lambda \leq 1$, and hence the space $C^{0,0,\lambda,0}_{\delta,T}$ is nonseparable for $0 < \lambda \leq 1$, because the set of points $x_0 \in \mathbb{R}^n$ is uncountable. Precisely the element of the type $u_{(x_0,t_0)}$ cannot be approximated in $C^{0,0,\lambda,0}_{\delta,T}$ by functions from $C^{0,0,\lambda',0}_{\delta',T}$ if $0 < \lambda < \lambda' \leq 1$, $\delta' > \delta$.*

3. The de Rham complex over anisotropic weighted Hölder spaces $C^{2s+k,s,\lambda,0}_\delta$

Denote by Λ^q the bundle of exterior differential forms of degree $0 \leq q \leq n$ over $\mathbb{R}^n$. Given domain $\mathcal{X}$ in $\mathbb{R}^n$, we write $\Omega^q(\mathcal{X})$ for the space of all differential forms of degree q with C^∞-differentiable coefficients on $\mathcal{X}$. These spaces form the so-called de Rham complex $\Omega^\cdot(\mathcal{X})$ on $\mathcal{X}$, whose differentials are given by exterior derivatives d. it is convenient to use the notation $du := d^q u$ for $u \in \Omega^q(\mathcal{X})$ and to set $d^q = 0$ if $q < 0$ and $q \geq n$ (see, for instance, [20]). We omit the index q for graded operators if this does not lead to misunderstanding. The superscript q in the signs of the functional spaces always means that we consider differential forms the degree q with the coefficients of the corresponding function class.

As usual, denote by d^* the formal adjoint operator for the operator d; more exactly, $d^* g = (d^{q-1})^* g$ for $g \in \Omega^q(\mathbb{R}^n)$. By the definition,

$$d \circ d \equiv 0, \quad d^* \circ d^* \equiv 0. \tag{25}$$

For a differential operator A acting on sections of the vector bundle Λ^q over $\mathbb{R}^n$, denote by $C^{s,\lambda}_{\delta,\Lambda^q} \cap \mathcal{S}_A$ the space of differential forms $u \in C^{s,\lambda}_{\delta,\Lambda^q}$ satisfying $Au = 0$ in the sense of distributions in $\mathbb{R}^n$. Obviously, this is a closed subspace in $C^{s,\lambda}_{\delta,\Lambda^q}$, and hence it is a Banach space with the induced norm.

Then, according to (25), the differential operator $d \oplus d^*$ induces linear bounded operators

$$d \oplus d^* : C^{s+1,\lambda}_{\delta,\Lambda^q} \to C^{s,\lambda}_{\delta+1,\Lambda^{q+1}} \cap \mathcal{S}_d \oplus C^{s,\lambda}_{\delta+1,\Lambda^{q-1}} \cap \mathcal{S}_{d^*}. \tag{26}$$

$$d \oplus d^* : \mathfrak{C}^{s+1,\lambda}_{\delta,\Lambda^q} \to \mathfrak{C}^{s,\lambda}_{\delta+1,\Lambda^{q+1}} \cap \mathcal{S}_d \oplus \mathfrak{C}^{s,\lambda}_{\delta+1,\Lambda^{q-1}} \cap \mathcal{S}_{d^*}. \tag{27}$$

The images $R^{s,\lambda}_{\delta+1,\Lambda^{q+1},\Lambda^{q-1}}$ and $\mathfrak{R}^{s,\lambda}_{\delta+1,\Lambda^{q+1},\Lambda^{q-1}}$ of the linear bounded operators (26) and (27), respectively, we described in [14]. Namely, let $H^q_{\leq m}$ stand for the set of all differential forms of degree q with harmonic polynomials of degree $\leq m$ as the coefficients, let $L^2(\mathcal{X})$ be the standard Hilbert Lebesgue space of functions on $\mathcal{X}$ with the scalar product $(\cdot,\cdot)_{L^2(\mathcal{X})}$.

Theorem 6. *Let $n \geq 2$, $s \in \mathbb{Z}_+$, $0 < \lambda < 1$. If $\delta > 0$ and $\delta + 1 - n \notin \mathbb{Z}_+$, then operators* (26) *u* (27) *are Fredholm. Moreover,*

(1) (26) *and* (27) *are isomorphisms if* $0 < \delta < n - 1$*;*

(2) (26) *and* (27) *are injections with closed images if* $n - 1 + m < \delta < n + m$ *for $m \in \mathbb{Z}_+$; more exactly, the image $R^{s,\lambda}_{\delta+1,\Lambda^{q+1},\Lambda^{q-1}}$ consists of all the pairs $f \in C^{s,\lambda}_{\delta+1,\Lambda^{q+1}} \cap \mathcal{S}_d$, $g \in C^{s,\lambda}_{\delta+1,\Lambda^{q-1}} \cap \mathcal{S}_{d^*}$ satisfying*

$$(f, dh)_{L^2_{\Lambda^{q+1}}(\mathbb{R}^n)} + (g, d^*h)_{L^2_{\Lambda^{q-1}}(\mathbb{R}^n)} = 0 \text{ for all } h \in H^q_{\leq m+1}, \tag{28}$$

and the image $\mathfrak{R}^{s,\lambda}_{\delta+1,\Lambda^{q+1},\Lambda^{q-1}}$ consists of all the pairs $f \in \mathfrak{C}^{s,\lambda}_{\delta+1,\Lambda^{q+1}} \cap \mathcal{S}_d$ $g \in \mathfrak{C}^{s,\lambda}_{\delta+1,\Lambda^{q-1}} \cap \mathcal{S}_{d^*}$, satisfying (28)*.*

Denote by $\Lambda^q(t)$ the induced bundle over the half-space $\mathbb{R}^{n+1}_{t\geq 0} = \mathbb{R}^n \times [0,+\infty)$ with the coordinates (x,t), i.e., the sections of $\Lambda^q(t)$ are differential forms over $\mathbb{R}^n$ depending on the parameter $t \in [0,+\infty)$:

$$U = \sum_{|I|=q} U_I(x,t) dx_I,\ I = (i_1, \dots i_q),\ 1 \leq i_j \leq n,\ 0 \leq q \leq n$$

where, as usual, $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_q}$, and the symbol $\wedge$ designates the exterior product of differential forms. Consider the induced de Rham complex $\{d^q, \Lambda^q(t)\}_{q=0}^n$ over $\mathbb{R}^{n+1}_{t\geq 0}$:

$$0 \to C^\infty_{\Lambda^0}(\mathbb{R}^{n+1}_{t\geq 0}) \xrightarrow{d_0} C^\infty_{\Lambda^1}(\mathbb{R}^{n+1}_{t\geq 0}) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} C^\infty_{\Lambda^n}(\mathbb{R}^{n+1}_{t\geq 0}) \to 0,$$

i.e.,

$$(d^q U)(x,t) = \sum_{j=1}^n \sum_{|I|=q} \frac{\partial U_I(x,t)}{\partial x_j} dx_j \wedge dx_I.$$

We may easily extend Theorem 6 to the bounded linear operators

$$d^q \oplus (d^q)^* : C^{2s+k+1,s,\lambda,0}_{\delta,T,\Lambda^q} \to C^{2s+k,s,\lambda,0}_{\delta+1,T,\Lambda^{q+1}} \cap \mathcal{S}_d \oplus C^{2s+k,\lambda,0}_{\delta+1,T,\Lambda^{q-1}} \cap \mathcal{S}_{d^*}, \quad k \in \mathbb{Z}_+, \tag{29}$$

(30) $d^q \oplus (d^q)^* : \mathfrak{C}^{2s+k+1,s,\lambda,0}_{\delta,T,\Lambda^q} \to \mathfrak{C}^{2s+k,s,\lambda,0}_{\delta+1,T,\Lambda^{q+1}} \cap \mathcal{S}_d \oplus \mathfrak{C}^{2s+k,\lambda,0}_{\delta+1,T,\Lambda^{q-1}} \cap \mathcal{S}_{d^*}, \quad k \in \mathbb{Z}_+.$

With this purpose, denote by $R^{2s+k,s,\lambda,0}_{\delta+1,T,\Lambda^{q+1},\Lambda^{q-1}}$ and $\mathfrak{R}^{2s+k,s,\lambda,0}_{\delta+1,T,\Lambda^{q+1},\Lambda^{q-1}}$ the images of the bounded linear operators (29) and (30), respectively.

Corollary 1. *Let $n \geq 2$, $s \in \mathbb{Z}_+$, $0 < \lambda < 1$. If $\delta > 0$ and $\delta + 1 - n \notin \mathbb{Z}_+$ then operators* (29) *are* (30) *are normally solvable. Moreover,*

(1) (29) *and* (30) *are isomorphisms if* $0 < \delta < n-1$*;*

(2) (29) *and* (30) *are injections with closed images if* $n - 1 + m < \delta < n + m$ *for $m \in \mathbb{Z}_+$; more exactly, the image $R^{2s+k-1,s,\lambda,0}_{\delta+1,T,\Lambda^{q+1},\Lambda^{q-1}}$ consists of all the pairs $f \in C^{2s+k,s,\lambda,0}_{\delta+1,T,\Lambda^{q+1}} \cap \mathcal{S}_d$, $g \in C^{2s+k,s,\lambda,0}_{\delta+1,T,\Lambda^{q-1}} \cap \mathcal{S}_{d^*}$ satisfying*

(31)
$$(f(\cdot,t), dh)_{L^2_{\Lambda^{q+1}}(\mathbb{R}^n)} + (g(\cdot,t), d^*h)_{L^2_{\Lambda^{q-1}}(\mathbb{R}^n)} = 0 \text{ for all } t \in [0,T] \text{ и } h \in H^q_{\leq m+1},$$

and the image $\mathfrak{R}^{2s+k,s,\lambda,0}_{\delta+1,T,\Lambda^{q+1},\Lambda^{q-1}}$ consists of all the pairs $f \in \mathfrak{C}^{2s+k,s,\lambda,0}_{\delta+1,T,\Lambda^{q+1}} \cap \mathcal{S}_d$, $g \in \mathfrak{C}^{2s+k,s,\lambda,0}_{\delta+1,T,\Lambda^{q-1}} \cap \mathcal{S}_{d^*}$ satisfying (31).

Proof. It is known that the Laplacian of the complex $\Omega^{\cdot}(\mathbb{R}^n)$ on q-forms is given by

(32) $$\Delta^q := d^*d + dd^* = -I_{k_q}\,\Delta,$$

where $k_q = \binom{n}{k}$, I_{k_q} is the identity $(k_q \times k_q)$-matrix, Δ is the usual Laplace operator applied componentwise the to the coefficients of the form.

Now we note that if $f = 0$ and $g = 0$ then (32) implies that the coefficients of the form $u_0(x,t)$ satisfying $du_0 = 0$, $d^*u_0 = 0$ in $\mathbb{R}^n$ are harmonic functions vanishing at infinity for each fixed $t \in [0,T]$ because $\delta > 0$. Then the Liouville theorem for harmonic functions means that the operator $d \oplus d^*$ is injective on $C^{2s+k+1,s,\lambda,0}_{\delta,T,\Lambda^q}$ for all $k \geq 1$, $s \in \mathbb{Z}_+$, $0 \leq q \leq n$ and $\delta > 0$.

By the definition, any element $u(x,t) \in C^{2s+k+1,s,\lambda,0}_{\delta,T,\Lambda^q}$ belongs to $C^{2s+k+1,\lambda}_{\delta,\Lambda^q}$ for each fixed $t \in [0,T]$. Hence, for $n - 1 + m < \delta < n + m$ with $m \in \mathbb{Z}_+$, Theorem 6 implies that for any pair (f,g) from $R^{2s+k,s,\lambda,0}_{\delta+1,T,\Lambda^{q+1},\Lambda^{q-1}}$ identity (31) is fulfilled.

To continue the proof of the theorem, we put

$$(E_q u)(x) = \int_{\mathbb{R}^n} u(y) \wedge e_q(x,y)$$

for a suitable q-form u, where

$$e_q(x,y) = \sum_{|I|=q} e(x-y)\,(\star dy_I)\,dx_I, \; e(x) = \begin{cases} \dfrac{1}{\pi}\ln|x|, & \text{if} \quad n = 2, \\ \dfrac{1}{\sigma_n}\dfrac{|x|^{2-n}}{2-n}, & \text{if} \quad n \geq 3, \end{cases}$$

i.e. e is the standard fundamental solution of the convolution type in $\mathbb{R}^n$, and e_q is its analogue for action on exterior differential forms (here σ_n is the area of the unit sphere in $\mathbb{R}^n$). Now, for $f \in C^{s,\lambda}_{\delta+1,T,\Lambda^{q+1}}$, $g \in C^{s,\lambda}_{\delta+1,T,\Lambda^{q-1}}$, let

(33) $$(\Phi_q\, f)(x) = \int_{\mathbb{R}^n} f(y) \wedge \phi_q(x,y), \; (\hat{\Phi}_q\, g)(x) = \int_{\mathbb{R}^n} g(y) \wedge \hat{\phi}_q(x,y)$$

where

$$\phi_q(x,y) = (d^{n-q-1})^*_y e_q(x,y), \; \hat{\phi}_q(x,y) = d^{n-q}_y e_q(x,y), \quad n \geq 2.$$

It was shown in [14, Theorem 3] that, under the hypothesis of 6, potentials (33) induce a bounded linear operator

$$(\Phi \oplus \hat{\Phi}) : C^{s-1,\lambda}_{\delta+1,\Lambda^{q+1}} \oplus C^{s-1,\lambda}_{\delta+1,\Lambda^{q-1}} \to C^{s+1,\lambda}_{\delta,\Lambda^q}, \tag{34}$$

such that

$$d(\Phi f + \hat{\Phi} g) = f, \quad d^*(\Phi f + \hat{\Phi} g) = g \text{ in } \mathbb{R}^n, \tag{35}$$

if $f \in C^{s,\lambda}_{\delta+1,\Lambda^{q+1}} \cap \mathcal{S}_d$ and $g \in C^{s,\lambda}_{\delta+1,\Lambda^{q-1}} \cap \mathcal{S}_{d^*}$ satisfy (28) for $n-1+m < \delta < n+m$, $s \in \mathbb{Z}_+$.

Therefore, for each pair $f \in C^{2s+k,s,\lambda,0}_{\delta+1,T,\Lambda^{q+1}} \cap \mathcal{S}_d$ and $g \in C^{2s+k,s,\lambda,0}_{\delta+1,T,\Lambda^{q-1}} \cap \mathcal{S}_{d^*}$, satisfying (31) for $n-1+m < \delta < n+m$, the differential form $u(x,t) = \Phi f(\cdot,t) + \hat{\Phi} g(\cdot,t)$ belongs to $C^{2s+k+1,\lambda}_{\delta+1,\Lambda^q}$, depends on the parameter $t \in [0,T]$, and is the unique solution to the following system of equations:

$$\begin{cases} d(\Phi f(\cdot,t) + \hat{\Phi} g(\cdot,t))(x) = f(x,t) & \text{in} \quad \mathbb{R}^n \times [0,T], \\ d^*(\Phi f(\cdot,t) + \hat{\Phi} g(\cdot,t))(x) = g(x,t) & \text{in} \quad \mathbb{R}^n \times [0,T]. \end{cases} \tag{36}$$

On the other hand, by the definition of $C^{2s+k+1,s,\lambda,0}_{\delta,T}$, for such a pair $f \in C^{2s+k,s,\lambda,0}_{\delta+1,T,\Lambda^{q+1}} \cap \mathcal{S}_d$ and $g \in C^{2s+k,s,\lambda,0}_{\delta+1,T,\Lambda^{q-1}} \cap \mathcal{S}_{d^*}$ we have $\partial^j_t f \in C^{2(s-j)+k,s-j,\lambda,0}_{\delta+1,T,\Lambda^{q+1}} \cap \mathcal{S}_d$, $\partial^j_t g \in C^{2(s-j)+k,s-j,\lambda,0}_{\delta+1,T,\Lambda^{q-1}} \cap \mathcal{S}_{d^*}$ and the identity

$$(\partial^j_t f(\cdot,t), dh)_{L^2_{\Lambda^{q+1}}(\mathbb{R}^n)} + (\partial^j_t g(\cdot,t), d^* h)_{L^2_{\Lambda^{q-1}}(\mathbb{R}^n)} = 0$$

holds for all $t \in [0,T]$ and $h \in H^q_{\leq m+1}$ if $n-1+m < \delta < n+m$. Therefore, by Theorem 6, for all suitable δ, we have

$$\sup_{t\in[0,T]} \|\partial^j_t [\Phi f(\cdot,t) + \hat{\Phi} g(\cdot,t)]\|_{C^{2(s-j)+k+1,\lambda}_{\delta,T,\Lambda^q}} =$$

$$\sup_{t\in[0,T]} \|\Phi \partial^j_t f(\cdot,t) + \hat{\Phi} \partial^j_t g(\cdot,t)\|_{C^{2(s-j)+k+1,\lambda}_{\delta,T,\Lambda^q}} \leq$$

$$\|\Phi + \hat{\Phi}\|_{j,k} \left(\sup_{t\in[0,T]} \|\partial^j_t f(\cdot,t)\|_{C^{2(s-j)+k,\lambda}_{\delta+1,T,\Lambda^{q+1}}} + \sup_{t\in[0,T]} \|\partial^j_t g(\cdot,t)\|_{C^{2(s-j)+k,\lambda}_{\delta+1,T,\Lambda^{q-1}}} \right)$$

if $0 \leq j \leq s$, where $\|\Phi + \hat{\Phi}\|_{j,k}$ stands for the norm of the operator

$$\Phi \oplus \hat{\Phi} : C^{2(s-j)+k,\lambda}_{\delta+1,T,\Lambda^{q+1}} \oplus C^{2(s-j)+k,\lambda}_{\delta+1,T,\Lambda^{q-1}} \to C^{2(s-j)+k+1,\lambda}_{\delta,T,\Lambda^q}. \tag{37}$$

Hence, if a differential form $u(x,t)$ is a solution to (36) then it in fact belongs to $C^{2s+k+1,s,\lambda,0}_{\delta,T,\Lambda^q}$. Moreover, operators (34) induce continuous linear operators

$$\Phi \oplus \hat{\Phi} : C^{2s+k,s,\lambda,0}_{\delta+1,T,\Lambda^{q+1}} \oplus C^{2s+k,s,\lambda,0}_{\delta+1,T,\Lambda^{q-1}} \to C^{2s+k+1,s,\lambda,0}_{\delta,T,\Lambda^q}.$$

If $0 < \delta < n-1$, then the image $R^{2s+k,s,\lambda,0}_{\delta+1,T,\Lambda^{q+1},\Lambda^{q-1}}$ is closed because it coincides with $C^{2s+k,s,\lambda,0}_{\delta+1,T,\Lambda^{q+1}} \cap \mathcal{S}_d \oplus C^{2s+k,s,\lambda,0}_{\delta+1,T,\Lambda^{q-1}} \cap \mathcal{S}_{d^*}$. Then mapping (29) is an isomorphism.

If $n-1+m < \delta < n+m$ then $\delta+1 > n/2$ and Lemma 1 implies that the image $R^{2s+k,s,\lambda,0}_{\delta+1,T,\Lambda^{q+1}}$ is closed, too.

Finally, the assertions about mapping (30) follow from Theorem 5 and the continuity of the operators d, d^*, Φ, $\hat{\Phi}$ on the scale $C^{2s+k,s,\lambda,0}_{\delta,T}$. □

Since the operators d^0, $(d^0)^*$, $(d^1)^*$ represent the gradient operator and the rotation and divergence operators in $\mathbb{R}^3$ respectively, Corollary 1 in particular clarifies the behavior of the well-known Leray-Helmholtz projection (from the theory of the Navier-Stokes equations) on the scale of anisotropic weighted Hölder spaces under consideration (cf. [7, Ch. 1, §2] on the Lebesgue spaces or [14, Corollary 1]).

More precisely, it is not hard to see that the introduced anisotropic weighted spaces are "physically" acceptable for studying the Navier–Stokes equations for suitable powers δ.

Lemma 1. *If $1 \leq p < +\infty$ and $\delta > n/p$ then there is a constant $c_{\delta,p} > 0$ such that*

$$\sup_{t\in[0,T]} \|u(\cdot,t)\|_{L^p(\mathbb{R}^n)} \leq c_{\delta,p} \, \|u\|_{C^{0,0,0,0}_{\delta,T}} \text{ for all } t \in [0,T] \text{ and all } u \in C^{0,0,0,0}_{\delta,T}.$$

Next, denote by Σ^q the closure of $\mathcal{D}^q(\mathbb{R}^n) \cap \mathcal{S}_{d^*}$ in $L^2_{\Lambda^q}(\mathbb{R}^n)$. The orthogonal projection $\Pi_q : L^2_{\Lambda^q}(\mathbb{R}^n) \to \Sigma^q$ is called the Leray–Helmholtz projection.

Denote by $\Sigma^{2s+k,s,\lambda,0}_{\delta,\Lambda^q}$ the closure of the set $C^\infty([0,T], \mathcal{D}_{\Lambda^q}(\mathbb{R}^n)) \cap \mathcal{S}_{d^*}$ in $C^{2s+k,s,\lambda,0}_{\delta,\Lambda^q}$.

Corollary 2. *Let $n \geq 2$, $\delta > 1$, $\delta - n \notin \mathbb{Z}_+$, $k, s \in \mathbb{Z}_+$, $0 < \lambda < 1$. Then the following decomposition holds for each $0 \leq q \leq n$:*

$$I = d^*\hat{\Phi} + d\Phi \tag{38}$$

on $C^{2s+k,s,\lambda,0}_{\delta,\Lambda^q}$ and $\mathfrak{C}^{2s+k,s,\lambda,0}_{\delta,\Lambda^q}$ with bounded linear operators $d^*\hat{\Phi}$ and $d\Phi$ in these spaces. If $\delta > n/2$ then decomposition (38) is $L^2_{\Lambda^q}(\mathbb{R}^n)$-orthogonal on $\mathfrak{C}^{2s+k,s,\lambda,0}_{\delta,\Lambda^q}$, and the projection Π_q coincides with $d^*\hat{\Phi}$ on $\mathfrak{C}^{2s+k,s,\lambda,0}_{\delta,\Lambda^q}$, and it maps this space continuously to $\Sigma^{2s+k,s,\lambda,0}_{\delta,\Lambda^q}$.

Proof. Recall that $d^q = 0$ for $q < 0$ and $q \geq n$. As we have seen in proving Corollary 1, if $\delta > 1$, $\delta - n \notin \mathbb{Z}_+$, then the linear operators $d^*\hat{\Phi}$ and $d\Phi$ are bounded in $C^{2s+k,s,\lambda,0}_{\delta,\Lambda^q}$ and in $\mathfrak{C}^{2s+k,s,\lambda,0}_{\delta,\Lambda^q}$. Since $e(x-y)$ is the fundamental solution of the Laplace operator and it has convolution type, we have for all $u \in C^{2s+k,s,\lambda,0}_{\delta,\Lambda^q}$, $\delta > 1$, $\delta - n \notin \mathbb{Z}_+$:

$$\Delta^q \Big(u(x,t) - d^*\hat{\Phi}u(x,t) - d\Phi u(x,t) \Big) = 0 \text{ in } \mathbb{R}^n$$

in the sense of distributions for all $t \in [0,T]$. On the other hand, the form

$$\Big(u - d^*\hat{\Phi}u - d\Phi u \Big)(x,t)$$

has harmonic coefficients and equals to zero at infinity with respect to the space variables if $\delta > 1$. Then, by the Liouville Theorem, it is identical zero, i.e., decomposition (38) holds on $C^{2s+k,s,\lambda,0}_{\delta,\Lambda^q}$ and $\mathfrak{C}^{2s+k,s,\lambda,0}_{\delta,\Lambda^q}$.

It follows from Lemma 1 that, if $\delta > n/2$, $\delta - n \notin \mathbb{Z}_+$ then $C^{2s+k,s,\lambda,0}_{\delta,\Lambda^q}$ is continuously embedded into $L^2_{\Lambda^q}(\mathbb{R}^n)$ and, for each form $u \in \mathfrak{C}^{2s+k,s,\lambda,0}_{\delta,\Lambda^q}$, the forms $d^*\hat{\Phi}u$ and $d\Phi u$ are $L^2_{\Lambda^q}(\mathbb{R}^n)$-orthogonal because, in this case, we can integrate by parts and $d \circ d = 0$. Indeed,

(39)

$$(d^*\hat{\Phi}u, d\Phi u)_{L^2_{\Lambda^q}(\mathbb{R}^n)} = \lim_{k\to\infty} (d^*\hat{\Phi}u, d\Phi u_k)_{L^2_{\Lambda^q}(\mathbb{R}^n)} = \lim_{k\to\infty} \lim_{R\to+\infty} (d\hat{\Phi}u, d\Phi u_k)_{L^2_{\Lambda^q}(B_R)}$$

for any sequence $\{u_k\} \subset C^\infty([0,T], \mathcal{D}(\mathbb{R}^n))$ approximating u in $\mathfrak{C}^{2s+k,s,\lambda,0}_{\delta,\Lambda^q}$. On the other hand, as $d \circ d = 0$, integrating by parts yields

(40)

$$|(d\hat{\Phi}u, d\Phi u_k)_{L^2_{\Lambda^q}(B_R)}| = \Big| \int_{\partial B_R} *\Phi u_k \wedge d\hat{\Phi}u \Big| \leq \sigma_n \, R^{n-2\delta} \|\Phi u_k\|_{C^{0,0,0,0}_{\delta-1,\Lambda^{q-1}}} \|d\hat{\Phi}u\|_{C^{0,0,0,0}_{\delta,\Lambda^q}},$$

where $*$ is the so-called Hodge operator on $\mathbb{R}^n$:

$$\star : \Omega^q(\mathbb{R}^n) \to \Omega^{n-q}(\mathbb{R}^n),$$

defined (by linearity) with the use of the relation $dx^I \wedge (\star dx^I) = dx$ for all multi-indices $I = (i_1, \ldots, i_q)$ with $1 \leq i_1 < \ldots < i_q \leq n$. Now, since $2\delta > n$ and the sequence $\{\|\Phi u_k\|_{C^{0,0,0,0}_{\delta-1,\Lambda^{q-1}}}\}$ is bounded, formulas (39) and (40) enable us to conclude that

$$(d^*\hat{\Phi}u, d\Phi u)_{L^2_{\Lambda^q}(\mathbb{R}^n)} = 0 \text{ for all } u \in \mathfrak{C}^{2s+k,s,\lambda,0}_{\delta,\Lambda^q}.$$

Finally, $d^* \circ d^* = 0$, and , as we have seen in the proof of Corollary 1, the operator $d^*\hat{\Phi}$ maps continuously $\mathfrak{C}^{2s+k,s,\lambda,0}_{\delta,\Lambda^q}$ into $\Sigma^{2s+k,s,\lambda,0}_{\delta,\Lambda^q}$, which was to be proved. □

It follows from the proof of Corollary 2 that decomposition (38) is $L^2_{\Lambda^q}(\mathbb{R}^n)$-orthogonal also on $C^{2s+k,s,\lambda,0}_{\delta,\Lambda^q}$, if $2s + k \geq 1$ and $\delta > n/2$.

At the end, we observe that, though the scale of anisotropic weighted Hölder spaces $C^{2s+k,s,\lambda,0}_{\delta,\Lambda^q}$ is appropriate for studying the operator equations induced by the differentials of the de Rham complex, it is not so good for the investigating parabolic equations. Namely, the parabolic potentials (constructed with the use of the fundamental solution to the heat operator) may act discontinuously on the scale. On the other hand, it was shown in [11, §4] that second-order parabolic equations and the corresponding potentials can be better investigated on the scale $C^{2s+k,s,\lambda,\frac{\lambda}{2}}_{\delta}$, whereas the continuity of the "elliptic potentials" Φ and $\hat{\Phi}$ fails and we have to correct the scale by using the graph norms of the de Rham differentials (see [11, §3] for a rather narrow interval of the weight indices).

References

[1] S.L. Sobolev, *Some Applications of Functional Analysis to Mathematical Physics*, Nauka, Moscow, 1988. Zbl 0662.46001

[2] N.M. Gunther, *La théorie du potentiel et ses Applications aux problémes fondamentaux de la physique mathématique*, Gauthier-Villars, Paris, 1934. Zbl 0009.11301

[3] D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equations of second order*, Springer-Verlag, Berlin, 1983. Zbl 0562.35001

[4] O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Ural'tseva , *Linear and Quasilinear Equations of Parabolic Type*, Nauka, Moscow, 1967. Zbl 0164.12302

[5] O.A. Ladyzhenskaya, N.N. Ural'tseva, *Linear and Quasilinear Equations of Elliptic Type*, Nauka, Moscow, 1973. Zbl 0269.35029

[6] S. Krantz, *Intrinsic Lipschitz classes on manifolds with applications to complex function theory and estimates for the $\overline{\partial}$ and $\overline{\partial}_b$ equations*, Manuscr. Math., **24**:4 (1978), 351–378. Zbl 0382.32012

[7] O.A. Ladyzhenskaya, *Mathematical questions of the dynamics of a viscous incompressible fluid*, Nauka, Moscow, 1970. Zbl 0215.29004

[8] A. Bertozzi, A. Majda, *Vorticity and Incompressible Flows*, Cambridge University Press, Cambridge, 2002. Zbl 0983.76001

[9] L.I. Nicolaescu, *Lectures on the Geometry of Manifolds*, World Scientific, London, 2007. Zbl 1155.53001

[10] T. Behrndt, *On the Cauchy problem for the heat equation on Riemannian manifolds with conical singularities*, Q. J. Math., **64**:4 (2011), 981–1007. Zbl 1285.58008

[11] A. Shlapunov, N. Tarkhanov, *An Open Mapping Theorem for the Navier-Stokes Equations*, Advances and Applications in Fluid Mechanics, **21**:2 (2018), 127–246.
[12] R. McOwen, *The behavior of the Laplacian on weighted Sobolev spaces*, Comm. Pure Appl. Math., **32** (1979), 783–795. Zbl 0426.35029
[13] R. Lockhart, R. McOwen, *Elliptic differential operators on noncompact manifolds.* Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV, **12**:3 (1985), 409–447. Zbl 0615.58048
[14] K.V. Sidorova (Gagelgans), A.A. Shlapunov, *On the Closure of Smooth Compactly Supported Functions in Weighted Holder Spaces*, Math. Notes, **105**:4 (2019), 604-–617. Zbl 07083432
[15] N. Tarkhanov, *The Cauchy Problem for Solutions of Elliptic Equations*, Akademie-Verlag, Berlin, 1995. Zbl 0831.35001
[16] V. Mazya, J. Rossmann, *Schauder estimates for solutions to boundary value problems for second order elliptic systems in polyhedral domains*, Appl. Anal., **83**:3 2004, 271–308. Zbl 1106.35015
[17] V.A. Kondrat'ev, *Boundary problems for parabolic equations in closed domains*, Trans. Mosc. Math. Soc. **15** (1966), 450–504. Zbl 0162.15203
[18] N.V. Krylov, *Lectures on elliptic and parabolic equations in Hölder spaces*, Graduate Studies in Math., **12**, AMS, Providence, RI, 1996. Zbl 0865.35001
[19] L. Hörmander, *The analysis of linear partial differential operators. I: Distribution theory and Fourier analysis*, Springer–Verlag, Berlin–Heidelberg, 1983. Zbl 0521.35001
[20] N. Tarkhanov, *Complexes of differential operators*, Kluwer Ac. Publ., Dordrecht, 1995. Zbl 0852.58076

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