S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 17, стр. 566–584 (2020) УДК 517, 515.168
DOI 10.33048/semi.2020.17.036 MSC 58A10, 58A12, 46E30, 55N20

THE SOBOLEV–POINCARÉ INEQUALITY AND THE $L_{q,p}$-COHOMOLOGY OF TWISTED CYLINDERS

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ABSTRACT. We establish a vanishing result for the $L_{q,p}$-cohomology ($q \geq p$) of a twisted cylinder, which is a generalization of a warped cylinder. The result is new even for warped cylinders. We base on the methods for proving the (p,q)-Sobolev–Poincaré inequality developed by L. Shartser.
Keywords: differential form, Sobolev–Poincaré inequality, $L_{q,p}$-cohomology, twisted cylinder, homotopy operator.

1. INTRODUCTION

The $L_{q,p}$-*cohomology* $H^k_{q,p}(M)$ of a Riemannian manifold (M,g) is, by definition, the quotient of the space of closed p-integrable differential k-forms by the exterior differentials of q-integrable k-forms. If $p = q$ then $L_{q,p}$-cohomology is usually referred to simply as L_p-cohomology and the index p is used instead of p,p in all the notations.

A *twisted product* $X \times_h Y$ of two Riemannian manifolds (X, g_X) and (Y, g_Y) is the direct product manifold $X \times_g Y$ endowed with a Riemannian metric of the form

$$g := g_X + h^2(x,y)g_Y, \tag{1.1}$$

where $h : X \times Y \to \mathbb{R}$ is a smooth positive function (see [5]). If X is a half-interval $[a,b)$ then the twisted product $X \times_h Y$ is called a *twisted cylinder*.

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The work of the second author was carried out in the framework of the State Contract of the Sobolev Institute of Mathematics (Project 0314—2019—0006).
Received February, 25, 2020, published April, 16, 2020.

We refer to an m-dimensional Riemannian manifold (M, g_M) as an *asymptotic twisted product* (respectively, as an *asymptotic twisted cylinder*) if, outside an m-dimensional compact submanifold, it is bi-Lipschitz equivalent to a twisted product (respectively, to a twisted cylinder).

In this paper, we prove some vanishing results for the $L_{q,p}$-cohomology of twisted cylinders $[a, b) \times_h N$ for a positive smooth function $h : [a, b) \times N \to \mathbb{R}$ in the case where the base N is a closed manifold and $p \geq q > 1$, $\frac{1}{p} - \frac{1}{q} < \frac{q-1}{q(\dim N+1)}$.

If in (1.1) the function h depends only on x then we obtain the familiar notion of a *warped product* (see [1]). Twisted products were the object of recent investigations [4, 6, 8, 9, 10, 16, 21]. The $L_{q,p}$-cohomology of warped cylinders $[a, b) \times_h N$, i.e., of product manifolds $[a, b) \times N$ endowed with a warped product metric

$$g = dt^2 + h^2(t)g_N,$$

where g_N is the Riemannian metric of N and $h : [a, b) \to \mathbb{R}$ is a positive smooth function, was studied by Gol′dshtein, Kuz′minov, and Shvedov [11], Kuz′minov and Shvedov [19, 20] (for $p = q$), and Kopylov [17, 18] for $p, q \in [1, \infty)$, $\frac{1}{p} - \frac{1}{q} < \frac{1}{\dim N+1}$. In particular, in [18], the second author found a sufficient condition for the L_{qp}-cohomology of a warped cylinder to be nontrivial (and even infinite-dimensional) in terms of Hardy's inequality.

The main result of the paper (Theorem 7.1) states that the $L_{q,p}$-cohomology $H^k_{q,p}(C^h_{a,b}N)$ of the twisted cylinder $C^h_{a,b}N$ with $q \geq p \geq 1$ and $\frac{1}{p} - \frac{1}{q} < \frac{q-1}{q(\dim N+1)}$ is zero provided that the de Rham cohomology $H^k_{\mathrm{DR}}(N)$ of the base N is trivial and some integral conditions on the twisting function involving p, q and an auxiliary parameter $\overline{p}$ are fulfilled.

The paper is organized as follows: In Section 2, we recall some basic definitions concerning the $L_{q,p}$-cohomology of Riemannian manifolds. Section 3 describes the representations of differential forms on a twisted cylinder obtained in [10] and analogous to the representations of forms on a warped product proposed by Gol′dshtein, Kuz′minov, and Shvedov in [12]. In Section 4, we develop a version of the weighted Sobolev–Poncaré inequality for convex sets in $\mathbb{R}^n$ by introducing a homotopy operator and consider some of its consequences; the exposition is based on the ideas of Shartser suggested in [22] and [23]. In Section 5, we consider a new homotopy operator A_α on differential forms defined on a convex domain in $\mathbb{R}^n$ and show that it guarantees the fulfillment of an inequality of Sobolev–Poincaré-type for $q \geq p \geq 1$ and $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$. In Section 6, using the ideas of Shartser's article [23], we "glue" local homotopy operators on a twisted cylinder to obtain a global homotopy operator. In Section 7, we use this global homotopy operator for proving our above-metioned main result on the triviality of the $L_{q,p}$-cohomology of a twisted cylinder (Theorem 7.1), and in Section 8, we extend this theorem to asymptotic twisted cylinders (Theorem 8.2). Section 9 contains some examples.

2. Basic Definitions

We recall the main definitions and notations.

Below we tacitly assume all manifolds to be oriented.

Let M be a smooth oriented Riemannian manifold. Denote by $\mathcal{D}^k(M) := C_0^\infty(M, \Lambda^k)$ the space of all smooth differential k-forms with compact support contained in $M \backslash \partial M$ denote by $L^1_{loc}(M, \Lambda^k)$ the space of locally integrable differential forms.

Denote by $L^p(M,\Lambda^k)$ the Banach space of locally integrable differential k-forms endowed with the norm $\|\theta\|_{L^p(M,\Lambda^k)} := \left(\int_M |\theta|^p dx\right)^{\frac{1}{p}} < \infty$ (as usual, we identify forms coinciding outside a set of measure zero). Of course, we can add a positive (smooth) weight $\sigma : M \to \mathbb{R}$ and thus integrate $|\theta|^p\sigma^p$ to obtain the weighted L^p-space $L^p(M,\Lambda^k,\sigma)$.

Definition 2.1. We call a differential $(k+1)$-form $\theta \in L^1_{loc}(M,\Lambda^{k+1})$ *the weak exterior derivative* (or *differential*) of a differential k-form $\phi \in L^1_{\rm loc}(M,\Lambda^k)$ and write $d\phi = \theta$ if

$$\int_M \theta\wedge\omega = (-1)^{k+1}\int_M \phi\wedge d\omega$$

for any $\omega \in \mathcal{D}^{n-k}(M)$.

Remark 2.2. Note that the orientability of M is not substantial in this definition since one can take integrals over orientable domains on M instead of integrals over M.

We then introduce an analog of Sobolev spaces for differential k-forms, i.e., the space of q-integrable forms with p-integrable weak exterior derivative:

$$\Omega^k_{q,p}(M) = \left\{\, \omega \in L^q(M,\Lambda^k) \mid d\omega \in L^p(M,\Lambda^{k+1})\right\}.$$

This is a Banach space for the graph norm

$$\|\omega\|_{q,p} = \left(\|\omega\|^2_{L^q(M,\Lambda^k)} + \|d\omega\|^2_{L^p(M,\Lambda^{k+1})}\right)^{1/2}.$$

The space $\Omega^k_{q,p}(M)$ is a reflexive Banach space for any $1 < q,p < \infty$. This can be proved using standard arguments of functional analysis.

We now define our basic ingredients (for three parameters r,q,p).

Definition 2.3. Put
(a) $Z^k_{p,r}(M) = \mathrm{Ker}[d : \Omega^k_{p,r}(M) \to L^r(M,\Lambda^{k+1})]$.
(b) $B^k_{q,p}(M) = \mathrm{Im}[d : \Omega^{k-1}_{q,p}(M) \to L^p(M,\Lambda^k)]$.

The subspace $Z^k_{p,r}(M)$ does not depend on r and is a closed subspace in $L^p(M,\Lambda^k)$ (see Lemma [14, Lemma 2.4(i)]). This allows us to use the notation $Z^k_p(M)$ for all $Z^k_{p,r}(M)$. Note that $Z^k_p(M) \subset L^p(M,\Lambda^k)$ is always a closed subspace but that is in general not true for $B^k_{q,p}(M)$. Denote by $\overline{B}^k_{q,p}(M)$ its closure in the L^p-topology. Observe also that since $d\circ d = 0$, one has $\overline{B}^k_{q,p}(M) \subset Z^k_p(M)$. Thus,

$$B^k_{q,p}(M) \subset \overline{B}^k_{q,p}(M) \subset Z^k_p(M) = \overline{Z}^k_p(M) \subset L^p(M,\Lambda^k).$$

Definition 2.4. Suppose that $1 \le q,p \le \infty$. The $L_{q,p}$*-cohomology* of (M,g) is defined as the quotient

$$H^k_{q,p}(M) := Z^k_p(M)/B^k_{q,p}(M)\,,$$

and the *reduced* $L_{q,p}$*-cohomology* of (M,g) is, by definition, the space

$$\overline{H}^k_{q,p}(M) := Z^k_p(M)/\overline{B}^k_{q,p}(M)\,.$$

Since $B^k_{p,q}$ is not always closed, the L_p-cohomology is in general a (non-Hausdorff) semi-normed space, while the reduced L_p-cohomology is a Banach space.

Below $|X|$ stands for the volume of a Riemannian manifold (X, g).

It follows from the results of [13] that, under suitable assumptions on p, q, the $L_{q,p}$-cohomology of a Riemannian manifold M can be expressed in terms of smooth forms.

Let $C^\infty(M, \Lambda^k)$ be the space of smooth k-forms on M.

Introduce the notations:

$$\begin{aligned}
C^\infty L^p(M, \Lambda^k) &:= C^\infty(M, \Lambda^k) \cap L^p(M, \Lambda^k);\\
C^\infty L^p(M, \Lambda^k, \sigma) &:= C^\infty(M, \Lambda^k) \cap L^p(M, \Lambda^k, \sigma);\\
C^\infty \Omega^k_{q,p}(M) &:= C^\infty(M, \Lambda^k) \cap \Omega^k_{q,p}(M);\\
C^\infty H^k_{q,p}(M) &:= \frac{C^\infty(M, \Lambda^k) \cap Z^k_p(M)}{C^\infty(M, \Lambda^k) \cap B^k_{q,p}(M)};\\
C^\infty \overline{H}^k_{q,p}(M) &:= \frac{C^\infty(M, \Lambda^k) \cap Z^k_p(M)}{C^\infty(M, \Lambda^k) \cap \overline{B}^k_{q,p}(M)}.
\end{aligned}$$

Theorem 2.5. [13, Theorem 12.5 and 12.8, Corollary 12.9]. *Let (M, g) be a n-dimensional Riemannian manifold and suppose the fulfillment of one of the following conditions:*

- *$p, q \in (1, \infty)$ and $\frac{1}{p} - \frac{1}{q} \le \frac{1}{n}$;*
- *$p, q \in [1, \infty)$ and $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$.*

Then the cohomology $H^*_{q,p}(M)$ can be represented by smooth forms, and thus $H^*_{q,p}(M) = C^\infty H^*_{q,p}(M)$.

More exactly, any closed form in $Z^k_p(M)$ is cohomologous to a smooth form in $L^p(M)$. Furthermore, if two smooth closed forms $\alpha, \beta \in C^\infty(M, \Lambda^k) \cap Z^k_p(M)$ are cohomologous modulo $d\Omega^{k-1}_{q,p}(M)$ then they are cohomologous modulo $dC^\infty\Omega^{k-1}_{q,p}(M)$.

Similarly, any reduced cohomology class can be represented by a smooth form.

3. Differential Forms on a Twisted Cylinder

From now on, $C^h_{a,b}N$ is the twisted cylinder $[a, b) \times_h N$, that is, the product of a half-interval $[a, b)$ and a closed smooth n-dimensional Riemannian manifold (N, g_N) equipped with the Riemannian metric $dt^2 + h^2(t, x)g_N$, where $h : [a, b) \times N \to \mathbb{R}$ is a smooth positive function.

Every differential form on $[a, b) \times N$ admits a unique representation of the form $\omega = \omega_A + dt \wedge \omega_B$, where the forms ω_0 and ω_1 do not contain dt (cf. [12]). It means that ω_0 and ω_1 can be viewed as one-parameter families $\omega_A(t)$ and $\omega_B(t)$, $t \in I$, of differential forms on N.

The modulus of a form ω of degree k on $C^h_{a,b}N$ is expressed via the moduli of $\omega_A(t)$ and $\omega_B(t)$ on N as follows:

$$|\omega(t, x)|_{C^h_{a,b}N} = \left[h^{-2k}(t, x)|\omega_A(t, x)|^2_N + h^{-2(k+1)}(t, x)|\omega_B(t, x)|^2_N\right]^{1/2} \tag{3.1}$$

Consequently,

(3.2)

$$\|\omega\|_{L^p(C^h_{a,b}N,\Lambda^k)}=\left[\int\limits_a^b\int\limits_N\left(h^{2(\frac{n}{p}-k)}(t,x)|\omega_A(t,x)|^2_N+h^{2(\frac{n}{p}-k+1)}(t,x)|\omega_B(t,x)|^2_N\right)^{\frac{p}{2}}dxdt\right]^{\frac{1}{p}}.$$

Put

$$f_{k,p}(t)=\min_{x\in N}\left\{h^{\frac{n}{p}-k}(t,x)\right\}$$

and

$$F_{k,p}(t)=\max_{x\in N}\left\{h^{\frac{n}{p}-k}(t,x)\right\}.$$

4. The Weighted Sobolev–Poincare Inequality for Convex Sets in $\mathbb{R}^n$

Denote by $\Omega^*_{loc}(M)$ the space all locally integrable differential forms with locally integrable weak differential.

Suppose that $D\subset R^n$ is a convex set and $\psi_y : D\times[0,1]\to D$, $\psi_y(x,t):=tx+(1-t)y$, is the homotopy induced by the convex structure. For a k-form $\omega\in\Omega^k_{loc}(D)$ the pullback $\psi^*_y\omega$ can be written as

$$\psi^*_y\omega(x,t)=\left(\psi^*_y\omega\right)_0(x,t)+dt\wedge\left(\psi^*_y\omega\right)_1(x,t),$$

where $\left(\psi^*_y\omega\right)_0$ and $\left(\psi^*_y\omega\right)_1$ do not contain dt.

For each $y\in D$ define a homotopy operator

$$K_y:\Omega^k_{loc}(D)\to\Omega^{k-1}_{loc}(D)$$

as follows:

$$K_y\omega(x):=\int_0^1\left(\psi^*_y\omega\right)_1(t)\,dt$$

It is easy to see that K_y takes smooth forms to smooth forms. It is proved in [15] that $K_yd\omega+dK_y\omega=\omega$ The following proposition is a generalization of results from [2] and Shartser's thesis [22] (see also [23]) to the weighted case and to unbounded convex domains.

Proposition 4.1. *Suppose that D is a convex set in $\mathbb{R}^n$, $q\geq p\geq 1$, and $\beta : D\to\mathbb{R}$ is a positive smooth function.*

If the inequality

$$C(k,p,q,n,\beta):=\int_0^1\sup_{z\in D}\|\beta(x)\mathbf{1}_{tx+(1-t)D}(z)\|_{L^q(D,dx)}t^k(1-t)^{-n/p}dt<\infty$$

holds then the inequality

$$\left\|\beta(x)\left\|\frac{K_yd\omega(x)}{|x-y|}\right\|_{L^p(D,dy)}\right\|_{L^q(D,dx)}\leq C(k,p,q,n,\beta)\,\|d\omega\|_{L^p(D,\Lambda^{k+1})}.$$

is valid for every $\omega\in\Omega^k_{loc}(D)$ such that $d\omega\in L^p(D,\Lambda^{k+1})$. Here $\mathbf{1}_{xt+(1-t)D}$ is the characteristic function of the set $xt+(1-t)D$.

Proof. By the definition of K_y, we have

$$\left\|\beta(x)\left\|\frac{K_yd\omega(x)}{|x-y|}\right\|_{L^p(D,dy)}\right\|_{L^q(D,dx)}=\left\|\beta(x)\left\|\int_0^1\frac{\left(\psi^*_yd\omega\right)_1(x,t)}{|x-y|}dt\right\|_{L^p(D,dy)}\right\|_{L^q(D,dx)}$$

$$\leq \int_0^1 \left\| \beta(x) \left\| \frac{\left(\psi_y^* d\omega\right)_1 (x,t)}{|x-y|} \right\|_{L^p(D,dy)} \right\|_{L^q(D,dx)} dt$$

$$\leq \int_0^1 \left\{ \int_D \beta^q(x) \left[\int_D \frac{\left|\left(\psi_y^* d\omega\right)_1 (x,t)\right|^p}{|x-y|^p} dy \right]^{q/p} dx \right\}^{1/q} dt.$$

As usual, we identify the tangent space to $\mathbb{R}^n$ at any of its points with $\mathbb{R}^n$. By easy calculations,

$$\left|\left(\psi_y^* d\omega\right)_1 (x,t)\right| \leq |x-y|\, t^k \left| d\omega\left(\psi_y(x,t)\right)\right|.$$

Therefore,

$$\int_0^1 \left\{ \int_D \beta^q(x) \left[\int_D \frac{\left|\left(\psi_y^* d\omega\right)_1 (x,t)\right|^p}{|x-y|^p} dy \right]^{q/p} dx \right\}^{1/q} dt$$

$$\leq \int_0^1 \left\{ \int_D \beta^q(x) \left[\int_D t^{kp} \left| d\omega\left(\psi_y(x,t)\right)\right|^p dy \right]^{q/p} dx \right\}^{1/q} dt$$

$$= \int_0^1 \left\{ \int_D \beta^q(x) \left[\int_D t^{kp} \left| d\omega\left(tx + (1-t)y\right)\right|^p dy \right]^{q/p} dx \right\}^{1/q} dt := I.$$

The change of variables $z = tx + (1-t)y$ in the inner integral yields

$$I = \int_0^1 \left\{ \int_D \beta^q(x) \left[\int_{tx+(1-t)D} |d\omega(z)|^p\, dz \right]^{q/p} dx \right\}^{1/q} t^k (1-t)^{-n/p} dt$$

Since D is convex, the set $tx + (1-t)D$ is contained in D for all $x \in D$ and $t \in [0,1]$. Using Minkowski's integral inequality, we infer

$$\left\{ \int_D \beta^q(x) \left[\int_{tx+(1-t)D} |d\omega(z)|^p\, dz \right]^{q/p} dx \right\}^{1/q}$$

$$= \left\{ \int_D \beta^q(x) \left[\int_D \mathbf{1}_{tx+(1-t)D}(z)\, |d\omega(z)|^p\, dz \right]^{q/p} dx \right\}^{1/q}$$

$$= \left\{ \left(\int_D \left[\int_D \beta^p(x) \mathbf{1}_{tx+(1-t)D}(z)\, |d\omega(z)|^p\, dz \right]^{q/p} dx \right)^{p/q} \right\}^{1/p}$$

$$= \left\{ \left\| \int_D \beta^p(x) \mathbf{1}_{tx+(1-t)D}(z)\, |d\omega(z)|^p\, dz \right\|_{L^{q/p}(D,dx)} \right\}^{1/p}$$

$$\leq \left\{ \int_D \left\| \beta^p(x) \mathbf{1}_{tx+(1-t)D}(z)\, |d\omega(z)|^p \right\|_{L^{q/p}(D,dx)} dz \right\}^{1/p}$$

$$= \left\{ \int_D \left(\int_D \beta^q(x) \mathbf{1}_{tx+(1-t)D}(z) \, |d\omega(z)|^q \, dx \right)^{p/q} dz \right\}^{1/p}$$
$$= \left\{ \int_D \left(\int_D \beta^q(x) \mathbf{1}_{tx+(1-t)D}(z) dx \right)^{p/q} |d\omega(z)|^p \, dz \right\}^{1/p}$$
$$\leq \left(\sup_{z\in D} \int_D \beta^q(x) \mathbf{1}_{tx+(1-t)D}(z) dx \right)^{1/q} \left(\int_D |d\omega(z)|^p \, dz \right)^{1/p}$$
$$= \sup_{z\in D} \|\beta(x)\mathbf{1}_{tx+(1-t)D}(z)\|_{L^q(D,dx)} \, \|d\omega\|_{L^p(D,\Lambda^{k+1})} \, .$$

The proposition follows. □

Estimate

$$C(k,p,q,n,\beta) = \int_0^1 \sup_{z\in D} \|\beta(x)\mathbf{1}_{tx+(1-t)D}(z)\|_{L^q(D,dx)} t^k (1-t)^{-n/p} dt$$

in particular cases.

Corollary 4.2. *Suppose that D is a convex set of finite measure in $\mathbb{R}^n$, $q \geq p \geq 1$, $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$, and the weight $\beta(x) \equiv 1$. Then*

$$C(k,p,q,n,1) \leq |D|^{1/q} \int_0^1 t^{k-n/q} (1-t)^{-n/p} \min(t^{n/q}, (1-t)^{n/q}) dt.$$

Remark 4.3. It is easy to see that the integral of the corollary exists because of the conditions imposed on p and q.

Proof. Using the change of variables $u = tx$, we obtain

$$\int_0^1 \sup_{z\in D} \left\|\mathbf{1}_{tx+(1-t)D}(z)\right\|_{L^q(D,dx)} t^k (1-t)^{-n/p} dt$$
$$= \int_0^1 \sup_{z\in D} \left\|\mathbf{1}_{u+(1-t)D}(z)\right\|_{L^q(tD,du)} t^{k-n/q} (1-t)^{-n/p} dt.$$

Note that $|tD \cap \{u + (1-t)D\}| \leq |D| \min(t^n, (1-t)^n)$. It follows that

$$\left\|\mathbf{1}_{u+(1-t)D}(z)\right\|_{L^q(tD,du)} \leq |D|^{1/q} \min(t^{n/q}, (1-t)^{n/q});$$
$$C(k,p,q,n,1) \leq |D|^{1/q} \int_0^1 t^{k-n/q} (1-t)^{-n/p} \min(t^{n/q}, (1-t)^{n/q}) dt$$

□

Corollary 4.4. *Suppose that U is a convex set of finite measure $|U|$ in $\mathbb{R}^n$, $D = [a,b) \times U$, $q \geq p \geq 1$, $\frac{1}{p} - \frac{1}{q} < \frac{q-1}{q(n+1)}$, and $\beta : [a,b) \to \mathbb{R}$ is an integrable positive function. If $\|\beta\|_{L^q([a,b))} < \infty$ then*

$$C(k,p,q,n,\beta) \leq |U|^{1/q} \, \|\beta\|_{L^q([a,b))} \, .$$

Proof. If $x \in D$ then $x = (\tau, w)$, where $\tau \in [a,b)$ and $w \in U$. Using the special type of the weight $\beta(x) := \beta(\tau)$ and representing $z \in D$ as $z = (\eta, \zeta)$ with $\eta \in [a,b)$ and $\zeta \in U$, we obtain

$$\int_0^1 \sup_{z\in D} \left\|\beta(x)\mathbf{1}_{tx+(1-t)D}(z)\right\|_{L^q(D,dx)} t^k (1-t)^{-\frac{n+1}{p}} dt$$

$$\leq \int_0^1 \sup_{a\leq\eta<b}\left(\int_a^b \beta^q(\tau)\mathbf{1}_{t\tau+(1-t)[a,b)}(\eta)d\tau\right)^{\frac{1}{q}} \sup_{\zeta\in U}\left(\int_U \mathbf{1}_{tw+(1-t)U}(\zeta)dw\right)^{\frac{1}{q}} t^k(1-t)^{-\frac{n+1}{p}}dt,$$

where $x=(\tau,w)$.

Using the change of variables $u=tw$ and the estimate

$$|tU\cap\{u+(1-t)U\}|\leq |U|\min(t^n,(1-t)^n),$$

we finally get

$$\int_0^1 \sup_{a\leq\eta<b}\left(\int_a^b \beta^q(\tau)\mathbf{1}_{t\tau+(1-t)[a,b)}(\eta)d\tau\right)^{\frac{1}{q}} \sup_{\zeta\in U}\left(\int_U \mathbf{1}_{tw+(1-t)U}(\zeta)dw\right)^{\frac{1}{q}} t^k(1-t)^{-\frac{n+1}{p}}dt$$
$$\leq |U|^{1/q}\|\beta\|_{L^q([a,b))}\int_0^1 t^{k-n/q}(1-t)^{-(n+1)/p}\min(t^{n/q},(1-t)^{n/q})dt$$

The conditions on p and q imply the finiteness of the last integral. □

Corollary 4.4 is a key ingredient in the proof of out main result, Theorem 7.1. Unfortunately, for being able to "separate" the variable t, we have to impose the stronger constraint $\frac{1}{p}-\frac{1}{q}<\frac{1}{n+1}-\frac{1}{q(n+1)}$ than the condition $\frac{1}{p}-\frac{1}{q}<\frac{1}{n+1}$ given by Proposition 4.1.

5. A New Homotopy Operator for $q\geq p$. The Case of a Convex Domain in $\mathbb{R}^n$

In the previous section, we considered the homotopy operator on Ω^*_{loc} of the form

$$A_\alpha=\int_D \alpha(y)K_y\omega(x)dy$$

for a convex set D in $\mathbb{R}^n$. We will need to modify A for obtaining some estimates.

Consider the same operator K_y as in the previous section:

$$\psi_y(x,t)=tx+(1-t)y,\quad K_y\omega(x)=\int_0^1(\psi_y)_1^*\omega dt.$$

Recall that $dK_y\omega+K_yd\omega=\omega$. Choose a smooth positive function $\alpha:D\to\mathbb{R}$ such that $\int_D\alpha(x)dx=1$ and put

$$A_\alpha\omega(x):=\int_D\alpha(y)K_y\omega(x)dy,\quad \omega\in\Omega^*_{\mathrm{loc}}.$$

By a straightforward calculation,

$$dA_\alpha\omega=d\left(\int_D\alpha(y)K_y\omega(x)dy\right)=\int_D\alpha(y)d_xK_y\omega(x)dy;$$

$$A_\alpha d\omega=\int_D\alpha(y)K_yd\omega(x)dy;$$

$$dA_\alpha\omega+A_\alpha d\omega=\int_D\alpha(y)\left[d_xK_y\omega(x)+K_yd\omega(x)\right]dy=\int_D\alpha(y)\omega(x)dy=\omega.$$

In particular, if $d\omega=0$ then

$$dA_\alpha\omega=\omega.$$

The definition of A_α easily implies the following

Proposition 5.1. *The homotopy operator A_α takes smooth forms to smooth forms.*

Definition 5.2. Call a smooth positive function $\alpha : D \to \mathbb{R}$ an *admissible weight* for a convex domain $D \subset \mathbb{R}^n$ and $p \geq 1$ if

$$\int_D \alpha(x)dx = 1; \quad \|\alpha\|_{L^{p'}(D)} < \infty; \quad \|\alpha(y)|y|\,\|_{L^{p'}(D)} < \infty.$$

For $p \geq 1$, we as usual put

$$p' = \begin{cases} \frac{p}{p-1} & \text{if } p > 1, \\ \infty & \text{if } p = 1 \end{cases}$$

Theorem 5.3. *Suppose that $q \geq p \geq 1$, $D \subset \mathbb{R}^n$ is a convex set, $\beta : D \to \mathbb{R}$ is a positive smooth function, and $\alpha : D \to \mathbb{R}$ is an admissible weight. If*

$$C_1(k,p,q,n,\beta) := \int_0^1 \sup_{z\in D} \|\beta(x)\mathbf{1}_{tx+(1-t)D}(z)\|_{L^q(D,dx)} t^k (1-t)^{-n/p} dt < \infty;$$

$$C_2(k,p,q,n,\beta) := \int_0^1 \sup_{z\in D} \||x|\beta(x)\mathbf{1}_{tx+(1-t)D}(z)\|_{L^q(D,dx)} t^k (1-t)^{-n/p} dt < \infty$$

then for any $\omega \in C^\infty L^p(D,\Lambda^k)$ we have

$$\|A_\alpha\omega\|_{L^q(D,\Lambda^{k-1},\beta)} \leq C(k,p,q,\alpha,\beta,n)\|\omega\|_{L^p(D,\Lambda^k)}$$

where

$$C(k,p,q,\alpha,\beta,n) = \|\alpha(y)|y|\|_{L^{p'}(D)} C_1(k,p,q,n,\beta) + \|\alpha\|_{L^{p'}(D)} C_2(k,p,q,n,\beta).$$

Proof. Put $\xi := A_\alpha\omega$. If $p > 1$ then, by Hölder's inequality, we infer

$$\|A_\alpha\omega\|_{L^q(D,\Lambda^{k-1},\beta)} = \left\|\beta(x)\int_D \alpha(y)K_y\omega(x)dy\right\|_{L^q(D,\Lambda^{k-1},dx)}$$
$$\leq \left\|\beta(x)\left\|\frac{K_y\omega(x)}{|x-y|}\right\|_{L^p(D,\Lambda^{k-1},dy)} \|\alpha(y)|x-y|\|_{L^{p'}(D,dy)}\right\|_{L^q(D,\Lambda^{k-1},dx)}.$$

The above estimate also obviously holds for $p = 1$.

By the triangle inequality,

$$\|\alpha(y)|x-y|\|_{L^{p'}(D,dy)} \leq |x|\|\alpha(y)\|_{L^{p'}(D,dy)} + \|\alpha(y)|y|\,\|_{L^{p'}(D,dy)}.$$

Therefore,

$$\|A_\alpha\omega\|_{L^q(\beta,D,\Lambda^{k-1})}$$
$$\leq \|\alpha(y)|y|\,\|_{L^{p'}(D,dy)} \left\|\beta(x)\left\|\frac{K_y\omega(x)}{|x-y|}\right\|_{L^p(D,\Lambda^{k-1},dy)}\right\|_{L^q(D,\Lambda^{k-1},dx)}$$
$$+ \|\alpha(y)\|_{L^{p'}(D,dy)} \left\|\beta(x)|x|\left\|\frac{K_y\omega(x)}{|x-y|}\right\|_{L^p(D,\Lambda^{k-1},dy)}\right\|_{L^q(D,\Lambda^{k-1},dx)}.$$

By Proposition 4.1,

$$\left\|\beta(x)\left\|\frac{K_y\omega(x)}{|x-y|}\right\|_{L^p(D,\Lambda^{k-1},dy)}\right\|_{L^q(D,\Lambda^{k-1},dx)} \leq C_1(k,p,q,n,\beta)\,\|\omega\|_{L^p(D,\Lambda^k)};$$

$$\left\|\beta(x)|x|\left\|\frac{K_y\omega(x)}{|x-y|}\right\|_{L^p(D,\Lambda^{k-1},dy)}\right\|_{L^q(D,\Lambda^{k-1},dx)} \leq C_2(k,p,q,n,\beta)\,\|\omega\|_{L^p(D,\Lambda^k)}.$$

The theorem is proved. □

Corollary 5.4. *Suppose that $q \ge p \ge 1$, $D \subset \mathbb{R}^n$ is a convex set, $\alpha : [a,b) \to \mathbb{R}$ is an admissible weight, $\beta, \gamma : D \to \mathbb{R}$ are positive smooth functions. If the conditions*

$$C_1(k,\overline{p},q,n,\beta) := \int_0^1 \sup_{z\in D} \|\beta(x)\mathbf{1}_{tx+(1-t)D}(z)\|_{L^q(D,dx)} t^k (1-t)^{-n/\overline{p}} dt < \infty;$$

$$C_2(k,\overline{p},q,n,\beta) := \int_0^1 \sup_{z\in D} \||x|\beta(x)\mathbf{1}_{tx+(1-t)D}(z)\|_{L^q(D,dx)} t^k (1-t)^{-n/\overline{p}} dt < \infty;$$

$$Q(k,\overline{p},p,\gamma) := \left\|\gamma^{-1}\right\|_{L^{p\overline{p}/(p-\overline{p})}(D)} < \infty$$

are fulfilled for some $\overline{p}$, $1 \le \overline{p} \le p$ (for $\overline{p} = p$, we put $\frac{p\overline{p}}{p-\overline{p}} = \infty$), then the inequality

$$\|A_\alpha\omega\|_{L^q(D,\Lambda^{k-1},\beta)} \le C(k,p,q,\alpha,\beta,\gamma,n)\|\omega\|_{L^p(D,\Lambda^k,\gamma)},$$

where

$$C(k,p,q,\alpha,\beta,\gamma,n) = Q(k,\overline{p},p,\gamma)\, C(k,\overline{p},q,n,\alpha,\beta),$$

holds for any $\omega \in C^\infty L^p(D,\Lambda^k)$.

Proof. By Theorem 5.3,

$$\|A_\alpha\omega\|_{L^q(D,\Lambda^{k-1},\beta)} \le C(k,\overline{p},q,n,\alpha,\beta)\, \|\omega\|_{L^{\overline{p}}(D,\Lambda^k)}\,.$$

If $\overline{p} < p$ then, using Hölder's inequality, we have

$$\|\omega\|_{L^{\overline{p}}(D,\Lambda^k)} \le \|\gamma\omega\|_{L^p(D,\Lambda^k)} \left\|\gamma^{-1}\right\|_{L^{p\overline{p}/(p-\overline{p})}(D)}\,. \tag{5.1}$$

Inequality (5.1) also holds for $\overline{p} = p$.

The corollary follows. □

Corollary 5.5. *Suppose that $q \ge p \ge 1$, $\frac{1}{p} - \frac{1}{q} < \frac{q-1}{q(n+1)}$, U is a bounded convex set in $\mathbb{R}^n$, $D = [a,b) \times U$, $\alpha : [a,b) \to \mathbb{R}$ is an admissible weight, and $\beta, \gamma : [a,b) \to \mathbb{R}$ are positive smooth functions. If the conditions $\|\beta\|_{L^q([a,b))}<\infty$, $\|\tau\beta(\tau)\|_{L^q([a,b))}<\infty$, and $\|\gamma^{-1}\|_{L^{p\overline{p}/(p-\overline{p})}([a,b))} < \infty$ are fulfilled for some $\overline{p}$, $1 \le \overline{p} \le p$ (for $\overline{p} = p$, we put $\frac{p\overline{p}}{p-\overline{p}} = \infty$), then the inequality*

$$\|A_\alpha\omega\|_{L^q(D,\Lambda^{k-1},\beta)} \le \text{const}\, \|\omega\|_{L^p(D,\Lambda^k,\gamma)}$$

with some constant depending k,p,q,n,α,β, and γ holds for any $\omega \in C^\infty L^p(D,\Lambda^k,\gamma)$.

Proof. Suppose that a number $\overline{p} \le p$ satisfies the conditions of the corollary.

If $x \in D$ then $x = (\tau, w)$, where $\tau \in [a,b)$ and $w \in U$. By Corollary 4.4, since $\frac{1}{\overline{p}} - \frac{1}{q} < \frac{q-1}{q(n+1)}$ and $\|\beta\|_{L^q([a,b))} < \infty$, we have

$$\int_0^1 \sup_{z\in D} \left\|\beta(\tau)\mathbf{1}_{tx+(1-t)D}(z)\right\|_{L^q(D,dx)} t^k (1-t)^{-(n+1)/\overline{p}} dt$$
$$\le |U|^{1/q}\|\beta\|_{L^q([a,b))} \int_0^1 t^{k-n/q}(1-t)^{-(n+1)/p} \min(t^{n/q},(1-t)^{n/q}) dt.$$

On the other hand, since $\|\tau\beta(\tau)\|_{L^q([a,b))} < \infty$, we have by Corollary 4.4:

$$\int_0^1 \sup_{z\in D} \left\|\, |x|\beta(\tau)\mathbf{1}_{tx+(1-t)D}(z)\right\|_{L^q(D,dx)} t^k (1-t)^{-(n+1)/\overline{p}} dt$$
$$= \int_0^1 \sup_{z\in D} \left\| \sqrt{\tau^2+w^2}\beta(\tau)\mathbf{1}_{tx+(1-t)D}(z)\right\|_{L^q(D,dx)} t^k (1-t)^{-(n+1)/\overline{p}} dt$$

$$\le \sqrt{2}\int_0^1 \sup_{z\in D}\left\| (|\tau|+|w|)\beta(\tau)\mathbf{1}_{tx+(1-t)D}(z)\right\|_{L^q(D,dx)} t^k(1-t)^{-(n+1)/\overline{p}}dt$$
$$\le \sqrt{2}\int_0^1 \sup_{z\in D}\left\| |\tau|\beta(\tau)\mathbf{1}_{tx+(1-t)D}(z)\right\|_{L^q(D,dx)} t^k(1-t)^{-(n+1)/\overline{p}}dt$$
$$+ \sqrt{2}\int_0^1 \sup_{z\in D}\left\| |w|\beta(\tau)\mathbf{1}_{tx+(1-t)D}(z)\right\|_{L^q(D,dx)} t^k(1-t)^{-(n+1)/\overline{p}}dt$$
$$\le \sqrt{2}|U|^{1/q}\|\tau\beta(\tau)\|_{L^q([a,b))}\int_0^1 t^{k-n/q}(1-t)^{-(n+1)/\overline{p}}\min(t^{n/q},(1-t)^{n/q})dt$$
$$+ \sqrt{2}\sup_{w\in U}|w|\,\|\beta\|_{L^q([a,b))}\int_0^1 t^{k-n/q}(1-t)^{-(n+1)/\overline{p}}\min(t^{n/q},(1-t)^{n/q})dt<\infty.$$

The relations $\|\tau\beta(\tau)\|_{L^q([a,b))}<\infty$ and $\||x|\beta(\tau)\|_{L^q(D)}<\infty$ enable us to apply Corollary 5.4 and obtain the desired assertion. □

6. Globalization: the Sobolev–Poincare Inequality on a Cylinder

Here we globalize the Sobolev–Poincaré inequality to cylinders. The main assertion of the section is

Theorem 6.1. *Suppose that M is the cylinder $[a,b)\times N$, where N is a closed manifold of dimension n, $q\ge p\ge 1$, $\frac{1}{p}-\frac{1}{q}<\frac{q-1}{q(n+1)}$, and $\beta,\gamma:[a,b)\to\mathbb{R}$ be positive smooth functions. Let ω be an exact k-form in $C^\infty L^p(M,\Lambda^k,\gamma)$. If the conditions $\|\beta\|_{L^q([a,b))}<\infty$, $\|t\beta(t)\|_{L^q([a,b))}<\infty$, and $\|\gamma^{-1}\|_{L^{p\overline{p}/(p-\overline{p})}([a,b))}<\infty$ are fulfilled for some $\overline{p}$, $1\le\overline{p}\le p$ (for $\overline{p}=p$, we put $\frac{p\overline{p}}{p-\overline{p}}=\infty$), then there exists a $(k-1)$-form $\xi\in C^\infty L^q(M,\Lambda^{k-1},\beta)$ such that*

$$d\xi=\omega \quad \text{and} \quad \|\xi\|_{L^q(M,\Lambda^{k-1},\beta)}\le \text{const}\,\|\omega\|_{L^p(M,\Lambda^k,\gamma)}. \tag{6.1}$$

Let $\tilde{\mathcal{U}}=\{\tilde{U}_x\}$, $x\in N$, be a coordinate open cover of the base N. At each point $x\in N$, consider a geodesic ball U_x that is geodesically convex (small balls are geodesically convex, see [7, Proposition 4.2]) and such that its closure (a compact set) is contained in $\tilde{U}_x$. Then $\mathcal{U}'=\{\tilde{U}_x\}$ is an open cover of N. Extract a finite subcover $\mathcal{U}=\{U_i\}$, $i=1,\dots,l$, from U^0. Since $\mathcal{U}$ consists of geodesic balls, it is a *good cover*, i.e., all finite intersections $U_I=U_{i_0}\cap\dots\cap U_{i_{s-1}}$, $I=(i_0,\dots,i_{s-1})$, are bi-Lipschitz diffeomorphic to convex open sets with compact closure in $\mathbb{R}^n$. With such a cover $\mathcal{U}$, we associate the corresponding cover $\mathcal{V}=\{V_i=[a,b)\times U_i\}$, $i=1,\dots,l$, of M and put $V_I=V_{i_0}\cap\dots\cap V_{i_{s-1}}$ for $I=(i_0,\dots,i_{s-1})$. Then each intersection V_I is bi-Lipschitz diffeomorphic to a cylinder of the form $[a,b)\times U_{\mathbb{R}^n}$, where $U_{\mathbb{R}^n}$ is a convex set with compact closure in $\mathbb{R}^n$. By analogy with [23], we put

$$K^{k,0}:=C^\infty(M,\Lambda^k);\quad K^{k,s}:=\bigoplus_{i_0<\dots<i_{s-1}}C^\infty(V_I,\Lambda^k).$$

Given $\varkappa\in K^{r,s}$, denote by $\varkappa_I$, $I=(i_0,\dots,i_{s-1})$, $i_0<\dots<i_s$, the components of $\varkappa$. Define a coboundary operator $\delta:K^{k,s}\to K^{k,s+1}$ as follows:

$$(\delta\varkappa)_J=\left.\left(\sum_{r=0}^{s}(-1)^r\varkappa_{j_0\dots\hat{j}_r\dots j_s}\right)\right|_{V_J},\quad J=(j_0,\dots,j_s).$$

Let $L^q(K^{k,s})$ be the space of elements $\varkappa \in K^{k,s}$ with the finite norm

$$\|\varkappa\|_{L^q(K^{k,s},\beta)} = \sum_{i_0<\dots i_{s-1}} \|\varkappa_I\|_{L^q(V_I,\Lambda^k,\beta)}.$$

As usual, if $\varkappa \in K^{k,s}$ has components $\varkappa_I$, $I = (i_0, \dots, i_{s-1})$, $i_0 < \dots < i_s$, and ν is a permutation of the set $\{0, \dots, s-1\}$ then $\alpha_{\nu(I)} = \alpha_I \operatorname{sign} \nu$.

The following proposition is a modification for our case of [23, Proposition 3.6], which is in turn an adaptation of [3, Propositions 8.3 and 8.5].

Proposition 6.2. *$(K^{k,\bullet}, \delta)$ is an exact complex. Moreover, if $\lambda \in L^q(K^{k,s+1}, \beta)$ satisfies $\delta\lambda = 0$ then there exists $\varkappa \in L^q(K^{k,s}, \beta)$ such that $\lambda = \delta\varkappa$ and*

- $\|\varkappa\|_{L^q(K^{k,s},\beta)} \le \operatorname{const} \|\lambda\|_{L^q(K^{k,s+1},\beta)}$
- $\|d\varkappa\|_{L^q(K^{k+1,s},\beta)} \le \operatorname{const} \big(\|\lambda\|_{L^q(K^{k,s+1},\beta)} + \|d\lambda\|_{L^q(K^{k+1,s+1},\beta)}\big)$.

Proof. The fact that $(K^{k,\bullet}, \delta)$ is an exact complex was established in [3, Propositions 8.3 and 8.5] but we will give the standard argument for completeness. If $\varkappa \in L^q(K^{k,s}, \beta)$ then

$$\begin{aligned}(\delta(\delta\varkappa))_{i_0\cdots_{s+1}} &= \sum_r (-1)^i (\delta\varkappa)_{i_0\dots\hat{i}_r\dots i_{s+1}} \\ &= \sum_{l<r} (-1)^r(-1)^l \varkappa_{i_0\dots\hat{i}_l\dots\hat{i}_r\dots i_{s+1}} + \sum_{l<r} (-1)^r(-1)^{l-1} \varkappa_{i_0\dots\hat{i}_l\dots\hat{i}_r\dots i_{s+1}} = 0.\end{aligned}$$

Suppose that $\lambda \in L^q(K^{k,s+1}, \beta)$ is such that $\delta\lambda = 0$. Let $\tilde{\rho}_j$ be a partition of unity subordinate to the cover $\{U_i\}$ of N. Then the functions $\rho_j : M \to \mathbb{R}$, $\rho_j(t, x) = \tilde{\rho}_j(x)$ for all $(t, x) \in M = [a, b) \times N$, constitute a partition of unity subordinate to the cover $\{V_i\}$ of M. Put

$$\varkappa_{i_0\dots i_{s-1}} := \sum_j \rho_j \lambda_{j i_0\dots i_{s-1}}. \tag{6.2}$$

Show that $\delta\varkappa = \lambda$.

We have

$$(\delta\varkappa)_{i_0\dots i_s} = \sum_r (-1)^r \varkappa_{i_0\dots\hat{i}_r\dots i_s} = \sum_{r,j} (-1)^r \rho_j \lambda_{j i_0\dots\hat{i}_r\dots i_s}.$$

Since λ is a cocycle,

$$(\delta\lambda)_{j i_0\dots i_s} = \lambda_{i_0\dots i_s} + \sum_r (-1)^{r+1} \lambda_{j i_0\dots\hat{i}_r\dots i_s} = 0$$

Hence,

$$(\delta\varkappa)_{i_0\dots i_s} = \sum_j \rho_j \sum_r (-1)^r \lambda_{j i_0\dots\hat{i}_r\dots i_s} = \sum_j \rho_j \lambda_{i_0\dots i_s} = \lambda_{i_0\dots i_s}.$$

Thus, $(K^{k,\bullet}, \delta)$ is indeed an exact complex.

The element $\varkappa$ defined by (6.2) admits the estimates of the norms mentioned in the proposition.

Indeed, we infer

$$\|\varkappa\|_{L^q(K^{k,s},\beta)} = \sum_{i_0<\dots<i_{s-1}} \Big\| \sum_j \rho_j \lambda_{j i_0\dots i_{s-1}} \Big\|_{L^q(U_I,\Lambda^k,\beta)}$$

$$\leq \sum_{i_0<\dots<i_{s-1}} \sum_j \|\rho_j \lambda_{ji_0\dots i_{s-1}}\|_{L^q(U_I,\Lambda^k,\beta)}$$
$$\leq \sum_{i_0<\dots<i_{s-1}} \sum_j \|\lambda_{ji_0\dots i_{s-1}}\|_{L^q(U_{j,I},\Lambda^k,\beta)} \leq \|\lambda\|_{L^q(K^{k,s+1},\beta)},$$

which gives the first estimate of the proposition.

Let us prove the second estimate. We have

$$d\varkappa_{i_0\dots i_{s-1}} = \sum_j (d\rho_j \wedge \lambda_{ji_0\dots i_{s-1}} + \rho_j d\lambda_{ji_0\dots i_{s-1}}).$$

Therefore,

$$\|\varkappa\|_{L^q(K^{k+1,s},\beta)} = \sum_{i_0<\dots<i_{s-1}} \left\| \sum_j d\rho_j \wedge \lambda_{ji_0\dots i_{s-1}} + \rho_j d\lambda_{ji_0\dots i_{s-1}} \right\|_{L^q(U_I,\Lambda^{k+1},\beta)}$$
$$\leq \sum_{i_0<\dots<i_{s-1}} \sum_j \left(\|d\rho_j \wedge \lambda_{ji_0\dots i_{s-1}}\|_{L^q(U_I,\Lambda^{k+1},\beta)} + \|\rho_j d\lambda_{ji_0\dots i_{s-1}}\|_{L^q(U_I,\Lambda_{k+1}),\beta} \right)$$
$$\leq \text{const} \sum_{i_0<\dots<i_{s-1}} \sum_j \left(\|\lambda_{ji_0\dots i_{s-1}}\|_{L^q(U_I,\Lambda^k,\beta)} + \|d\lambda_{ji_0\dots i_{s-1}}\|_{L^q(U_I,\Lambda^{k+1},\beta)} \right)$$
$$= \text{const} \left(\|\lambda\|_{L^q(K^{k,s+1},\beta)} + \|d\lambda\|_{L^q(K^{k+1,s+1},\beta)} \right).$$

□

Now, applying the general scheme of [23], we first construct some elements $\xi^s \in L^q(K^{k-s-1,s+1},\beta)$ and then elements $x^s \in L^q(K^{k-s-1,s},\beta)$ such that $\xi = x^0 \in C^\infty L^q(M,\Lambda^{k-1},\beta)$ is an element satisfying the claim of Theorem 6.1.

Construction of the elements $\xi^s \in L^q(K^{k-s-1,s+1},\beta)$.

Put $\xi^{-1} = \omega$ and define (by induction) ξ^s by setting its component $(\xi^s)_I$ to be a solution to the equation

$$d\xi^s_I = (\delta\xi^{s-1})_I \tag{6.3}$$

in V_I, $I = (i_0,\dots,i_s)$ such that

$$\|\xi^s_I\|_{L^q(V_I,\Lambda^{k-s-1},\beta)} \leq \text{const}\, \|(\delta\xi^{s-1})_I\|_{L^q(V_I,\Lambda^{k-s},\beta)} \tag{6.4}$$

for $0 \leq s \leq k-1$.

Note that such a solution always exists due to the local Sobolev–Poincaré inequality (Corollary 5.5) since V_I is bi-Lipschitz diffeomorphic to a cylinder over a convex subset in $\mathbb{R}^n$ with compact closure.

We have the following estimate of the weighted q-norm of ξ^s:

Proposition 6.3. *If $I = (i_0,\dots,i_s)$ then*

$$\|\xi^s_I\|_{L^q(V_I,\Lambda^{k-s-1},\beta)} \leq \text{const}\, \|\omega\|_{L^q(M,\Lambda^k,\gamma)}.$$

Proof. Use induction on s. For $s = 0$, the assertion follows from the local Sobolev–Poincaré inequality. Let now $s > 0$. We infer

$$\|\xi^s_I\|_{L^q(V_I,\Lambda^{k-s-1},\beta)} \leq \text{const}\, \|(\delta\xi^{s-1})_I\|_{L^q(V_I,\Lambda^{k-s},\beta)}$$
$$\leq \text{const} \sum_{r=0}^{s} \|\xi^{s-1}_{i_0\dots\hat{i}_r\dots i_{s-1}}\|_{L^q(V_I,\Lambda^{k-s},\beta)}$$

$$
\begin{aligned}
&\le \text{const} \sum_{r=0}^{s} \|\xi^{s-1}_{i_0\dots\hat{i}_r\dots i_{s-1}}\|_{L^q(V_{i_0\dots\hat{i}_r\dots i_{s-1}},\Lambda^{k-s},\beta)} \\
&\le \text{const} \sum_{r=0}^{s} \|\omega\|_{L^p(M,\Lambda^k,\gamma)} \le \text{const}\, \|\omega\|_{L^p(M,\Lambda^k,\gamma)}
\end{aligned}
$$

□

Note that ξ^{k-1} is a collection of 0-forms satisfying the condition $d\delta\xi^{k-1} = 0$. Thus, the functions $(\delta\xi^{k-1})_I$ are constants on each set V_I, $I = (i_0, \dots, i_k)$. The global constant functions $(\delta\xi^{k-1})_I$ on M belong to $L^q(M, \beta)$ due to the hypotheses on β.

The following assertion is Theorem 3.10 in [23]:

Lemma 6.4. *There exists $c \in K^{0,k}$ with constant components c_I, $I = (i_0, \dots, i_{k-1})$, such that*

$$
(\delta c)_I = \sum_{r=0}^{k} (-1)^r c_{i_0\dots\hat{i}_r\dots i_k} (\delta\xi^{k-1})_I, \quad I = (i_0, \dots, i_k).
$$

In addition, there exist numbers $b_{I,L} \in \mathbb{R}$, $I = (i_0, \dots, i_{k-1})$, $L = (i_0, \dots, i_k)$, such that

$$
c_I = \sum_L b_{I,L} (\delta\xi^{k-1})_L,
$$

where $b_{i,L}$ depend on the chosen cover $\mathcal{U}$ of N.

We have

Proposition 6.5. *The constants c_I of Lemma 6.4 satisfy the estimate*

$$
\|c_I\|_{L^q(V_I,\beta)} \le \text{const}\, \|\omega\|_{L^p(M,\Lambda^k,\gamma)}
$$

Proof. By Lemma 6.4, each c_I is representable as $c_I = \sum_L b_{I,L}(\delta\xi^{k-1})_L$. Hence,

$$
\|c_I\|_{L^q(V_I,\beta)} \le \sum_L |b_{I,L}| \|(\delta\xi^{k-1})_L\|_{L^q(V_I,\beta)}.
$$

Since $(\delta\xi^{k-1})_L$ is a globally defined constant function on M as in the proof of Proposition 6.3, we have

$$
\begin{aligned}
\|(\delta\xi^{k-1})_L\|_{L^q(V_I,\beta)} &= \frac{\|\beta\|_{L^q([a,b))}(\text{vol}(U_I))^{1/q}}{\|\beta\|_{L^q([a,b))}(\text{vol}(U_L))^{1/q}} \|(\delta\xi^{k-1})_L\|_{L^q(V_L,\beta)} \\
&= \frac{(\text{vol}(U_I))^{1/q}}{(\text{vol}(U_L))^{1/q}} \|(\delta\xi^{k-1})_L\|_{L^q(V_L,\beta)} \le \text{const}\, \|\omega\|_{L^p(M,\Lambda^k,\gamma)}.
\end{aligned}
$$

This gives the estimate of the proposition. □

Construction of the elements $x^s \in L^q(K^{k-s-1,s}, \beta)$.

Let us now glue all the forms ξ^s, $s-0, \dots, k-1$, into a global form ξ satisfying (6.1). Construct by induction elements $x^s \in L^q(K^{k-s-1,s}, \beta)$, $s = k-1, \dots, 1, 0$, such that $\xi = x^0$ is a desired form on M.

Put $\tilde{\xi}^{k-1}_I = \xi^{k-1}_I - c_I$, where c_I is as in Lemma 6.4, $I = (i_0, \dots, i_{k-1})$. We have $d\tilde{\xi}^{k-1}_I = d\xi^{k-1}_I$ and $\delta\tilde{\xi}^{k-1}_I = 0$. By Proposition 6.2, there exists $x^{k-1} \in L^q(K^{0,k-1}, \beta)$ such that $\delta x^{k-1} = \tilde{\xi}^{k-1}$ and

$$
\|dx^{k-1}\|_{L^q(K^{1,k-1},\beta)} \le \text{const}\, \|\tilde{\xi}^{k-1}\|_{L^q(K^{0,k},\beta)},
$$

$$\|x^{k-1}\|_{L^q(K^{0,k-1},\beta)} \leq \text{const}\left(\|\tilde{\xi}^{k-1}\|_{L^q(K^{0,k},\beta)} + \|d\tilde{\xi}^{k-1}\|_{L^q(K^{1,k},\beta)}\right).$$

Propositions 6.3 and 6.5 yield

$$\|x^{k-1}\|_{L^q(K^{0,k-1},\beta)} \leq \text{const}\,\|\omega\|_{L^p(M,\Lambda^k,\gamma)} \tag{6.5}$$

and

$$\begin{aligned} \|dx^{k-1}\|_{L^q(K^{1,k-1},\beta)} &\leq \text{const}\left(\|\omega\|_{L^p(M,\Lambda^k,\gamma)} + \|\delta\xi^{k-2}\|_{L^q(K^{1,k},\beta)}\right) \\ &\leq \text{const}\left(\|\omega\|_{L^p(M,\Lambda^k,\gamma)} + \|\xi^{k-2}\|_{L^q(K^{1,k-1},\beta)}\right) \leq \text{const}\,\|\omega\|_{L^p(M,\Lambda^k,\gamma)}. \end{aligned} \tag{6.6}$$

Suppose that $x^{k-(r-1)}$ is already constructed. By Proposition 6.2, there exists x^{k-r} such that

$$\delta x^{k-r} = \xi^{k-r} - dx^{k-r+1},$$

where

$$\|x^{k-r}\|_{L^q(K^{r-1,k-r},\beta)} \leq \text{const}\,\|\xi^{k-r} - dx^{k-r+1}\|_{L^q(K^{r-1,k-r+1},\beta)} \tag{6.7}$$

and

$$\begin{aligned} &\|dx^{k-r}\|_{L^q(K^{r,k-r},\beta)} \\ &\leq \text{const}\left(\|\xi^{k-r} - dx^{k-r+1}\|_{L^q(K^{r-1,k-r+1},\beta)} + \|d\xi^{k-r}\|_{L^q(K^{r,k-r+1},\beta)}\right) \\ &\quad\leq \text{const}\big(\|\xi^{k-r}\|_{L^q(K^{r-1,k-r+1},\beta)} + \|dx^{k-r+1}\|_{L^q(K^{r-1,k-r+1},\beta)} \\ &\qquad\qquad + \|\delta\xi^{k-r-1}\|_{L^q(K^{r,k-r+1},\beta)}\big) \\ &\qquad\qquad\leq \text{const}\left(\|\omega\|_{L^p(M,\Lambda^k,\gamma)} + \|dx^{k-r+1}\|_{L^q(K^{r-1,k-r+1},\beta)}\right). \end{aligned} \tag{6.8}$$

Here the last inequality stems from the fact that

$$\delta(\xi^{k-r} - dx^{k-r+1}) = \delta\delta x^{k-r} = 0.$$

The above considerations imply the following

Proposition 6.6. *The forms x^s admit the estimates:*
(1) $\|x^{k-r}\|_{L^q(K^{r-1,k-r},\beta)} \leq \text{const}\,\|\omega\|_{L^p(M,\Lambda^k,\gamma)}$;
(2) $\|dx^{k-r}\|_{L^q(K^{r,k-r},\beta)} \leq \text{const}\,\|\omega\|_{L^p(M,\Lambda^k,\gamma)}$.

Proof. Use induction on r. For $r = 1$, (1) and (2) are just estimates (6.5) and (6.6). Assume that $r > 1$. For proving estimate (2), observe that, by the induction hypothesis and (6.8),

$$\begin{aligned} \|dx^{k-r}\|_{L^q(K^{r,k-r},\beta)} &\leq \text{const}\left(\|\omega\|_{L^p(M,\Lambda^k,\gamma)} + \|dx^{k-r+1}\|_{L^q(K^{r-1,k-r+1},\beta)}\right) \\ &\leq \text{const}\,\|\omega\|_{L^p(M,\Lambda^k,\gamma)}. \end{aligned}$$

Now, Proposition 6.3 and estimates (6.7) and (2) yield

$$\begin{aligned} &\|x^{k-r}\|_{L^q(K^{r-1,k-r},\beta)} \leq \text{const}\,\|\xi^{k-r} - dx^{k-r+1}\|_{L^q(K^{r-1,k-r+1},\beta)} \\ &\leq \text{const}\left(\|\xi^{k-r}\|_{L^q(K^{r-1,k-r+1},\beta)} + \|dx^{k-r+1}\|_{L^q(K^{r-1,k-r+1},\beta)}\right) \leq \text{const}\,\|\omega\|_{L^p(M,\Lambda^k,\gamma)}. \end{aligned}$$

□

Finally, put $\xi = x^0$. Then $d\xi = \omega$. Indeed, we have

$$\delta(\omega - dx^0) = \delta\omega - d\delta x^0 = \delta\omega - d(\xi^0 - dx^1) = \delta\omega - d\xi^0 = 0.$$

Since $\delta(\omega - dx^0)_i = (\omega - dx^0)|_{V_i}$, we infer that $\omega = dx_0$ on M. By Proposition 6.6,

$$\|\xi\|_{L^q(M,\Lambda^{k-1},\beta)} = \|x^0\|_{L^q(K^{k-1,0},\beta)} \leq \text{const}\,\|\omega\|_{L^p(M,\Lambda^k,\gamma)}.$$

Theorem 6.1 is completely proved.

7. $L_{q,p}$-Cohomology of a Twisted Cylinder

Theorem 7.1. *Suppose that N is a closed manifold of dimension n, $H^k_{\mathrm{DR}}(N) = 0$, $q \geq p \geq 1$, and $\frac{1}{p} - \frac{1}{q} < \frac{q-1}{q(n+1)}$. If*

$$\|\max(F_{k-2,q}, F_{k-1,q})\|_{L^q([a,b))} < \infty, \quad \|t\max(F_{k-2,q}, F_{k-1,q})(t))\|_{L^q([a,b))} < \infty,$$

and

$$\|\{\min(f_{k-1,p}, f_{k,p})\}^{-1}\|_{L^{\frac{p\overline{p}}{p-\overline{p}}}([a,b))} < \infty$$

for some $\overline{p}$, $1 \leq \overline{p} \leq p$ (for $\overline{p} = p$, we put $\frac{p\overline{p}}{p-\overline{p}} = \infty$), then $H^k_{q,p}(C^h_{a,b}N) = 0$.

Proof. Let $\overline{M}$ be the cylinder $[a,b)\times N$ with the usual product metric. By the Künneth formula for the de Rham cohomology, we have

$$H^k_{\mathrm{DR}}(\overline{M}) = H^k_{\mathrm{DR}}(N) = 0.$$

Let $\omega \in C^\infty L^p(C^h_{a,b}N, \Lambda^k)$. Using expression (3.2) for the norm and the definition of $f_{l,p}$, we infer

$$\begin{aligned}
(7.1)\quad & \|\omega\|_{L^p(\overline{M},\Lambda^k,\min(f_{k-1,p},f_{k,p}))} \\
&= \left[\int_a^b \{\min(f_{k-1,p}(t), f_{k,p}(t))\}^p \int_N (|\omega_A(t,x)|^2_N + |\omega_B(t,x)|^2_N)^{\frac{p}{2}} dxdt\right]^{\frac{1}{p}} \\
&\leq \left[\int_a^b \int_N \big(h^{2(\frac{n}{p}-k)}(t,x)|\omega_A(t,x)|^2_N + h^{2(\frac{n}{p}-k+1)}(t,x)|\omega_B(t,x)|^2_N\big)^{\frac{p}{2}} dxdt\right]^{\frac{1}{p}} \\
&= \|\omega\|_{L^p(C^h_{a,b}N,\Lambda^k)}.
\end{aligned}$$

Thus, $\omega \in C^\infty L^p(\overline{M}, \Lambda^k, \min(f_{k-1,p}, f_{k,p}))$. Since the de Rham cohomology $H^k_{\mathrm{DR}}(\overline{M})$ is trivial, ω is exact, and we can apply Theorem 6.1, by which there exists $\xi \in C^\infty L^q(\overline{M}, \Lambda^k, \max(F_{k-2,q}(t), F_{k-1,q}(t))$ with

$$(7.2)\qquad \|\xi\|_{L^q(\overline{M},\Lambda^{k-1},\max(F_{k-2,q},F_{k-1,q}))} \leq \mathrm{const}\|\omega\|_{L^p(\overline{M},\Lambda^k,\min(f_{k-1,p},f_{k,p}))}.$$

For this form ξ, we have

$$\begin{aligned}
(7.3)\quad & \|\xi\|_{L^q(C^h_{a,b}N,\Lambda^{k-1})} \\
&= \left[\int_a^b \int_N \big(h^{2(\frac{n}{q}-k+1)}(t,x)|\xi_A(t,x)|^2_N + h^{2(\frac{n}{q}-k+2)}(t,x)|\xi_B(t,x)|^2_N\big)^{\frac{q}{2}} dxdt\right]^{\frac{1}{q}} \\
&\leq \left[\int_a^b \{\max(F_{k-2,q}(t), F_{k-1,q}(t))\}^q \int_N (|\xi_A(t,x)|^2_N + |\xi_B(t,x)|^2_N)^{\frac{q}{2}} dxdt\right]^{\frac{1}{q}} \\
&= \|\xi\|_{L^q(\overline{M},\Lambda^{k-1},\max(F_{k-2,q},F_{k-1,q}))}.
\end{aligned}$$

Combining (7.1),(7.2), and (7.3), we obtain

$$\|\xi\|_{L^q(C^h_{a,b}N,\Lambda^{k-1})} \leq \mathrm{const}\,\|\omega\|_{L^p(C^h_{a,b}N,\Lambda^k)}.$$

Thus, $C^\infty H^k_{q,p}(C^h_{a,b}N) = 0$, and hence, by Theorem 2.5, also $H^k_{q,p}(C^h_{a,b}N) = 0$. □

Remark 7.2. As was observed in the introduction, in [18], the second author established sufficient conditions for the L_{qp}-cohomology of a warped cylinder to be infinite-dimensional in terms of "almost duality" and Hardy's inequality. In [18], the base manifold must have nontrivial L_p- and L_q-cohomology.

8. $L_{q,p}$-Cohomology of an Asymptotic Twisted Cylinder

Recall the following definition, given in [10]:

Definition 8.1. We refer to a pair (M, X) consisting of an m-dimensional manifold M and an m-dimensional compact submanifold X with boundary as an *asymptotic twisted cylinder* $AC^h_{a,b}\partial X$ if $M\backslash(X\backslash\partial X)$ is bi-Lipschitz diffeomorphically equivalent to the twisted cylinder $C^h_{a,b}\partial X$.

For asymptotic twisted cylinders, Theorem 7.1 gives:

Theorem 8.2. *Let* $(M, X) = AC^h_{a,b}\partial X$ *be an asymptotic twisted cylinder with* $\dim M = \dim X = m = n+1$. *Assume that* $q \geq p \geq 1$, $\frac{1}{p} - \frac{1}{q} < \frac{q-1}{qm}$, *and* $H^k_{\mathrm{DR}}(\partial X) = 0$. *If*

$$\|\max(F_{k-2,q}, F_{k-1,q})\|_{L^q([a,b))} < \infty, \quad \|t\max(F_{k-2,q}, F_{k-1,q})(t))\|_{L^q([a,b))} < \infty,$$

and

$$\|\{\min(f_{k-1,p}, f_{k,p})\}^{-1}\|_{L^{\frac{p\overline{p}}{p-\overline{p}}}([a,b))} < \infty$$

for some $\overline{p}$, $1 \leq \overline{p} \leq p$ *(for* $\overline{p} = p$, *we put* $\frac{p\overline{p}}{p-\overline{p}} = \infty$*), then* $H^k_{q,p}(M) = 0$.

Proof. Since bi-Lipschitz diffeomorphisms preserve L_{p_1} and L_{p_2} and extension by zero gives a topological isomorphism between the spaces $W^*_{p_1,p_2}(C^h_{a,b}\partial X)$ and $W^*_{p_1,p_2}(M)$ for all p_1, p_2, we have a topological isomorphism

$$H^*_{p_1,p_2}(M) \cong H^*_{p_1,p_2}(C^h_{a,b}\partial X)$$

for all p_1, p_2. The theorem now follows from Theorem 7.1. □

9. Examples

Let us analyze the conditions of the last theorems for comparatively simple cases. Suppose that N is the n-dimensional sphere S^n. Then $H^k_{\mathrm{DR}}(N) = 0$ for any $k \neq 0, n$. By the hypothesess of the theorems, $q \geq p \geq 1$ and $\frac{1}{p} - \frac{1}{q} < \frac{q-1}{q(n+1)}$. Put

$$s(t) := \max_{x\in S^n} h(t,x) \quad \text{and} \quad g(t) := \min_{x\in S^n} h(t,x).$$

Then, by definition,

$$I_{1,q,k} := \max(F_{k-2,q}, F_{k-1,q}) = \max(s^{\frac{n}{q}-k+2}, s^{\frac{n}{q}-k+1}),$$

$$I_{2,q,k}(t) := t\max(F_{k-2,q}, F_{k-1,q})(t)) = t\max(s^{\frac{n}{q}-k+2}(t), s^{\frac{n}{q}-k+1}(t))$$

and

$$I_{3,p,k} := \{\min(f_{k-1,p}, f_{k,p})\}^{-1} = \{\min(g^{\frac{n}{p}-k+1}, g^{\frac{n}{p}-k})\}^{-1}.$$

By the hypotheses of the theorems, we must check the integrability of these three functions in the corresponding degrees under the above-mentioned restrictions on p and q.

Suppose for simplicity that $s(t)$ and $g(t)$ are smooth increasing functions tending to ∞ as $t \to b-0$. Denote the maximal integrability intervals for s^u and g^v by $(-\infty, \alpha)$ and $(-\infty, \beta)$, i.e s^u is integrable on $[a, b)$ for every $u < \alpha$ and is not integrable for every $u > \alpha$ and similarly for g^v. Let also α_1 be the supremum of μ such that $ts^\mu(t)$ is integrable on $[a, b)$.

For this case $I_{1,q,k} = s^{\frac{n}{q}-k+2}$, $I_{2,q,k}(t) = ts^{\frac{n}{q}-k+2}(t)$, and $I_{3,p,k} = g^{k-\frac{n}{p}}$.

The conditions of the theorems are fulfilled if

$$\frac{n}{q} - k + 2 < \min(\alpha, \alpha_1), \quad \frac{n}{p} - k > -\beta.$$

Note that these inequalities cannot hold simultaneously if $b = \infty$. In this case, α, α_1, and β are all negative, whence $\frac{n}{p} - k > \frac{n}{q} - k + 2$. We thus have $\frac{1}{p} - \frac{1}{q} > \frac{2}{n}$, which contradicts the hypotheses.

Examine more closely the case of $0 \leq a < b < \infty$. The function t is bounded, and hence $\alpha = \alpha_1$. Therefore, the inequalities for $I_{1,q,k}$ and $I_{3,p,k}$ can be combined into one inequality

$$\frac{k-2+\alpha}{n} < \frac{1}{q} \leq \frac{1}{p} < \frac{k-\beta}{n}.$$

It means that the additional condition $k - 2 + \alpha \leq k - \beta$, i.e., $\alpha + \beta \leq 2$, must be fulfilled.

The last condition is $\frac{1}{p} - \frac{1}{q} < \frac{q-1}{q(n+1)}$, i.e., $p \leq q < \frac{np}{n+1-p}$.

Summarizing, we conclude that for known integrability limits α and β, we need to check two simple conditions for p and q:

$$\alpha + \beta \leq 2, \ p < q < \frac{np}{n+1-p}$$

and the inequality

$$\frac{k-2+\alpha}{n} < \frac{1}{q} \leq \frac{1}{p} < \frac{k-\beta}{n}.$$

for the degree k.

Under these conditions, the cohomology of the twisted cylinder $C^h_{[a,b)}S^n$ vanishes.

For example, if $s(t) = g(t) = (b-t)^{-2}$ then $\alpha = \beta = 1/2$. For $p = 2$ we have $2 \leq q < 2\frac{n}{n-1}$.

The last inequality yields

$$\frac{k-3/2}{n} < \frac{1}{q} \leq \frac{1}{2} < \frac{k-1/2}{n}. \tag{9.1}$$

Let q be an arbitrary number in $(2, \frac{8}{3})$. Then the second inequality $q < \frac{2n}{n-1}$ gives us the constraint $n < q/(q-2)$. Since $q/(q-2) < 4$, we can take $n = 4$. We have $1/2 < (2k-1)/8$, i.e., $k > 2$. For $k = 3$, the leftmost inequality gives us the valid condition $3/8 < 1/q$. Note that if $q = 2$ then always $q < \frac{2n}{n-1}$. If n is even and $k = \frac{n}{2} + 1$ then all inequalities in (9.1) are fulfilled. Thus, we have

$$H^3_{q,2}\big(C^h_{[a,b)}S^4\big) = 0 \quad \text{if } q \in \left[2, \frac{8}{3}\right)$$

and

$$H^{l+1}_{2,2}\big(C^h_{[a,b)}S^{2l}\big) = 0, \quad l \geq 2.$$

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