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ALL TIGHT DESCRIPTIONS OF 3-PATHS IN PLANE GRAPHS WITH GIRTH AT LEAST 8

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Abstract. Lebesgue (1940) proved that every plane graph with minimum degree δ at least 3 and girth g (the length of a shortest cycle) at least 5 has a path on three vertices (3-path) of degree 3 each. A description is tight if no its parameter can be strengthened, and no triplet dropped.

Borodin et al. (2013) gave a tight description of 3-paths in plane graphs with $\delta \geq 3$ and $g \geq 3$, and another tight description was given by Borodin, Ivanova and Kostochka in 2017.

In 2015, we gave seven tight descriptions of 3-paths when $\delta \geq 3$ and $g \geq 4$. Furthermore, we proved that this set of tight descriptions is complete, which was a result of a new type in the structural theory of plane graphs. Also, we characterized (2018) all one-term tight descriptions if $\delta \geq 3$ and $g \geq 3$. The problem of producing all tight descriptions for $g \geq 3$ remains widely open even for $\delta \geq 3$.

Recently, eleven tight descriptions of 3-paths were obtained for plane graphs with $\delta = 2$ and $g \geq 4$ by Jendrol', Maceková, Montassier, and Soták, four of which descriptions are for $g \geq 9$. In 2018, Aksenov, Borodin and Ivanova proved nine new tight descriptions of 3-paths for $\delta = 2$ and $g \geq 9$ and showed that no other tight descriptions exist.

The purpose of this note is to give a complete list of tight descriptions of 3-paths in the plane graphs with $\delta = 2$ and $g \geq 8$.

Keywords: Plane graph, structure properties, tight description, 3-path, minimum degree, height, weight, girth.

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1. Introduction

Throughout the paper, G is a plane graph. Let $\delta(G)$ be the minimum vertex degree and let $w_k(G)$ be the minimum degree-sum of a path on k vertices in G. We will drop the argument when G is clear from context. The degree of a vertex v or a face f, that is the number of edges incident with v or f, is denoted by $d(v)$ or $d(f)$, respectively. A *k-vertex* is a vertex v with $d(v) = k$. By k^+ or k^- we denote any integer not smaller or not greater than k, respectively. Hence, a k^+-vertex v satisfies $d(v) \geq k$, etc. An edge uv is an *(i,j)-edge* if $d(u) \leq i$ and $d(v) \leq j$. A path uvw is a *path of type* (i,j,k) or *(i,j,k)-path* if $d(u) \leq i$, $d(v) \leq j$, and $d(w) \leq k$.

Already in 1904, Wernicke [29] proved that every G with $\delta = 5$ has a $(5,6)$-edge, and Franklin [17] strengthened this to the existence of at least two 6^--neighbors of a 5^--vertex; this implies that $w_3 \leq 17$, which bound is sharp. Recently, we proved [8] that there is also a $(5,6,6)$-path, which is tight, and there are no tight descriptions of 3-paths for $\delta = 5$ other than $\{(6,5,6)\}$ and $\{(5,6,6)\}$. (A description of paths is *tight* if no its parameter can be strengthened and no term dropped).

In 1996, Jendrol' and Madaras [25] ensured for $\delta = 5$ that $w_4 \leq 23$, which is sharp. Recently, we found [9] the first tight description of 4-paths for $\delta = 5$, and then Batueva, Borodin and Ivanova [4] found all remaining nine such tight descriptions.

It follows from Lebesgue's [28] results in 1940 that each G with $\delta \geq 3$ satisfies $w_2 \leq 14$. For 3-connected plane graphs, Kotzig [27] proved a precise result: $w_2 \leq 13$.

In 1972, Erdős (see [18]) conjectured that Kotzig's bound $w_2 \leq 13$ holds for all plane graphs with $\delta \geq 3$. Barnette (see [18]) announced to have proved this conjecture, but the proof has never appeared in print. The first published proof of Erdős' conjecture is due to Borodin [5]. More generally, Borodin [6] proved that every G with $\delta \geq 3$ contains a $(3,10)$-, or $(4,7)$-, or $(5,6)$-edge, which description is tight.

In 1993, Ando, Iwasaki, Kaneko [3] proved that every 3-connected G satisfies $w_3 \leq 21$, which is sharp due to the Jendrol' construction in [19]. This was refined by Borodin [7] in 1997 as follows: every 3-connected G has: (i) either $w_3 \leq 18$ or a vertex of degree ≤ 15 adjacent to two 3-vertices, and (ii) either $w_3 \leq 17$ or $w_2 \leq 7$. Here, the bounds $w_3 \leq 21$ and $w_3 \leq 17$ were known to be tight long ago, and the sharpness of $w_3 \leq 18$ was recently confirmed by Borodin et al. [14].

In 1997, Jendrol' [20] gave an approximate description of 3-paths: every G with $\delta \geq 3$ and $g \geq 3$ has a 3-path of one of the following types: $(10,3,10)$, $(7,4,7)$, $(6,5,6)$, $(3,4,15)$, $(3,6,11)$, $(3,8,5)$, $(3,10,3)$, $(4,4,11)$, $(4,5,7)$, or $(4,7,5)$.

In 2013, Borodin et al. [14] gave the first tight description of 3-paths: every G with $\delta \geq 3$ and $g \geq 3$ has a 3-path of one of the following types: $(3,4,11)$, $(3,7,5)$, $(3,10,4)$, $(3,15,3)$, $(4,4,9)$, $(6,4,8)$, $(7,4,7)$, $(6,5,6)$. Another similar tight description for $\delta \geq 3$ and $g \geq 3$ was given by Borodin, Ivanova and Kostochka [15].

In 2015, we [10] gave seven tight descriptions of 3-paths when $\delta \geq 3$ and $g \geq 4$. Furthermore, we proved that this set of descriptions is complete, which was a result of a new type in the structural theory of plane graphs. Also, we [12] characterized all one-term tight descriptions if $\delta \geq 3$ and $g \geq 3$. The problem of producing all tight descriptions for $g \geq 3$ remains widely open even for $\delta \geq 3$. Other results on k-paths with $k \geq 3$ and $\delta \geq 3$ can be found in surveys Borodin–Ivanova [11], Cranston–West [16] and Jendrol'–Voss [26].

Aksenov, Borodin and Ivanova [1] proved precise upper bounds for w_3 in several natural classes of plane graphs with $\delta = 2$ and $5 \leq g \leq 7$ and disproved a conjecture by Jendrol' and Maceková [21] concerning the case $g = 5$.

Recently, eleven tight descriptions of 3-paths were obtained for $\delta = 2$ and $g \geq 4$ by Jendrol', Maceková, Montassier, and Soták [21–24], four of which descriptions are for $g \geq 9$ (for details, see Theorems 1 and 2 below).

Aksenov, Borodin and Ivanova [2] gave the following complete list of tight descriptions of 3-paths for $\delta = 2$ and $g \geq 9$.

Theorem 1 ([2])**.** *There exist precisely these tight descriptions of* 3*-paths in plane graphs with minimum degree* 2 *and girth* g *at least* 9*:*
(A) $g \geq 16$*:* $\{(2,2,2)\}$ (folklore)*;*
(B) $11 \leq g \leq 15$*:*
$\{(2,2,3)\}$ ([22]) *and*
$\{(2,3,2)\}$*;*
(C) $g = 10$*:*
$\{(2,2,3),(2,3,2)\}$ ([21], the tightness shown in [22])*,*
$\{(2,4,2)\}$ ([22])*,*
$\{(2,3,3)\}$*,* $\{(2,2,4),(3,2,3)\}$*, and*
$\{(3,2,4)\}$*;*
(D) $g = 9$*:*
$\{(2,2,5),(2,3,2)\}$ ([23])*,*
$\{(2,5,2),(2,2,3)\}$*,* $\{(2,2,5),(3,2,3)\}$*,* $\{(2,5,3)\}$*,* $\{(2,3,5)\}$*, and*
$\{(3,2,5)\}$.

In [13] we described all tight descriptions of 3-paths centered at 2-vertices whenever $g \geq 6$.

The purpose of this note is to completely resolve the case $g \geq 8$, as follows.

Theorem 2. *There exist these and only these five tight descriptions of* 3*-paths in plane graphs with minimum degree* 2 *and girth at least* 8*:*
(i) $\{(2,2,5),(2,3,2)\}$ ([22])*;*
(ii) $\{(2,5,2),(2,2,3)\}$*;*
(iii) $\{(3,2,5)\}$ ([13])*;*
(iv) $\{(2,3,5)\}$*;*
(v) $\{(2,5,3)\}$.

2. Proving Theorem 2

Since $\{(2,2,5),(2,3,2)\}$ is a tight description ([22]), it follows that $\{(2,3,5)\}$ is also a description, since both a $(2,2,5)$-path and a $(2,3,2)$-path are also $(2,3,5)$-paths. So, we first prove that $\{(2,5,2),(2,2,3)\}$ is a description. This implies that $\{(2,5,3)\}$ is also a description, since both a $(2,5,2)$-path and a $(2,2,3)$-path are also $(2,5,3)$-paths.

Next, we show that all descriptions $\{(2,5,2),(2,2,3)\}$, $\{(2,3,5)\}$ and $\{(2,5,3)\}$ are tight.

Finally, we show that there are no tight descriptions for $g \geq 8$ other than those five listed in Theorem 2.

2.1. **Proving that** $\{(2,5,2),(2,2,3)\}$ **is a description.** We need the following refinement of the already mentioned Borodin's [6] tight description $\{(3,10),(4,7),(5,6)\}$ of 2-paths (that is, edges) in plane graphs with $\delta \geq 3$.

Lemma 1 ([13]). *Every plane graph with minimum degree at least* 3 *has at least one of the following:*

(a) *a* 3*-face incident with a* $(3,10)$*-, or* $(4,7)$*-, or* $(5,6)$*-edge;*

(b) *a* 4*-face incident either with two* 3*-vertices and another* 5^-*-vertex or with a* 3*-vertex, two* 4*-vertices and the forth vertex of degree at most* 5*;*

(c) *a* 5*-face incident with four* 3*-vertices and the fifth vertex of degree at most* 5*, where all parameters are best possible.*

Suppose on the contrary that G does not obey the description $\{(2,5,2),(2,2,3)\}$.

We consider the graph G^* with $\delta(G^*) \geq 3$ obtained from G by contracting all 2-vertices and look at its 5^--faces f^* implied by Lemma 1. Note that $g(G^*) \geq 3$ since $g(G) \geq 8$ and due to the absence $(2,2,2)$-paths in G.

If $f^* = v_1v_2v_3$ with $d(v_1) \leq 5$, then then the boundary $\partial(f)$ of the pre-image f of f^* in G must have at most two 2-vertices at the pair of edges since $g(G) \geq 8$, which is impossible due to the absence $(2,2,3)$-paths in G. On the other hand, both edges v_1v_2 and v_1v_3 cannot contain a 2-vertices due to the absence of $(2,5,2)$-paths in G. This implies $d(f) \leq 3+2+1 < 8$, a contradiction.

Now suppose $d(f^*) = v_1v_2v_3v_4$. It follows from Lemma 1(b) that f^* must have a 3-vertex, say v_1, opposite to a 5^--vertex, v_3, in $\partial(f)$. It is not hard to see that to avoid $(2,5,2)$-paths and $(2,2,3)$-paths, our f can have at most one 2-vertex at the pair of edges v_1v_2, v_1v_4 and at most two 2-vertices at v_2v_3, v_3v_4. However, then $d(f) \leq 4+1+1 < 8$, a contradiction.

Finally, suppose $d(f^*) = v_1 \dots v_5$ with $d(v_1) = \dots = d(v_4) = 3$ and $d(v_5) \leq 5$. Now at most one 2-vertex may be put on every edge in $\partial(f)$ and, moreover, at most one 2-vertex can appear at any two consecutive edges in $\partial(f)$. This implies $d(f) \leq 5 + 2 \times 1 < 8$, a contradiction.

2.2. **Proving the tightness of** $\{(2,5,2),(2,2,3)\}$**,** $\{(2,3,5)\}$ **and** $\{(2,5,3)\}$**.** To show the tightness of $\{(2,5,2),(2,2,3)\}$, we have to prove that neither $\{(2,4,2), (2,2,3)\}$ nor $\{(2,5,2),(2,2,2)\}$ is a description. To reject the former option, it suffices to put two 2-vertices on every edge of the icosahedron and note that the graph H_1 obtained has girth 9 but no $(4,4,4)$-paths, since each its 3-path goes through 5-vertex.

To reject the latter, we reproduce the graph obtained in Jendrol' et al. (see Fig. 5 in [22]). Take concentric cycles $W_8 = w_1 \dots w_8$, $XY_{16} = x_1y_1 \dots x_8y_8$, $Z_8 = z_1 \dots z_8$, and add a path with two internal 2-vertices between w_i to x_i and also between y_i and z_i whenever $1 \leq i \leq 8$. It remains to observe that H_2 obtained has no $(2,5,2)$-paths (and, in particular, no $(2,2,2)$-paths) and $g(H_2) = 8$.

The tightness of $\{(2,5,3)\}$ follows similarly from the same graphs H_1 and H_2. Indeed, we cannot strengthen $\{(2,5,3)\}$ to $\{(2,4,3)\}$, since each 3-path in H_1 goes through 5-vertex, and to $\{(2,5,2)\}$, since H_2 has no vertex adjacent to two 2-vertices.

Finally, to see the tightness of $\{(2,3,5)\}$, it suffices to observe that $\{(2,3,4)\}$ is invalid due to H_1, while $\{(2,2,5)\}$ fails to describe the graph H_3 that is a result of deleting all edges joining 10-vertices to 10-vertices from the triangulation obtained from the icosahedron by putting a 3-vertex into each face and joining it to the boundary vertices of this face. Note that H_3 is bipartite, with every edge joining a 3-vertex to a 5-vertex.

2.3. **Proving the non-existence of tight descriptions other than those five in Theorem 2.** Suppose $D = \{(x_1, y_1, z_1), \ldots, (x_k, y_k, z_k)\}$ is a tight description of 3-paths in plane graphs with $\delta = 2$ and $g \geq 8$. By symmetry, we can assume that $x_i \leq z_i$ whenever $1 \leq i \leq k$. It follows from the graph H_1 above that D must have an entry, say y_1 or z_1, not smaller than 5.

CASE 1. $y_1 \geq 5$. Now $(x_1, y_1, z_1) = (2, 5^+, 2)$ since $\{(2, 5, 3)\}$ is already a tight description. It follows from H_2 (which has no $(2, 5, 2)$-paths) that, say, $(x_2, y_2, z_2) = (2^+, 2^+, 3^+)$.

Since $\{(2, 5, 2), (2, 2, 3)\}$ is known to be a tight description, it follows that $D = \{(2, 5, 2), (2, 2, 3)\}$.

CASE 2. $z_1 \geq 5$. Now $(x_1, y_1, z_1) = (2, 2, 5^+)$ since $\{(3, 2, 5)\}$ and $\{(2, 3, 5)\}$ are tight descriptions. Let H_4 be a graph obtained from the dodecahedron by putting a 2-vertex on each edge. Clearly, H_4 satisfies $g(H_4) = 10$ and has only $(3, 2, 3)$- and $(2, 3, 2)$-paths. This implies that D must have a term, say (x_2, y_2, z_2), such that either $(x_2, y_2, z_2) = (3^+, 2^+, 3^+)$ or $(x_2, y_2, z_2) = (3^+, 2^+, 3^+)$.

In the first case, we have $D = \{(2, 2, 5), (3, 2, 3)\}$ since $\{(2, 2, 5), (3, 2, 3)\}$ is known to be a tight description. In the second case, we similarly deduce that $D = \{(2, 2, 5), (2, 3, 2)\}$, as desired.

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