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ON ALTERNATING SUBGROUP A_5 IN AUTOTOPISM GROUP OF FINITE SEMIFIELD PLANE

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ABSTRACT. We discuss well-known hypothesis that the full collineation group of any finite non-Desarguesian semifield plane is solvable. We continue to investigate the semifield planes of odd order which admit an autotopism subgroup isomorphic to alternating group A_5. It is proved that a semifield plane of any odd order does not admit A_5 in autotopism group.

Keywords: semifield plane, autotopism group, alternating group.

The paper continues the investigations of finite semifield planes that started in [1, 2, 3]. For more detailed information we recommend the monograph [4] and the review [5].

According to [4], a *semifield* is a set S with binary algebraic operations $+$ and $*$ satisfying the conditions:

1) $\langle S, +\rangle$ is an abelian group with a zero 0;

2) $\langle S^*, *\rangle$ is a loop ($S^* = S \setminus \{0\}$);

3) S satisfies the distributive laws $a*(b+c) = a*b+a*c$, $(b+c)*a = b*a+c*a$ for all $a, b, c \in S$.

A finite projective plane which is coordinatized by a semifield admits the large groups of central *collineations* (automorphisms) because it is a translation plane and a dual translation plane. A finite semifield plane can be determined, as any translation plane, with the use of a linear space over a finite field and a special family of linear transformations, so-called *spread set*. We will consider a coordinatizing

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semifield of order p^n (p is prime) as a n-dimensional linear space W over the field $\mathbb{Z}_p$. A subset of linear transformations

$$R = \{\theta(y) \mid y \in W\} \subset GL_n(p) \cup \{0\}$$

is called a spread set [4], if θ is an injection which satisfies the conditions:

1) $\theta(0) = 0$ is the zero matrix;
2) the identity matrix E has a pre-image $e \in W$;
3) θ is an additive mapping, $\theta(x + y) = \theta(x) + \theta(y)$ for all $x, y \in W$.

Then the space W with the multiplicative law

$$x * y = x \cdot \theta(y), \qquad x, y \in W,$$

is a semifield and it coordinatizes a semifield projective plane π of order p^n. Note, that if a spread set is a field $R \simeq GF(p^n)$, then π is a Desarguesian plane (classical projective plane), its structure and the properties are well-known [4].

Let π be a non-Desarguesian semifield plane of order p^n. Its full collineation group is $Aut\,\pi = T \leftthreetimes G$, where $T = \{\tau_{a,b} \mid a, b \in W\}$ is a group of translations,

$$\tau_{a,b} : (x, y) \to (x + a, y + b), \quad x, y \in W,$$

G is a translation complement, that is a stabilizer of the point $(0, 0)$. The automorphisms from G are defined by linear transformations of the linear space $W \oplus W$:

$$\alpha : (x, y) \to (x, y) \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Let $\Lambda < G$ in a group of *autotopisms*, which are the collineations fixing a triangle with the translation line $[\infty]$ as a side and the translation point (∞) as a vertex. Then $G = \Omega \leftthreetimes \Lambda$, where the group Ω of all elations with the center (∞) is isomorphic to a certain subgroup of the translation group T. So, a full collineation group of a finite semifield plane is determined by the structure of an autotopism group.

It is well-known the hypothesis ([4], p. 178) that a full collineation group of any finite non-Desarguesian plane is solvable (see also [6], question 11.76, 1990). To date, the hypothesis has been confirmed for some classes of semifield planes, but there is no general approach to solving this problem. Clearly, the problem is reduced to the solvability of an autotopism group [4]. According to Feit–Thompson theorem that any group of odd order is solvable, it is necessary to study the subgroups of even order in an autotopism group.

If we assume that an autotopism group is non-solvable, then the composition factors must be isomorphic to the known simple groups. A direct search of all variants from the list leads to a rather large numbers of situations. The paper [3] discuss the existence of an autotopism subgroup isomorphic to the alternating group A_5 (which is a subgroup of many simple nonabelian groups). We obtained a matrix representation of a spread set for such a semifield plane of any odd order p^N.

Theorem 1. *Let π be a semifield plane of odd order p^N (p be prime) with an autotopism group Λ including a subgroup $H \simeq A_5$. Then $N = 4n$ and the plane π can be determined by a $8n$-dimensional linear space over $\mathbb{Z}_p$ with a spread set $R \subset GL_{4n}(p) \cup \{0\}$ consisting of $(4n \times 4n)$-matrices*

$$\theta(V_1, U_1, V_2, U_2) = \begin{pmatrix} \mu(U_2) & -\psi(V_2) & \psi(U_1) & \varphi(V_1) \\ \psi(V_2) & \mu(U_2) & -\psi(V_1) & \varphi(U_1) \\ -\psi(U_1) & \psi(V_1) & \mu(U_2) & \varphi(V_2) \\ V_1 & U_1 & V_2 & U_2 \end{pmatrix}. \tag{1}$$

Here $V_1, U_1, V_2 \in Q_1$, $U_2 \in Q_2$; Q_1, Q_2 are spread sets in $GL_n(p) \cup \{0\}$; ψ, μ, φ are injective linear mappings from Q_1, Q_2 respectively to $GL_n(p) \cup \{0\}$, and $\mu(E) = E$, $\varphi(E) \neq E$, $\psi(E) = -E$ (E is an identity matrix).

A semifield plane with the spread set (1) exists if and only if all non-zero matrices of this set are nonsingular. It is shown in [3] that if $p \equiv 1(\bmod 4)$, then the spread set (1) contains a non-zero singular matrix. So, a semifield plane for this characteristic p admits no autotopism subgroup isomorphic to A_5. The report [7] announced the similar result if $N = 4$ or $N = 8$. And the present paper concludes the discussion of this issue.

Theorem 2. *A non-Desarguesian semifield plane of order p^N, where $p > 2$ is a prime, does not admit a subgroup of autotopisms isomorphic to the alternating group A_5.*

Proof. Using the notations of the Theorem 1, we assume that $\varphi(E) = P$. The mappings μ, φ, ψ are linear, so for any $t \in \mathbb{Z}_p$ we have $\mu(tE) = tE$, $\psi(tE) = -tE$ and $\varphi(tE) = tP$. Consider the matrix $\theta(xE, yE, E, 0)$ $(x, y \in \mathbb{Z}_p)$:

$$\begin{pmatrix} 0 & E & -yE & xP \\ -E & 0 & xE & yP \\ yE & -xE & 0 & P \\ xE & yE & E & 0 \end{pmatrix};$$

multiply its second «row» to y and add to the third «row»:

$$\begin{pmatrix} 0 & E & -yE & xP \\ -E & 0 & xE & yP \\ 0 & -xE & xyE & (1+y^2)P \\ xE & yE & E & 0 \end{pmatrix};$$

multiply the first «row» to x and add to the third «row»:

$$\begin{pmatrix} 0 & E & -yE & xP \\ -E & 0 & xE & yP \\ 0 & 0 & 0 & (1+x^2+y^2)P \\ xE & yE & E & 0 \end{pmatrix}.$$

It is well-known [4] that any element of a finite field is a sum of two squares; so, there exist $x, y \in \mathbb{Z}_p$ with the property $x^2 + y^2 = -1$. Thus, the matrix $\theta(xE, yE, E, 0)$ is singular. Theorem 2 is proved.

We emphasize that the proof of Theorem 2 is based on the result of Theorem 1 which required very complicated calculations. To sum up, we note that all other finite groups containing A_5 as a subgroup are excluded from the group of autotopisms of a semifield plane of any odd order.

In conclusion, we present the results obtained by the author using the described method and presented at the research seminar of the Department of higher algebra of MSU.

Theorem 3. *A non-Desarguesian semifield plane of order p^4, where $p > 2$ is a prime and $p \equiv 1(\bmod 4)$, does not admit a subgroup of autotopisms isomorphic to $SL(2,5)$.*

Theorem 4. *A non-Desarguesian semifield plane of order p^N, where $p > 2$ is a prime and $N \not\equiv 0(\bmod 8)$, does not admit a subgroup of autotopisms isomorphic to $Sz(8)$.*

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