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INTEGRATION OF SYSTEMS OF TWO SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER THAT ADMIT FOUR ESSENTIAL OPERATORS

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Abstract. We discuss an algorithm for integrating systems of two second-order ordinary differential equations (ODE) with a small parameter that admit approximate Lie algebras with four essential generators. The algorithm is a modification of the method of consecutive order reduction and is based on using operators of invariant differentiation. A special attention is given to the peculiarities of its application in dependence of the structural properties of Lie algebras of approximate symmetries.

Keywords: system of two second-order ordinary differential equations with a small parameter, approximate Lie algebra of generators, operator of invariant differentiation, invariant representation, differential invariant, integration of equations.

1. Introduction

Integrating rth-order ordinary differential equations (ODE) with r point symmetries is a classical problem in group analysis. It was solved for scalar second-order ODE (see, e.g., [1]), third-order ODE (see [2]), systems of two second-order ODE (see, e.g., [3, 4]).

Recently, an approach for integrating systems of p-order ODE of the form

$$u^{j,(p)} = f^j\left(t, u^1, \dots, u^n, u^{1,(1)}, \dots, u^{n,(p-1)}\right), \quad j = 1, \dots, n, \tag{1}$$

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that admit r-dimensional ($r = np$) Lie algebras L_r of operators

$$X_\alpha = \xi_\alpha(t, u^1, \dots, u^n)\frac{\partial}{\partial t} + \sum_{j=1}^{n} \eta^j_\alpha(t, u^1, \dots, u^n)\frac{\partial}{\partial u^j}, \quad \alpha = 1, \dots, r, \tag{2}$$

was suggested in [5]. This approach is based on using an operator of invariant differentiation (OID) (see, e.g., [6]) of the admitted Lie algebra L_r. In [7], the first author introduced the OID for the approximate Lie algebra, proposed an algorithm for its constructing, and considered the application of this OID to order reduction of systems of ODE of the form

$$u^{j,(p)}(t) = f^j_{(0)}(t, u, u', \dots, u^{(p-1)}) + \varepsilon f^j_{(1)}(t, u, u', \dots, u^{(p-1)}), \quad j = 1, \dots, n, \quad \varepsilon \ll 1, \tag{3}$$

that admit approximate Lie algebras of operators (up to $o\,(\varepsilon)$[1])

$$\begin{aligned} X_{\alpha_0} &= X_{\alpha_0,(0)} + \varepsilon X_{\alpha_0,(1)}, \quad \alpha_0 = 1, \dots, r_0, \\ X_{\alpha_1} &= \varepsilon X_{\alpha_1,(0)}, \quad \alpha_1 = r_0 + 1, \dots, \widetilde{r}. \end{aligned}$$

In the present article, we study the applicability of the algorithm proposed in [7] to systems of the form

$$\begin{cases} \ddot{x} = f_0\,(t, x, y, \dot{x}, \dot{y}) + \varepsilon f_1\,(t, x, y, \dot{x}, \dot{y})\,, \\ \ddot{y} = g_0\,(t, x, y, \dot{x}, \dot{y}) + \varepsilon g_1\,(t, x, y, \dot{x}, \dot{y}) \end{cases} \tag{4}$$

that admit approximate Lie algebras $\widetilde{L}$ with four essential operators. The peculiarities of the consecutive order reduction and integration of systems of ODE in dependence of the structural properties of the admitted approximate Lie algebras are described; the symmetries of reduced systems of ODE are considered and the possibilities of their applying are discussed.

2. An Invariant Representation of Systems of Two Second-Order ODE with a Small arameter

Suppose that system (4) admits four approximate operators

$$\begin{aligned} X_{\alpha_0} &= X_{\alpha_0,(0)} + \varepsilon X_{\alpha_0,(1)}, \quad \alpha_0 = 1, \dots, r_0, \\ X_{\alpha_1} &= \varepsilon X_{\alpha_1,(0)}, \quad \alpha_1 = r_0 + 1, \dots, 4, \end{aligned} \tag{5}$$

where

$$X_{\alpha,(p)} = \xi_{\alpha,(p)}(t, x, y)\frac{\partial}{\partial t} + \eta^1_{\alpha,(p)}(t, x, y)\frac{\partial}{\partial x} + \eta^2_{\alpha,(p)}(t, x, y)\frac{\partial}{\partial y}, \quad \alpha = 1, \dots, 4, \ p = 0, 1.$$

Assume that these operators are essential for some approximate Lie algebra $\widetilde{L}$, i.e., the set of the operators $\{X_{\alpha_0}, \varepsilon X_{\alpha_0}, X_{\alpha_1}\}$ obtained by neglecting the terms of order $o(\varepsilon)$ provides a basis of the approximate Lie algebra $\widetilde{L}$. In this case, the operators $X_{\alpha,(0)}$, $\alpha = 1, \dots, 4$, generate an "exact" Lie algebra L_4 which is admitted by corresponding "unperturbed" system of two second-order ODE

$$\begin{cases} \ddot{x} = f_0\,(t, x, y, \dot{x}, \dot{y})\,, \\ \ddot{y} = g_0\,(t, x, y, \dot{x}, \dot{y})\,. \end{cases} \tag{6}$$

[1]We use the following notations: the expression $f(z, \varepsilon) = o(\varepsilon)$ means $\lim\limits_{\varepsilon \to 0} \frac{f(z,\varepsilon)}{\varepsilon} = 0$; the expression $f \approx g$ means $f(z, \varepsilon) - g(z, \varepsilon) = o(\varepsilon)$.

The operators $X_{\alpha_0,(0)}$, $\alpha_0 = 1, \ldots, r_0$, generate a subalgebra L_{r_0} of the Lie algebra L_4 (see [8]). The dimension of the approximate Lie algebra $\widetilde{L}$ varies from five (if $r_0 = 1$) to eight (if $r_0 = 4$). The operators of the form X_{α_0} are called "zeroth-order"operators (with respect to ε), and operators X_{α_1} are "first-order". The corresponding operators $X_{\alpha_0,(0)}$ are called stable, and $X_{\alpha_1,(0)}$ are unstable with respect to the perturbation under consideration.

To represent system (4) in terms of the differential invariants of the admitted approximate Lie algebra, introduce the matrices $\Delta^{(2)}_{r_0,(0)}$, $\Delta^{(2)}_{r_0,(1)}$, $\Lambda^{(2)}_{4,(0)}$ consisting of the extended operators $\hat{X}^{(2)}_{\alpha_0,(0)}$, $\hat{X}^{(2)}_{\alpha_0,(1)}$, and $\hat{X}^{(2)}_{\alpha,(0)}$, $\alpha_0 = 1, \ldots, r_0$, $\alpha = 1, \ldots, 4$. For example, the matrix $\Delta^{(2)}_{r_0,(0)}$ has the form

$$\begin{pmatrix} \xi_{1,(0)} & \eta^1_{1,(0)} & \eta^2_{1,(0)} & \zeta^{1,(1)}_{1,(0)} & \zeta^{2,(1)}_{1,(0)} & \zeta^{1,(2)}_{1,(0)} & \zeta^{2,(2)}_{1,(0)} \\ \ddots & & & & & & \\ \xi_{r_0,(0)} & \eta^1_{r_0,(0)} & \eta^2_{r_0,(0)} & \zeta^{1,(1)}_{r_0,(0)} & \zeta^{2,(1)}_{r_0,(0)} & \zeta^{1,(2)}_{r_0,(0)} & \zeta^{2,(2)}_{r_0,(0)} \end{pmatrix},$$

where $\zeta^{i,(q)}_{\alpha,(p)}$ is the coordinate of the extension of the operator $X_{\alpha,(p)}$ to the qth-order derivative of $x^{(q)}$ if $i = 1$ or $y^{(q)}$ if $i = 2$ (see, e.g., [6]). In particular,

$$\zeta^{1,(q)}_{\alpha,(p)} = D^{(q)}_t \left(\eta^1_{\alpha,(p)} - \dot{x}\xi_{\alpha,(p)} \right) + \xi_{\alpha,(p)} x^{(q+1)}.$$

Constructing differential invariants of the approximate Lie algebra $\widetilde{L}$ is reduced to solving systems of first-order partial differential equations (see [9])

$$\begin{aligned} &\Omega_0 \colon \hat{X}^{(2)}_{\alpha,(0)} \left(I_{(0)} \right) = 0, \ \ \alpha = 1, \ldots, 4, \\ &\Omega_1 \colon \hat{X}^{(2)}_{\alpha_0,(0)} \left(I_{(1)} \right) + \hat{X}^{(2)}_{\alpha_0,(1)} \left(I_{(0)} \right) \approx 0, \ \ \alpha_0 = 1, \ldots, r_0. \end{aligned}$$

This system defines "zeroth-order" differential invariants (with respect to ε) of the form $I_k = I_{k,(0)} + \varepsilon I_{k,(1)}$, and "first-order" differential invariants (with respect to ε) of the form $J_s = \varepsilon J_{s,(0)}$. Furthermore, the functions $I_{k,(0)}$ are invariants of the Lie algebra L_4, the functions $I_{k,(1)}$ are particular solutions to system Ω_1, and the functions $J_{s,(0)}$ are particular solutions to the corresponding homogeneous system of equations

$$\hat{X}^{(2)}_{\alpha_0,(0)} \left(J_{s,(0)} \right) = 0, \ \ \alpha_0 = 1, \ldots, r_0,$$

and $J_{s,(0)}$ are functionally independent with $I_{k,(0)}$.

It is shown in [4] that if system (6) that corresponds to system (4) defines a regular manifold (see [6]) and is representable by differential invariants of L_4 then the conditions rg $\Lambda^{(1)}_{4,(0)} =$ rg $\Lambda^{(2)}_{4,(0)} = 4$ are fulfiled. In this case, the approximate Lie algebra $\widetilde{L}$ has three "zeroth-order" differential invariants (with respect to ε). Two of them, $I^1_2 = I^1_{2,(0)} + \varepsilon I^1_{2,(1)}$ and $I^2_2 = I^2_{2,(0)} + \varepsilon I^2_{2,(1)}$, are second-order, and one of them, $I = I_{(0)} + \varepsilon I_{(1)}$, is a first-order differential invariant (if rg $\Lambda^{(0)}_{4,(0)} = 3$) or an algebraic invariant (if rg $\Lambda^{(0)}_{4,(0)} = 2$).

If the conditions rg $\Delta^{(1)}_{r_0,(0)} = r_0$, rg $\Delta^{(2)}_{r_0,(0)} = r_0$ are also fulfilled then the approximate Lie algebra $\widetilde{L}$ has $4 - r_0$ "first-order" differential invariants (with respect to ε). Obviously, the number of "first-order" invariants depends on the number of stable operators and varies from zero for $r_0 = 4$ to three for $r_0 = 1$. Then, according to [9], system (4) is representable as

$$\left\{ \begin{aligned} I_2^1 &= F_0(I) + \varepsilon F_1\left(I_{(0)}, J_{1,(0)}, \dots, J_{4-r_0,(0)}\right), \\ I_2^2 &= G_0(I) + \varepsilon G_1\left(I_{(0)}, J_{1,(0)}, \dots, J_{4-r_0,(0)}\right). \end{aligned} \right. \tag{7}$$

Note that if the condition $\operatorname{rg} \Delta^{(k)}_{r_0,(0)} = r_0$ does not hold then Ω_1 gives additional equations that should be added to the system Ω_0. However, such cases do not appear for the approximate Lie algebras under consideration.

3. An Operator of Invariant Differentiation for an Approximate Lie Algebra

An operator of invariant differentiation (OID) λD_t for an approximate Lie algebra $\widetilde{L}$ that is admitted by system (16) is introduced by analogy with the case of an exact Lie algebra. This operator transforms one differential invariant of the Lie algebra into another one. Assume that the OID is linear with respect to ε.

It is shown in [7] that an OID for an approximate Lie algebra can be obtained from the condition

$$[\lambda D_t,\, X_\alpha] \approx 0. \tag{8}$$

Furthermore, if the matrix $\left(c^\gamma_{\alpha\beta}\right)$ composed of the structure constants $c^\gamma_{\alpha\beta}$ of the exact Lie algebra L_4 (where α, β give the row number and γ is the column number), satisfies the condition

$$\operatorname{rg}\left(c^\gamma_{\alpha\beta}\right) < 4 \tag{9}$$

then the factor λ of the OID can be found in the form

$$\lambda = \left(D_t(\Phi_0 + \varepsilon\Phi_1)\right)^{-1} \approx \frac{D_t\Phi_0 - \varepsilon D_t\Phi_1}{\left(D_t\Phi_0\right)^2}. \tag{10}$$

The functions Φ_0, Φ_1 are obtained as particular solutions to the corresponding systems of inhomogeneous first-order partial differential equations

$$\begin{aligned} &\Pi_0\colon \hat{X}^{(1)}_{\alpha,(0)}\Phi_0 = K_{\alpha,(0)}, \;\; \alpha = 1, \dots, 4, \\ &\Pi_1\colon \hat{X}^{(1)}_{\alpha_0,(0)}\Phi_1 + \hat{X}^{(1)}_{\alpha_0,(1)}\Phi_0 = K_{\alpha_0,(1)}, \;\; \alpha_0 = 1, \dots, r_0. \end{aligned} \tag{11}$$

These systems are obtained from (8) after splitting with respect to ε and a single integration. Methods for solving such systems require studying their completeness and consistency (see, e.g., [10]). Since $\operatorname{rg} \Lambda^{(1)}_{4,(0)} = 4$, the subsystem Π_0 is consistent for all $K_{\alpha,(0)}$. The subsystem Π_0 is complete if the constants $K_{\alpha,(0)}$ satisfy the system of algebraic equations

$$\sum_{\alpha=1}^{4} c^\alpha_{\beta\gamma} K_{\alpha,(0)} = 0, \;\; \beta,\, \gamma = 1, \dots, 4, \tag{12}$$

where $c^\alpha_{\beta\gamma}$ $(\alpha, \beta, \gamma = 1, \dots, 4)$ are the structure constants of the Lie algebra L_4. The subsystem Π_1 is consistent at the solutions of Π_0 if the rank of Π_1 is equal to r_0 (since $\operatorname{rg} \Delta^{(1)}_{r_0,(0)} = r_0$) and complete if the constants $K_{\alpha_0,(1)}$ satisfy the system of inhomogeneous equations

$$\sum_{\alpha_0=1}^{r_0} c^{\alpha_0}_{\beta_0\gamma_0} K_{\alpha_0,(1)} = -\sum_{\sigma=r_0+1}^{r_0+r} c^{\sigma}_{\beta_0\gamma_0} K_{\sigma-r_0,(0)}, \tag{13}$$

where $c^{\alpha_0}_{\beta_0\gamma_0}$, $c^{\sigma}_{\beta_0\gamma_0}$, $\alpha_0, \beta_0, \gamma_0 = 1, \ldots, r_0, \sigma = r_0+1, \ldots, r_0+4$, are the structure constants of the subalgebra L_{r_0} and the structure constants of the approximate Lie algebra $\widetilde{L}$ (that do not coincide with the structure constants of L_r and L_{r_0}) respectively.

Consider the sum $\Phi_0 + \varepsilon\Phi_1$, $\Phi_0 \neq \text{const}$ of the particular solutions Φ_0 and Φ_1. By [7], this sum is an invariant of the approximate Lie algebra with three essential operators which are linear combinations of operators (5) with the coefficients defined by (12) and (13).

The obtained OID is further used for constructing an approximate first integral of system (4) that is representable as (7). The main step of the proposed order reduction algorithm of a system of ODE with a small parameter is constructing an auxiliary system of differential relations connecting the differential invariants of the admitted Lie algebra, their full derivatives, and also the full derivatives of the constructed functions Φ_0 and Φ_1. This system can be regarded as a system of equations in full differentials, i.e. a system of first-order ODE, and its general solution gives an approximate first integral of the original system.

Remark. In the case of an unperturbed system of ODE, such an auxiliary system consists only of one first-order ODE. Such an equation is constructed for system (6) with four symmetries (see [4]) and for a system of rth-order ODE admitting an r-dimensional Lie algebra (see [5]).

Let us now construct a system of first-order ODE connected with system (4).

All nonsolvable four-dimensional Lie algebras are decomposable, i.e. they are representable as the direct sum of a three-dimensional simple Lie algebra and a one-dimensional subalgebra (see [11]). Moreover, condition (9) holds. Hence, for an approximate Lie algebra with four essential operators, there is always at least one OID with coefficient (10). Consider the result of the action of this OID at the invariants I, $J_1, \ldots, J_{4-r_0}$ of (7).

Applying the OID to I, we obtain the differential relation

$$\begin{aligned} &\left.\frac{D_t\Phi_0 - \varepsilon D_t\Phi_1}{(D_t\Phi_0)^2} D_t\left(I_{(0)} + \varepsilon I_{(1)}\right)\right|_{(7)} \\ &= H_0\left(I_{(0)}\right) + \varepsilon H_0'\left(I_{(0)}\right) I_{(1)} + \varepsilon H_1\left(I_{(0)}, J_{1,(0)}, \ldots, J_{4-r_0,(0)}\right). \end{aligned} \tag{14}$$

Rewrite it in the form

$$\begin{aligned} &\frac{D_t I_{(0)}}{D_t\Phi_0} - H_0\left(I_{(0)}\right) \\ &+\varepsilon\left(\frac{D_t I_{(1)}}{D_t\Phi_0} - \frac{D_t\Phi_1 D_t I_{(0)}}{(D_t\Phi_0)^2} - H_0'\left(I_{(0)}\right) I_{(1)} - H_1\left(I_{(0)}, J_{1,(0)}, \ldots, J_{4-r_0,(0)}\right)\right) = 0. \end{aligned}$$

One can show that this equation admits all the approximate operators X_α, $\alpha = 1, \ldots, r$. The proof of this assertion follows from the systems Ω_0, Ω_1, Π_0, Π_1, and the identities (see [6])

$$D_t(XF) - X(D_tF) = D_t\xi D_tF.$$

Applying the OID to J_k, we obtain

$$\frac{D_t\Phi_0 - \varepsilon D_t\Phi_1}{\left(D_t\Phi_0\right)^2} D_t\left(\varepsilon J_{k,(0)}\right)\Big|_{(7)} \approx \varepsilon Q_k\left(I_{(0)}, J_{1,(0)}, \dots, J_{4-r_0,(0)}\right),\ k = 1, \dots, 4 - r_0.$$

These approximate equalities are equivalent to the system of differential relations

$$\frac{D_t J_{k,(0)}}{D_t\Phi_0}\Big|_{(7)} = Q_k\left(I_{(0)}, J_{1,(0)}, \dots, J_{4-r_0,(0)}\right),\ k = 1, \dots, 4 - r_0. \tag{15}$$

It is easy to see that system (15) admits the operators $X_{\alpha_0,(0)}$ generating the subalgebra L_{r_0}.

In the next section, we will consider the solutions to the obtained system of the differential relations (14) and (15) and the symmetries of these solutions.

4. Order Reduction for Systems of Two Second-Order ODE with a Small Parameter

The general algorithm of order reduction for systems of ODE with a small parameter is described in [7] and consists of the following steps:

(1) calculating the approximate symmetries admitted by the system of ODE;
(2) representing the system of ODE in terms of the differential invariants of an admitted approximate Lie algebra;
(3) constructing an OID of the special form $\left(D_t(\Phi_0 + \varepsilon\Phi_1)\right)^{-1} D_t$;
(4) applying the obtained OID to the invariants of the lower orders and getting a system of first-order ODE;
(5) solving the obtained equations and constructing an approximate first integral of the initial system;
(6) adding the so-obtained relation to the equations of the initial system and excluding its differential consequences;
(7) repeating steps 3 – 6.

The fulillment of step 5 depends on the structure of the approximate Lie algebra of the initial system, and passage from step 7 to steps 3-6 requires an additional study of the symmetry properties of the reduced system. Let us examine the applicability of this algorithm to second-order systems of ODE with a small parameter depending on the structure of an admitted approximate Lie algebra $\widetilde{L}$, i.e., on the structure of the corresponding exact Lie algebra L_4 and the dimension and structure of the subalgebra L_{r_0}.

4.1. **Case 1.**

Suppose that system (4) admits an approximate Lie algebra $\widetilde{L}$ with four essential operators, and that the operators $X_{\alpha,(0)}$, $\alpha = 1, \dots, 4$, are stable with respect to the perturbation under consideration. Then the invariant representation (7) of system (4) does not contain "first-order" invariants, i.e., system (4) is representable as

$$\begin{cases} I_2^1 = F_0\left(I\right) + \varepsilon F_1\left(I_{(0)}\right), \\ I_2^2 = G_0\left(I\right) + \varepsilon G_1\left(I_{(0)}\right), \end{cases} \tag{16}$$

where I is a first-order differential invariant if $\mathrm{rg}\Lambda_4^{(0)} = 3$ or an algebraic invariant if $\mathrm{rg}\Lambda_4^{(0)} = 2$.

If the admitted approximate Lie algebra has an OID with factor (10) then an auxiliary system consists of one first-order ODE

$$\frac{\mathrm{d}I}{(H_0(I)+\varepsilon H_1(I))} \approx \mathrm{d}\left(\Phi_0+\varepsilon\Phi_1\right).$$

The general solution to this equation is a first integral of the initial system. Adding it to the initial system of ODE and excluding its differential consequences, we obtain a third-order system admitting an approximate Lie algebra with three essential operators which are the "zeroth-order" with respect to ε.

If the exact Lie algebra L_4 is nonsolvable then the reduced third-order system admits an approximate Lie algebra with three essential operators, where L_3 is a simple Lie algebra, i.e., it does not contain any ideals except trivial ones. The above-described procedure does not apply to this system, and no further reduction is possible.

If the exact Lie algebra L_4 is solvable then L_3 is its solvable ideal, and we can apply the above-described algorithm to the reduced system of equations and obtain a second-order system with two stable essential symmetries. After repeating the order reduction procedure, we obtain a first-order system with one stable symmetry. Repeating the procedure once again, we obtain an approximate solution to the initial system of ODE with a small parameter.

4.2. **Case 2.** Suppose that part of the essential operators of the approximate Lie algebra $\widetilde{L}$ admitted by (4) are "first-order" with respect to ε. 7 In this case, of importance is the representability of exact Lie algebra L_4 as a sequence of embedded subalgebras

$$L_r \supset L_{r-1} \supset \ldots \supset L_{r_0}, \ \dim L_{r-s}/L_{r-s-1} = 1, \tag{17}$$

In [15], the authors described all subalgebras for each four-dimensional Lie algebras. An analysis of these results showed that, for any solvable Lie algebra L_4 and any its subalgebra L_{r_0}, one can construct a sequence (17) containing the subalgebra L_{r_0}. For nonsolvable Lie algebras L_4 such sequences do not always exist. In the present work, we consider only systems of ODE where the exact Lie algebra L_4 corresponding to the admitted approximate Lie algebra $\widetilde{L}$ is solvable, and we use the results of [15] for them.

Then system (4) is represented by differential invariants as (7), where $J_{1,(0)}$ is the invariant of L_{r_0}. We can choose any other invariant $J_{s,(0)}$ as an invariant of corresponding subalgebra L_{r_0+s-1} from (17).

Then system (15) can be reduced to the "triangular" form

$$\left.\frac{D_t J_{s,(0)}}{D_t\Phi_0}\right|_{(7)} = Q_k\left(I_{(0)}, J_{1,(0)}, \ldots, J_{s,(0)}\right),\ s=1,\ldots,4-r_0, \tag{18}$$

which is invariant with respect to subalgebra L_{r_0+s-1}.

For order reduction of system (4), study the relationship between the derived algebra L_4' and its subalgebras from (17). If an ideal N_3 containing L_4' coincides with the subalgebra L_3 in sequence (17) then reduced system of ODE also has r_0 stable symmetries. If the ideal N_3 does not coincide with L_3 in (17) then the reduced system of ODE has r_0-1 stable symmetries. Then the procedure is repeated, the derived algebra L_3' is considered. Thus, the algorithm enables us to reduce the order of the original system r_0 times or more provided that the derived algebra (or the ideal containing it) is included in the sequence (17).

In the present article, as an example, we consider the case where system (4) admits an approximate Lie algebra with four essential operators, where three of the four essential operators operators of the corresponding exact Lie algebra L_4 are stable and one is unstable, and L_4. The remaining cases are examined similarly.

In this case, system (4) has the invariant representation

$$\left\{\begin{array}{l} I_2^1 \approx F_{(0)}\left(I\right) + \varepsilon F_{(1)}\left(I_{(0)},\, J_{1,(0)}\right), \\ I_2^2 \approx G_{(0)}\left(I\right) + \varepsilon G_{(1)}\left(I_{(0)},\, J_{1,(0)}\right). \end{array}\right. \tag{19}$$

Applying the OID to the invariants I and J_1 and expanding the results in a series in ε, we obtain the system

$$\left\{\begin{array}{l} \dfrac{D_t I_{(0)}}{D_t \Phi_0} = H_0\left(I_{(0)}\right), \\ \dfrac{D_t I_{(1)}}{D_t \Phi_0} - \dfrac{D_t I_{(0)}}{D_t \Phi_0}\dfrac{D_t \Phi_1}{D_t \Phi_0} = H_0'\left(I_{(0)}\right) I_{(1)} + H_1\left(I_{(0)}, J_{1,(0)}\right), \\ \dfrac{D_t J_{1,(0)}}{D_t \Phi_0} = Q\left(I_{(0)}, J_{1,(0)}\right). \end{array}\right. \tag{20}$$

Here $I_{(0)}$ is a differential invariant of L_4, $J_{1,(0)}$ is a differential invariant of the subalgebra L_3 generated by stable operators, and Φ_0 is a differential invariant of a three-dimensional ideal N_3 of L_4.

Consider two cases separately: N_3 coincides with L_3 and it does not.

4.2.1. *The Unstable Operator Is Not in L_4'.* In this case N_3, coincides with L_3. Then we can construct an OID such that $J_{1,(0)} = \psi(\Phi_0)$. Then system (20) has the form

$$\left\{\begin{array}{l} \dfrac{\mathrm{d} I_{(0)}}{\mathrm{d}\Phi_0} = H_0\left(I_{(0)}\right), \\ \dfrac{\mathrm{d}S}{\mathrm{d}\Phi_0} = H_0'\left(I_{(0)}\right) S + H_1\left(I_{(0)}, \psi(\Phi_0)\right), \quad S = I_{(1)} - H_0\left(I_{(0)}\right)\Phi_1, \\ J_{1,(0)} = \psi(\Phi_0). \end{array}\right.$$

The solution to this system of differential relations

$$W(I_{(0)} + \varepsilon I_{(1)}, \Phi_0 + \varepsilon \Phi_1) = C_1$$

is a first integral of the initial system, and $C_1 = C_{1,(0)} + \varepsilon C_{1,(1)}$.

Adding the obtained first integral to the equations of the initial system of ODE and excluding its differential consequences, we obtain a third-order system. It admits the operators $X_{\alpha,(0)} + \varepsilon X_{\alpha,(1)}$, $\alpha = 1, 2, 3$, or their linear combinations. In other words, the third-order system admits a Lie algebra with three stable essential operators, and the corresponding exact Lie algebra L_3 is solvable.

The described algorithm can be applied to the reduced system of equations to obtain a second-order system with two stable essential symmetries

After repeating the order reduction procedure, we obtain a first-order system with one stable symmetry.

Repeating the procedure once again, we obtain an approximate solution to the original system of ODE with a small parameter.

4.2.2. *The Unstable Operator Is in L'_4.* In this case, system (20) has the form

$$\begin{cases} \dfrac{\mathrm{d}I_{(0)}}{\mathrm{d}\Phi_0} = H_0\left(I_{(0)}\right), \\ \dfrac{\mathrm{d}S}{\mathrm{d}\Phi_0} = H_0'\left(I_{(0)}\right)S + H_1\left(I_{(0)}, J_{1,(0)}\right), \quad S = I_{(1)} - H_0\left(I_{(0)}\right)\Phi_1, \\ \dfrac{\mathrm{d}J_{1,(0)}}{\mathrm{d}I_{(0)}} = \dfrac{Q\left(I_{(0)}, J_{1,(0)}\right)}{H_0\left(I_{(0)}\right)}. \end{cases}$$

The last equation of this system admits the operator

$$K_{1,(0)}X_{1,(0)} + K_{2,(0)}X_{2,(0)} + K_{3,(0)}X_{3,(0)},$$

and hence it can be integrated by quadratures. After that, we insert the solution in the second equation and reduce this system to a system of two equations. The solution to the resulting system of equations gives a first integral of the initial system. Adding it to the equations of the system and removing the differential consequences of this approximate first integral, we obtain the third-order system of the form

$$\begin{cases} I_{(0)} + \varepsilon I_{(1)} = \varphi_0(\Phi_0 + \Phi_1) + \varepsilon\varphi_1(\Phi_0), \\ I_{2,(0)} + \varepsilon I_{2,(1)} = \psi_0(\Phi_0 + \Phi_1) + \varepsilon\psi_1(\Phi_0). \end{cases} \tag{21}$$

This system admits two "zeroth-order" operators $Y_{1,(0)} + \varepsilon Y_{1,(1)}$, $Y_{2,(0)} + \varepsilon Y_{2,(1)}$ and one "first-order" operator $\varepsilon Y_{3,(0)}$. The operators $Y_{1,(0)}$, $Y_{2,(0)}$ $Y_{3,(0)}$ constitute a basis in the ideal N_3.

Repeating the above arguments, consider a two-dimensional ideal N_2 containing the derived algebra of the Lie algebra N_3.

If $N_2 = \langle Y_{1,(0)}, Y_{2,(0)}\rangle$, as a result of applying the algorithm, we obtain a second-order system with two stable essential symmetries. After repeating the order reduction procedure, we obtain a first-order system with one stable symmetry. The next application of the procedure gives an approximate solution to the initial system of ODE with a small parameter.

If $N_2 = \langle Y_{1,(0)}, Y_{3,(0)}\rangle$, as a result of applying the algorithm we obtain a second-order system that admits one stable operator $Z_{1,(0)} + \varepsilon Z_{1,(1)}$ and one unstable operator $\varepsilon Z_{2,(0)}$. If the derived algebra of the Lie algebra N_2 contains no unstable operator, then, after the order reduction procedure, we obtain a first-order system. This system has one stable symmetry and hence, after repeating the procedure, we can obtain an approximate solution to the original system of ODE with a small parameter. If the derived algebra of N_2 contains an unstable operator then the first-order system obtained after the order reduction procedure has one unstable symmetry. In this case, it is possible to obtain the solution only for the unperturbed part of the equation.

Example. Consider the system of equations

$$\begin{cases} \ddot{x} = \dfrac{\dot{x}^2\dot{y}}{x} + \varepsilon\left(\dfrac{\dot{x}^2(t\dot{y} - y)}{x} + \dot{x}^2\dot{y} - \dfrac{2}{\dot{x}}\right), \\ \ddot{y} = \dfrac{\dot{x}}{x}\mathrm{e}^{\dot{y}} - \dfrac{\varepsilon\dot{x}}{x}\left(t\mathrm{e}^{\dot{y}} - (t\dot{y} - y)\mathrm{e}^{\dot{y}} - \dfrac{\dot{y}}{x}\right). \end{cases} \tag{22}$$

This system admits the operators

$$X_1 = (1+\varepsilon t)\frac{\partial}{\partial t}, \quad X_2 = \varepsilon x\frac{\partial}{\partial x}, \quad X_3 = (1+\varepsilon t)\frac{\partial}{\partial y},$$
$$X_4 = (t+\varepsilon t^2)\frac{\partial}{\partial t} + (y+\varepsilon t y)\frac{\partial}{\partial y}.$$

These operators are essential for a six-dimensional approximate Lie algebra.

For the corresponding exact Lie algebra L_4, there is a sequence of embedded subalgebras:

$$L_4 \supset < X_{1,(0)}, X_{3,(0)}, X_{4,(0)} > \supset < X_{1,(0)}, X_{3,(0)} > \supset < X_{1,(0)} > .$$

For an invariant representation of system (22), we use the invariants

$$I_1 = \dot{y} + \varepsilon(t\dot{y} - y), \quad I_2^1 = \frac{x\ddot{x} + 2\varepsilon x\dot{x}}{\dot{x}^2}, \quad I_2^2 = \frac{x\ddot{y} + \varepsilon t x\ddot{y}}{\dot{x}}, \quad I_0 = \varepsilon x.$$

Rewrite system (22) by means of an admitted approximate Lie algebra:

$$\begin{cases} I_2^1 \approx I_1 + \varepsilon I_{1,(0)} I_{0,(0)}, \\ I_2^2 \approx \mathrm{e}^{I_1} + \varepsilon \dfrac{I_{1,(0)}}{I_{0,(0)}}. \end{cases}$$

The function Φ for constructing an OID is obtained from the system

$$\begin{cases} \Phi_{0t} = 0, \\ x\Phi_{0x} + \dot{x}\Phi_{0\dot{x}} = K_2, \\ \Phi_{0y} = 0, \\ t\Phi_{0t} + y\Phi_{0y} - \dot{x}\Phi_{0\dot{x}} = K_4, \\ \Phi_{1t} = 0, \\ \Phi_{1y} = 0, \\ t\Phi_{1t} + y\Phi_{1y} - \dot{x}\Phi_{1\dot{x}} + t^2\Phi_{0t} + ty\Phi_{0y} - 2t\dot{x}\Phi_{0\dot{x}} + (y + t\dot{y})\Phi_{0\dot{y}} = 1. \end{cases}$$

Let, for example, $\Phi_0 = \ln x$, $\Phi_1 = \ln t$. Note that $\Phi_0 = \ln I_{0,(0)}$.

Applying the operator $\dfrac{x(t\dot{x} - \varepsilon x)}{t\dot{x}^2} D_t$ to the invariant I_1, we obtain

$$\begin{cases} \dfrac{D_t(\dot{y})}{D_t \ln x} = \mathrm{e}^{\dot{y}}, \\ \dfrac{D_t(t\dot{y} - y)}{D_t \ln x} = \mathrm{e}^{\dot{y}}(t\dot{y} - y) + \dot{y}\mathrm{e}^{-\ln x}. \end{cases}$$

Adding the solution to this system to the equations of the initial system (22) and excluding its differential consequences, we obtain the reduced system

$$\begin{cases} I_1 \approx -\ln(C_1 - \Phi_0) - \varepsilon H(\Phi_0), \ \ H(\Phi_0) = \dfrac{1}{C_1 - \Phi_0}\displaystyle\int (C_1 - \Phi_0)\mathrm{e}^{-\Phi_0}\ln(C_1 - \Phi_0)\mathrm{d}\Phi_0, \\ I_2^1 \approx I_1 - \varepsilon \mathrm{e}^{-\Phi_0}\ln(C_1 - \Phi_0), \end{cases}$$

which admits the operators X_1, X_3, and X_4. This procedure can be repeated and a solution to the initial system of ODE can be obtained. □

5. Conclusion

In this article, we have shown that the application of the order reduction algorithm to systems of two second-order ODE with a small parameter that admit approximate Lie algebras with four essential operators gives an approximate solution to these systems in the case when the corresponding exact four-dimensional Lie algebra is solvable.

Applying the algorithm to higher-order systems requires an additional investigation of the structure of an admitted approximate Lie algebra of operator $\tilde{L}$, i.e., of the structure of the corresponding exact Lie algebra L_4 and also the dimension and structure of the subalgebra L_{r_0}.

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