S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 17, стр. 61–76 (2020) УДК 512.54
DOI 10.33048/semi.2020.17.006 MSC 20F16, 20E36, 20E10

PRIMITIVE ELEMENTS AND AUTOMORPHISMS OF THE FREE METABELIAN GROUP OF RANK 3

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Abstract. The purpose of this article is twofold. On the one hand, we give an overview of the known results on primitive elements and automorphisms of a very specific free metabelian group M_3 of rank 3. On the other hand, we present new results, at the same time showing the work of non-standard research tools of studying of this group.
Keywords: free metabelian group, matrix group, primitive element.

1. Introduction

For every group G, by G' we denote the derived subgroup (commutant) of G, and G'' stands for the second derived subgroup of G. $\mathrm{Aut}(G)$ denotes the automorphism group of G, $\mathrm{IAut}(G)$ is the subgroup of $\mathrm{Aut}(G)$ consisting of all automorphisms inducing identical map on the abelianization $G_{ab} = G/G'$, and $\mathrm{Inn}(G)$ is the group of inner automorphisms of G.

For each positive n, let F_n be the free group of rank n. Then $M_n = F_n/F_n''$ is the free metabelian group of rank n, and $A_n = F_n/F_n' = M_n/M_n'$ is the free abelian group of rank n.

The study of the automorphisms of the free metabelian groups M_n began with the paper of S. Bachmuth [1]. He proved that $\mathrm{IAut}(M_2) = \mathrm{Inn}(M_2)$, hence every automorphism φ of M_2 is tame. In particular, the groups $\mathrm{IAut}(M_2)$ and $\mathrm{Aut}(M_2)$ are finitely generated. Recall that "tame" means that the considered automorphism is induced by some automorphism of the corresponding free group via the fixed standard homomorphism. An element f of F_n is said to be *primitive* if it can be

Roman'kov, V.A., Primitive elements and automorphisms of the free metabelian group of rank 3.

The research was supported with a grant from the Russian Science Foundation (project No. 19-71-10017).

Received November, 24, 2019, published February, 4, 2020.

included in a basis of F_n and similarly an element x of M_n is said to be *primitive* if it can be included in a basis of M_n. We say that a primitive element $g \in M_n$ is *tame* if it is the image of a primitive element of F_n under the standard homomorphism $F_n \to M_n$. More generally, a system of elements $g_1, ..., g_m$ of F_n is called *primitive* if it can be extended to a basis of F_n. A system of elements $h_1, ..., h_m$ of M_n is *primitive* if it can be extended to a basis of M_n, and *tame* if it is induced by a primitive system of F_n. In the natural way we define *tame* primitive elements and primitive systems of elements not only for M_n but for each relatively free group. Further we'll give more detailed descriptions of these notions.

S. Bachmuth in [1] introduced the *Bachmuth's representation* $\beta : \mathrm{IAut}(M_n) \to \mathrm{GL_n}(\mathbb{Z}\mathrm{A_n})$ which provides a good tool for investigation of the automorphism group $\mathrm{Aut}(M_n)$ via its subgroup $\mathrm{IAut}(M_n)$. This representation is a special case of the Magnus representation (see details below).

O. Chein [2] was first who constructed a non-tame automorphism η of M_3. S. Bachmuth and H.Y. Mochizuki [3] established that $\mathrm{Aut}(M_3)$ is not even finitely generated. This result was strenghtened by the author in [4] who proved that there is a non-tame primitive element $g \in M_3$, and gave a method of constructing of such the elements.

It turns out that any $\mathrm{Aut}(M_n)$ for $n \geq 4$ does not contain non-tame automorphisms. In other words, the standard homomorphism $\mathrm{Aut}(F_n) \to \mathrm{Aut}(M_n)$ for $n \geq 4$ is surjective. This result was proved independently by S. Bachmuth and H.Y. Mochizuki [6] and by the author [7], [8]. In particular one can give finite generating sets for $\mathrm{IAut}(M_n)$ and $\mathrm{Aut}(M_n)$ for any $n \geq 4$, as natural images of known finite generating sets of $\mathrm{IAut}(F_n)$ and $\mathrm{Aut}(F_n)$, respectively (see [9], [10], or [11] for description of these generating sets).

The automorphism group $\mathrm{Aut}(M_3)$ has a number of distinctive properties not shared by the automorphism groups $\mathrm{Aut}(M_n)$ of other finite ranks n. The most important of these is that it is not finitely generated. This phenomenon can be roughly explained as follows.

The properties of the automorphism groups $\mathrm{Aut}(M_n)$ for $n \geq 3$ are closely related to those of the matrix groups $\mathrm{GL}_{n-1}(R)$ over some commutative rings R that are derived from the Laurent polynomial rings $\Lambda_n = \mathbb{Z}[a_1^{\pm 1}, ..., a_n^{\pm 1}]$.

As is well known, $\mathrm{GL}_2(R)$ has a number of specific properties. We note here some of them that are of greatest significance for this area of investigations.

- By a theorem of A.A. Suslin [12], any group $\mathrm{GL}_n(\Lambda_k)$ coincides with the general group of elementary matrices $\mathrm{GE}_n(\Lambda_k)$ and so is finitely generated for all k and every $n \geq 3$. Recall that the group $\mathrm{GE}_n(R)$ over a ring R is generated by definition by all elementary transvections $t_{i,j}(\alpha) = e + \alpha e_{i,j}, i \neq j, \alpha \in R$, and by all elementary diagonal matrices $d_i(g)$ with element $g \in R^*$ in the ith position of the main diagonal and 1th on the other of its positions.
- By a theorem of S. Bachmuth and H.Y. Mochizuki [13], any group $\mathrm{GL}_2(\Lambda_k)$ does not coincide with $\mathrm{GE}_2(\Lambda_k)$ and is not finitely generated when $k \geq 2$.

By the way we note that the finite generation problem for $\mathrm{GL}_2(\Lambda_1)$ is still open.

There are two approaches to study the group $\mathrm{Aut}(M_3)$. S. Bachmuth and H.Y. Mochizuki [14] essentially used of structure theorem for $\mathrm{GL}_2(R)$ for appropriate ring R. The author [4] developed his own original method based on so called groups of matrix residues. This method has an advantage in this context: it provides not only non-tame automorphisms of M_3, but also non-tame primitive elements of M_3, that is more strongly. We highlight these two results as follows.

For a relatively free group G_n we denote by $\mathrm{TAut}(G_n)$ the subgroup of $\mathrm{Aut}(G_n)$ consisting of all tame automorphisms of G_n. Then

$\mathrm{TIAut}(G_n) = \mathrm{TAut}(G_n) \cap \mathrm{IAut}(G_n)$.

Theorem 1. *(S. Bachmuth, H.Y. Mochizuki* [14]*).* $\mathrm{Aut}(M_3) \neq \mathrm{TAut}(M_3)$ *and is not finitely generated.*

Let $G_n = F_n/V$ be a relatively free group of rank n. In general not every primitive system in G_n is induced by a primitive system of F_n. In other words, not every primitive system in G_n is tame (non-tame system is also called *wild*). In the obvious sense, we can talk about tame and wild primitive elements. Each tame basis in G_n corresponds to a tame automorphism of G_n, conversely, each wild basis corresponds to a wild automorphism of G_n.

Theorem 2. *(V.A. Roman'kov* [4]*).* M_3 *contains a wild primitive element.*

A similar result for any free metabelian Lie algebra of rank 3 over arbitrary field was established in [5].

In this paper we'll provide more information about primitive systems and elements in M_3. A part of this information is new. We'll give a couple of applications of the results described in the paper.

2. A non-standard homomorphism of $\mathrm{IAut}(M_3)$ into $\mathrm{GL}_2(\Lambda_2)$

Let $\{f_1, ..., f_n\}$ be a basis of F_n. We fix the standard homomorphisms $\pi'' : F_n \to M_n$, $\pi' : F_n \to A_n$ and $\pi : M_n \to A_n$, respectively. We denote the bases of M_n and A_n corresponding to $\{f_1, ..., f_n\}$ via these homomorphisms by $\{x_1, ..., x_n\}$ and $\{a_1, ..., a_n\}$ respectively.

For a group G by $\mathbb{Z}G$ we denote the group ring of G, and ΔG denotes the augmentation ideal of $\mathbb{Z}G$. We set $\Lambda_n = \mathbb{Z}A_n = \mathbb{Z}[a^{\pm 1}, ..., a_n^{\pm 1}]$, that is a Laurent polynomial ring. The augmentation ideal of Λ_n is denoted by Δ_n. The maps π, π' and π'' can be extended linearly to $\pi : \mathbb{Z}M_n \to \Lambda_n$, $\pi' : \mathbb{Z}F_n \to \Lambda_n$ and $\pi'' : \mathbb{Z}F_n \to \mathbb{Z}M_n$, respectively. The kernels of π' and π'' are the ideals of $\mathbb{Z}F_r$ generated by the elements $u - 1$ with $u \in F_r'$ and $u \in F_r''$, respectively.

Free Fox derivatives.

We use the partial derivatives introduced by Fox [15]. In our notation, these are defined as follows.

For $i = 1, ..., n$, the (left) Fox derivative associated with f_i is the linear map $D_i : \mathbb{Z}F_n \to \mathbb{Z}F_n$ satisfying the conditions

$$D_i(f_i) = 1, D_i(f_j) = 0 \text{ for } i \neq j, \tag{1}$$

and

$$D_i(uv) = D_i(u) + uD_i(v) \text{ for all } u, v \in F_n. \tag{2}$$

Obviously, an element $u \in F_n$ is trivial if and only if $D_i(u) = 0$ for all $i = 1, ..., n$. We also note that for an arbitrary element g of F_n and each $i = 1, ..., n$, $D_i(g^{-1}) = -g^{-1}D_i(g)$. An excellent brief introduction to the theory of Fox derivatives and their possible applications can be found in [16]. See also [17], [18].

The *trivialization* homomorphism $\varepsilon : \mathbb{Z}F_n \to \mathbb{Z}$ is defined on the generators of F_n by $f_i\varepsilon = 1$ for all $i = 1, ..., n$ and extended linearly to the group ring $\mathbb{Z}F_n$.

The Fox derivatives appear in another setting as well. ΔF_n is a free left $\mathbb{Z}F_n$−module with a free basis consisting of $\{f_1-1, ..., f_n-1\}$. This it leads us to the following formula which is called the *Main Identity* for the Fox derivatives:

$$\sum_{i=1}^{n} D_i(\alpha)(f_i - 1) = \alpha - \alpha\varepsilon, \tag{3}$$

where $\alpha \in \mathbb{Z}F_n$. Conversely, if for any element $f \in F_n$ and $\alpha_i \in \mathbb{Z}F_n$ we have equality

$$\sum_{i=1}^{r} \alpha_i(f_i - 1) = f - 1, \tag{4}$$

then $D_i(f) = \alpha_i$ for $i = 1, ..., n$.

Frequently, the free derivatives D_i are denoted by $\partial/\partial f_i$ for $i = 1, ..., n$. If $w = w(v_1, ..., v_m)$ is a group word in variables $v_1, ..., v_m$, we consider the values of the formal derivatives $\partial w/\partial v_i$, for $i = 1, ..., m$, as formal sums of words in the given variables.

In the case when all the v_i are elements of a group G, these values are considered as elements of the group ring $\mathbb{Z}G$. In other words, we operate in the free group F_m with basis $\{v_1, ..., v_m\}$ and apply the canonicall homomorphism $\mathbb{Z}F_m \to \mathbb{Z}G$.

Proposition 1. *(Chain Rule). If w and $v_1, ..., v_m$ are words in F_n, with $w = w(v_1, ..., v_m)$ and $v_i = v_i(f_1, ..., f_n)$ for $i = 1, ..., m$, then*

$$\frac{\partial}{\partial f_i}(w(v_1, ..., v_m)) = \sum_{k=1}^{m} \partial w/\partial v_k \cdot \partial v_k/\partial f_i \text{ for all } i = 1, ..., n. \tag{5}$$

More generally, we call a linear map $D : \mathbb{Z}F_n \to \mathbb{Z}F_n$ the *free Fox derivative* if D satisfies the property

$$D(uv) = D(u) + uD(v) \tag{6}$$

for all $u, v \in F_n$. Every such derivative has the form

$$D = \alpha_1 D_1 + ... + \alpha_n D_n, \tag{7}$$

where $\alpha_i = D(f_i)$ for $i = 1, ..., n$. By definition $(\alpha D)(u) = D(u)\alpha$ for any $\alpha \in \mathbb{Z}F_n, u \in F_n$. Conversely, we can define a derivative $D = \sum_{i=1}^{n} \alpha_i D_i$ for arbitrary tuple of elements $\alpha_i \in \mathbb{Z}F_n$.

Moreover, in a similar way we can define for arbitrary commutative associative ring K with 1 free derivatives $D_i : KF_n \to KF_n$ for $i = 1, ..., n$, and extend this notion as above. Analogs of the Main Identity and Chain Rule are obviously valid.

Free differential calculus has applications in a large number of areas in group theory. In 1950, R.C. Lyndon [19] described the cohomological dimensions of one relator groups. His analysis was based on some non-trivial results from free differential calculus. Another development in the theory of differentiations over free groups is due to J. Birman. She proved the following important result.

Theorem 3. *(Birman's inverse function theorem [20]). Let $\mathbf{y} = \{y_1, ..., y_n\}$ be a set of n elements of F_n. Let $J(\mathbf{y})$ denote the Jacobi matrix $(D_i(y_j))$. Then $\mathbf{y}$ is a basis of F_n if and only if $J(\mathbf{y})$ is invertible over $\mathbb{Z}F_n$.*

Note, that by M.S. Montgomery's result [21] a matrix over arbitrary group ring $\mathbb{Z}G$ is right-invertible if and only if it is left-invertible. Thus, termin "invertible" is correct.

We also note that U.U. Umirbaev [22] generalized this Birman's result by proving that a system of elements $\mathbf{f} = \{f_1, ..., f_l\}, l \leq r$, is a primitive system in F_r if and only if the Jacobi matrix $J(\mathbf{f}) = (D_i(f_j))$ is right-invertible, that there is a matrix B of size $r \times l$ over $\mathbb{Z}F_r$ for which $J(\mathbf{f}) \cdot B = E$, where E is the unit matrix of size l.

Futher we'll consider some generalizations of the free differential calculus. These generalizations allow us to build powerful tools used to study solvable groups.

Induced Fox derivatives: metabelian case

For every $j = 1, ..., n$ the free Fox derivative D_j induces a linear map $d_j : \mathbb{Z}M_n \to \Lambda_n$. These maps also are called the *free Fox derivatives* (or *the free partial derivatives*). For the sake of completeness, we briefly explain the details. By an calculation,

$$D_j([u, v]) = (1 - uvu^{-1})D_j(u) + (u - [u, v])D_j(v), \tag{8}$$

for all $u, v \in F_n$.

It follows that $D_j(w) \subseteq ker(\pi')$ for all $w \in F_n''$. Hence $D_j(ker(\pi'')) \subseteq ker(\pi')$ and D_j induces a linear map $d_j : \mathbb{Z}M_n \to \Lambda_n$ for $j = 1, ..., n$.

From the definition of Fox derivatives we have

$$d_j(x_j) = 1, \ d_j(x_i) = 0 \text{ for } i \neq j \tag{9}$$

and

$$d_j(uv) = d_j(u) + (u\pi)d_j(v) \text{ for all } u, v \in M_n. \tag{10}$$

The following lemma shows that the Fox derivatives determine elements of M_n completely.

Lemma 1. *Let u be an element in M_n. Then u is trivial if and only if $d_i(u) = 0$ for all $i = 1, ..., n$. Therefore, two elements u and v are equal in M_n if and only if $d_i(u) = d_i(v)$ for all $i = 1, ..., n$.*

The Main Identity for the Fox derivatives (3) has an analogue in the metabelian case:

$$\sum_{i=1}^{n} d_i(\alpha)(a_i - 1) = \alpha\pi - \alpha\bar{\varepsilon}, \tag{11}$$

where $\alpha \in \mathbb{Z}M_n$ and $\bar{\varepsilon} : \mathbb{Z}M_n \to \mathbb{Z}$ is the homomorphism of rings induced by $\varepsilon : \mathbb{Z}F_n \to \mathbb{Z}$, introduced above.

Suppose that for some elements $\alpha_i \in \mathbb{Z}M_n$

$$\sum_{i=1}^{n} \alpha_i(x_i - 1) = 0. \tag{12}$$

Then there is an element $u \in M_n'$ such that

$$d_i(u) = \alpha_i \text{ for } i = 1, ..., n. \tag{13}$$

The Chain Rule (5) has an analogue in the metabelian case:

$$d_i(w(v_1, ..., v_m)) = \sum_{k=1}^{m} \partial w/\partial v_k \cdot d_i(v_k) \text{ for } i = 1, ..., n. \tag{14}$$

Since M_n' is abelian, it may be regarded as a $\mathbb{Z}(M_n/M_n')-$module in the usual way, where the module action comes from conjugation in M_n. The epimorphism $\pi : M_n \to A_n$ induces an isomorphism from M_n/M_n' to A_n. So we may regard M_n' as a Λ_n-module. For $w \in M_n'$ and $\alpha \in \Lambda_n$, we write w^α to denote the image of w under the action of α (a notation which is consistent with our notation for conjugation). For $w \in M_n'$ and $\alpha \in \Lambda_n$, it is easily verified that for all $j \in \{1, ..., n\}$ we have

$$d_j(w^\alpha) = \alpha d_j(w). \tag{15}$$

Also note that, for $w \in M_n'$ and $u \in M_n$, we have

$$d_i(wu) = d_i(w) + d_i(u) \text{ for all } i = 1, ..., n. \tag{16}$$

Let $u \in M_n$ and write $u = u(x_1, ..., x_n)$. Let $w \in M_n'$ and $j \in \{1, ..., n\}$. Set

$$u' = u(x_1, ..., x_{j-1}, wx_j, x_{j+1}, ..., x_n) \tag{17}$$

and write

$$u = v_1 x_j^{\varepsilon(1)} v_2 x_j^{\varepsilon(2)} ... v_k x_j^{\varepsilon(k)} v_{k+1}, \tag{18}$$

where $\varepsilon(1), ..., \varepsilon(k) \in \{1, -1\}$ and where $v_1, ..., v_{k+1}$ are products of elements of $\{x_1, ..., x_{j-1}, x_{j+1}, ..., x_r\}$ and their inverses. Then,

$$u' = v_1(wx_j)^{\varepsilon(1)} v_2(wx_j)^{\varepsilon(2)} ... v_k(wx_j)^{\varepsilon(k)} v_{k+1} \tag{19}$$

$$= w^\alpha v_1 x_j^{\varepsilon(1)} v_2 x_j^{\varepsilon(2)} ... v_k x_j^{\varepsilon(k)} v_{k+1} = w^\alpha u, \tag{20}$$

where $\alpha \in \Lambda_n$, namely

$$\alpha = \sum_{i=1}^{k} \varepsilon(i)(v_1 x_j^{\delta(1)} ... v_i x_j^{\delta(i)} \pi), \tag{21}$$

where $\delta(i) = \frac{1}{2}(\varepsilon(i) - 1)$. It is straightforward to check that $\alpha = d_j(u)$.

If $w_1, ..., w_n \in M_n'$ then, using the fact that elements of M_n' commute, we obtain

$$u(w_1 x_1, ..., w_n x_n) = w_1^{\alpha_1} ... w_r^{\alpha_n} u, \tag{22}$$

where $\alpha_i = d_i(u)$ for $i = 1, ..., n$.

S. Bachmuth [1] proved the following metabelian version of the Birman's inverse function theorem: a system of elements $\mathbf{g} = \{g_1, ..., g_n\}$ is a basis of M_n if and only if the Jacobi matrix $J(\mathbf{g}) = (d_i(g_j))$ is invertible over Λ_n.

The free Fox derivatives are defined in the natural way for any group ring KM_n where K is a commutative associative ring with 1.

Let for any $m \in \mathbb{N}$, $K = \mathbb{Z}_m$. Then for every element $u \in (M_n')^m$ $d_i(u) = 0$ for all $i = 1, ..., n$. Moreover, if $v \notin (M_n')^m$ then there is $i \in \{1, ..., n\}$ for which $d_i(v) \neq 0$ in $\mathbb{Z}_m A_n$. Hence, we can consider the free Fox derivatives

$$d_i : \mathbb{Z}_m M_n/(M_n')^m \to \mathbb{Z}_m A_n, i = 1, ..., n, \tag{23}$$

with the similar properties.

There are the following three criteria for the primitivity of a system of elements in M_n.

Theorem 4. *(S. Bachmuth* [1], *E.I. Timoshenko* [23], [17] , *V.A. Roman'kov* [24], [4], [18]*). A system of elements* $\mathbf{g} = \{g_1, ..., g_m\}$*,* $m \leq n$*, of the free metabelian group* M_n *is primitive if and only if either of the following properties is satisfied:*

- *Criterion 1. The ideal generated in the ring* Λ_n *by all minors of size* $m \times m$ *in the Jacobi matrix* $J(\mathbf{g}) = (d_j(g_i))$ *coincides with the entire ring* Λ_n*.*
- *Criterion 2. There is a matrix* $B \in M_{n\times m}(\Lambda_n)$ *such that* $J(\mathbf{g}) \cdot B = E$*, where* E *is the unit matrix of size* m*.*
- *Criterion 3. There are free Fox derivatives* $D_i : M_n \to \Lambda_n, i = 1, ..., n$*, such that* $D_i(g_j) = \delta_{ij}$ *(the Kronecker delta).*

It turns out that the Fox derivatives play an important role in relation to the Magnus representation which is very effective tool used in group theory.

Magnus representation.

To introduce this approach we define for any group G and any module T over the group ring $\mathbb{Z}(G)$ a group of matrices

$$M(G,T) = \{\begin{pmatrix} g & t \\ 0 & 1 \end{pmatrix} | g \in G, t \in T\}. \tag{24}$$

One of the most effective approaches to studying free solvable groups is to use the *Magnus representation.* Originally W. Magnus established in [25] an embedding of a group $\bar{G}$ of type F_n/R' into the matrix group $\mathrm{M}(G, T_n)$, where $G = F_n/R$ is a finite group, and T_n is a free module over $\mathbb{Z}F_n$ with basis $\{t_1, ..., t_n\}$. This map is called the *Magnus embedding.* The finiteness restriction on G can be easily eliminated (see [11]).

The embedding $\mu : \bar{G} \to \mathrm{M}(G, T)$ is defined as follows. For any $\bar{g} \in \bar{G}$

$$\mu : \bar{g} \mapsto \{\begin{pmatrix} g & \sum_{i=1}^{n} d_i(\bar{g})t_i \\ 0 & 1 \end{pmatrix}, \tag{25}$$

where g is the image of $\bar{g}$ under the standard homomorphism $\bar{G} \to G$, and $d_i : \bar{G} \to \mathbb{Z}G$ is the i-th induced Fox derivations, $i = 1, ..., n$.

Definition 1. *The Magnus representation for* $\mathrm{Aut}(F_n)$ is the map

$$\mu : \mathrm{Aut}(F_n) \to \mathrm{GL_n(\mathbb{Z}F_n)} \tag{26}$$

assigning to $\varphi \in \mathrm{Aut}(F_n)$ the Jacobi matrix

$$J(\varphi) = (D_j(f_i\varphi)), \tag{27}$$

where D_j are the free Fox derivatives for $j = 1, ..., n$, and $D_j(f_i\varphi)$ is the $ij-$th entry of the matrix.

The Magnus representation μ for $\mathrm{Aut}(F_n)$ is injective since $f_i\varphi$ is recovered from $J(\varphi)$ by applying the Main Identity (3) for the free Fox derivatives to the $i-$th row for each $\varphi \in \mathrm{Aut}(F_n)$.

The Magnus representation is not homomorphism as is seen from the following assertion.

Proposition 2. *For $\varphi, \psi \in \mathrm{Aut}(F_n)$, the equality*

$$J(\varphi\psi) = J(\varphi)\psi \cdot J(\psi) \tag{28}$$

holds, where $J(\varphi)\psi$ means that ψ is applied to every entry in $J(\varphi)$.

In particular, it follows that the image of μ is contained in the group $\mathrm{GL}_n(\mathbb{Z}F_n)$ of invertible matrices.

The Jacobi matrix $J(\psi) \in M_n(\mathbb{Z}F_n)$ is naturally defined for any endomorphism $\psi : F_n \to F_n$ as $J(\psi) = (D_j(f_i\psi))$, where $D_j(f_i\psi)$ is the $ij-$th entry as before.

To obtain a genuine representation, a homomorphism, we need to change F_n to a group $\bar{G}$ of type F_n/R', where R is a normal subgroup in F_n. Let $G = F_n/R$ and $\bar{G} = F_n/R'$. Then $\bar{R} = R/R'$ is normal abelian subgroup of $\bar{G}$, which is considered as a module over the group ring $\mathbb{Z}G$. Let $\{x_1, ..., x_n\}$ be the generators of $\bar{G}$ corresponding to the basic elements $\{f_1, ..., f_n\}$ of F_n.

Definition 2. *The Magnus representation for* $\mathrm{Aut}(\bar{G})$ is the map

$$\mu_A : \mathrm{Aut}(\bar{G}) \to \mathrm{GL_n(\mathbb{Z}G)} \tag{29}$$

assigning to $\varphi \in \mathrm{Aut}(\bar{G})$ the Jacobi matrix $J(\varphi) = (d_j(x_i\varphi))$ of φ over $\mathbb{Z}G$. Here d_j are the induced free Fox derivatives on $\bar{G}$ with values in $\mathbb{Z}G$ with respect to generators x_i for $i, j = 1, ..., n$, and $d_j(x_i\varphi)$ is the $ij-$th entry in the matrix.

More generally, one can define similarly the *Magnus representation* for $\mathrm{End}(\bar{G})$

$$\mu_E : \mathrm{End}(\bar{G}) \to M_n(\mathbb{Z}G). \tag{30}$$

Property (28) gives for $\varphi, \psi \in \mathrm{End}(\bar{G})$ for which R/R' is invariant the following equality

$$J(\varphi\psi) = J(\varphi)\psi' \cdot J(\psi), \tag{31}$$

where ψ' means the induced by ψ endomorphism of G naturally extended to $\mathbb{Z}G$ and $M_n(\mathbb{Z}G)$.

For R a normal subgroup of a group H, let $\mathrm{IRAut}(H)$ denote the group of all automorphisms of H for which R is invariant, and which induce the identical map on the quotient H/R. Similarly, we define the semigroup $\mathrm{IREnd}(H)$ of all endomorphisms of H with given property.

Then the maps $\mu_{AR} : \mathrm{IRAut}(\bar{G}) \to \mathrm{GL_n(\mathbb{Z}G)}$ and $\mu_{ER} : \mathrm{IREnd}(\bar{G}) \to M_n(\mathbb{Z}G)$ induced by the maps μ_A and μ_E, respectively, are homomorphisms.

Bachmuth's embedding.

Let M_n be the free metabelian group of rank n with basis $\{x_1, ..., x_n\}$, and $A_n = M_n/M_n'$ be the abealization of M_n with the corresponding basis $\{a_1, ..., a_n\}$. The group ring $\mathbb{Z}A_n$ can be considered as a Laurent polynomial ring $\Lambda_n = \mathbb{Z}[a_1^{\pm 1}, ..., a_n^{\pm 1}]$. Let ψ be an endomorphism of M_n defined by $x_i\psi = c_i$ for $i = 1, ..., n$. We define the Jacobi matrix $J(\psi)$ corresponding to ψ as $J(\mathbf{c})$ where $\mathbf{c} = (c_1, ..., c_n)$. More exactly,

$$J(\psi) = (d_j(x_i\psi)),\ i, j = 1, ..., n, \tag{32}$$

where $d_j : \mathbb{Z}M_n \to \mathbb{Z}A_n$ are the induced free Fox partial derivatives for $j = 1, ..., n$, and $d_j(x_i\psi)$ is the $ij-$th entry in the matrix.

Let $\mathrm{IEnd}(M_n)$ denote the semigroup of all endomorphisms of M_n identical $modM_n'$. Note that $\mathrm{IEnd}(M_n) = \mathrm{IM_n'End}(M_n)$ in other denotions.

Proposition 3. *Let $\psi \in \mathrm{IEnd}(M_n)$ be arbitrary endomorphism identical $modM_n'$. Then the map*

$$\beta : \psi \to J(\psi) \tag{33}$$

gives an injective homomorphism (embedding) of $\mathrm{IEnd}(M_n)$ *into the semigroup of matrices of size n over the Laurent polynomial ring Λ_n. This homomorphism is called Bachmuth's embedding.*

Proof. Let $\psi, \xi \in \mathrm{IEnd}(M_n)$. The equality $J(\psi\xi) = J(\psi)J(\xi)$ follows from (31). It is a special case of the defined above homomorphism $\mu_{EM_n'}$. The injectivity of β has been established above too. □

Sometimes we call as *Bachmuth's embedding* its restriction to the group of all IA-automorphisms $\mathrm{IAut}(M_n)$. The following result describes the image $(\mathrm{IAut}(M_n)\beta$ in $\mathrm{GL}_n(\Lambda_n)$.

Proposition 4. *A matrix $A \in \mathrm{GL_n}(\Lambda_n)$ lies in the image* $\mathrm{IAut}(M_n)\beta$ *if and only if*

$$AB = B \textit{ for } B = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{pmatrix}. \tag{34}$$

Here $b_i = a_i - 1$ for $i = 1, ..., n$.

Proof. Let $\varphi \in \mathrm{IAut}(M_n)$, such that $x_i\varphi = u_i x_i$, where $u_i \in M_n'$ for $i = 1, ..., n$. Then

$$J(\varphi)B = (E + (d_j(x_i)))B = B \tag{35}$$

by the Main Identity (11) for the induced free Fox derivatives.

Conversely, suppose that for some matrix $A = E + C \in \mathrm{GL_n}(\mathbb{Z}\mathrm{A_n})$ we have $AB = B$, so $CB = 0$. Let $C = (c_{ij})$. Then $\sum_{j=1}^{n} c_{ij}b_j = 0$ for every $i = 1, ..., n$. It has been shown in Chapter 1 that there is a tuple of elements $\bar{u} = (u_1, ..., u_n) \in (M_n')^n$ such that $C = (d_j(u_i))$, where $d_j(u_i)$ is the $ij-$th entry. We define the endomorphism $\varphi \in \mathrm{IEnd}(M_n)$ by the map $x_i\varphi = u_i x_i$ for $i = 1, ..., n$. Then $J(\varphi) = A$. Since by our assumption there is the inverse matrix $A^{-1} \in \mathrm{GL_n}(\mathbb{Z}\mathrm{A_n})$ we can find other endomorphism $\psi \in \mathrm{IEnd}(M_n)$ for which $J(\psi) = A^{-1}$. Then $E = A^{-1}A = J(\psi)J(\varphi) = J(\psi\varphi)$ hence $\psi\varphi = id$ and so $\varphi \in \mathrm{IAut}(M_n)$. □

Aut(M_3).

Let $d_i : \mathbb{Z}M_n \to \Lambda_n$ for $i = 1, ..., n$, denote the partial free derivatives.

Let $\beta : \mathrm{IAut}(M_n) \to \mathrm{GL_n}(\Lambda_n)$ be the Bachmuth's embedding defined by the map $\varphi\beta = J(\varphi)$, where $\varphi \in \mathrm{IAut}(M_n)$ is any automorphism and $J(\varphi) = (d_j(x_i\varphi))$ is the Jacobi matrix.

By (35) the image of β is just the stabilizer of the vector $(b_1, ..., b_n)^t$, where $b_i = a_i - 1, t$ denotes transpose, and the matrix group acts by left multiplication.

We return now to $\mathrm{IAut}(M_3)$, noting that the method from [7], [8] used here works in the more general case of $\mathrm{IAut}(M_n)$ for $n \geq 3$.

Suppose that an IA-automorphism φ of M_3 is defined by the map

$$x_i\varphi = u_i x_i \text{ for } i = 1, 2, 3, \tag{36}$$

where $u_i \in M_3'$ for $i = 1, 2, 3$. We look at a conjugate of the Jacobi matrix $J(\varphi)$ of an automorphism $\varphi \in \mathrm{IAut}(M_3)$:

$$\begin{pmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & b_3 \end{pmatrix}^{-1} \cdot J(\varphi) \cdot \begin{pmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & b_3 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 + d_1(u_1) - b_1 b_3^{-1} d_1(u_3) & d_2(u_1) - b_1 b_3^{-1} d_2(u_3) & 0 \\ d_1(u_2) - b_2 b_3^{-1} d_1(u_3) & 1 + d_2(u_2) - b_2 b_3^{-1} d_2(u_3) & 0 \\ * & * & 1 \end{pmatrix}. \tag{37}$$

Denote by $A(\varphi)$ the north-west $2 \times 2-$submatrix in the right side matrix in (37).

We compute the kernel of the homomorphism

$$\theta : \mathrm{IAut}(M_3) \to \mathrm{GL}_2(\Lambda_3 + \mathrm{b}_3^{-1}\Lambda_3), \varphi \mapsto \mathrm{A}(\varphi), \tag{38}$$

from equation $A(\varphi) = E$. We get

$$u_1^{b_3} = u_3^{b_1}, u_2^{b_3} = u_3^{b_2}. \tag{39}$$

It follows that there exists an element w in M_3' such that $u_i = w^{b_i}$ for $i = 1, 2, 3$. This means that φ is the inner automorphism $\sigma_{w^{-1}}$ of M_3 corresponding to w^{-1}. Since the reverse inclusion is obvious, we conclude that $ker(\theta) = Inn_{M_3'}(M_3) \simeq M_3'$, the group of all inner automorphisms σ_w of M_3 corresponding to the elements $w \in M_3'$.

For a fixed positive number k, every element c of Λ_3 can be uniquely written in the form

$$c = c_k b_3^k + c_{k-1} b_3^{k-1} + ... + c_1 b_3 + c_0, \tag{40}$$

where $c_k \in \Lambda_3$ and $c_{k-1}, ..., c_0 \in \Lambda_2$. From this we get a unique expression

$$A(\varphi) = E + b_3^2 A_2(\varphi) + b_3 A_1(\varphi) + A_0(\varphi) + b_3^{-1} A_{-1}(\varphi), \tag{41}$$

where

$$A_1(\varphi), A_0(\varphi), A_{-1}(\varphi) \in M_2(\Lambda_2), A_2(\varphi) \in M_2(\Lambda_3). \tag{42}$$

We set

$$X = \begin{pmatrix} b_1 b_2 & -b_1^2 \\ b_2^2 & -b_1 b_2 \end{pmatrix}. \tag{43}$$

Let an automorphism $\chi \in \mathrm{IAut}(M_3$ is defined by $x_i\chi = x_i$ for $i = 1, 2$, and $x_3\chi = [x_1, x_2]x_3$. Then

$$A(\chi) = E + b_3^{-1} X. \tag{44}$$

Let φ be any IA-automorphism of M_3. Since the map θ defined by (38) is a homomorphism and the matrices $A(\varphi\chi)$ and $A(\chi\varphi)$ have the unique presentations of the type (41), we get equalities $A_{-1}(\varphi)A_{-1}(\chi) = A_{-1}(\chi)A_{-1}(\varphi) = 0$, because these matrices do not contain entries with denominators b_3^k for $k \geq 2$.

A straightforward computation shows that every matrix $B \in M_2(\Lambda_2)$ with the property $BX = XB = 0$ can be uniquely written as

$$B = \alpha X \text{ for } \alpha \in \Lambda_2. \tag{45}$$

It follows that for any $\varphi \in \mathrm{IAut}(M_3)$

$$A_{-1}(\varphi) = \alpha X \text{ for } \alpha = (\alpha)\varphi \in \Lambda_2. \tag{46}$$

Furthemore, since $A_{-1}(\varphi)X, XA_{-1}(\varphi)$, and $XA_1(\varphi)X$ are summands of the $b_3^{-1}-$components of $A_{-1}(\varphi\chi), A_{-1}(\chi\varphi)$, and $A_{-1}(\chi\varphi\chi)$ consequently there are the elements $\beta = \beta(\varphi), \gamma = \gamma(\varphi)$ and $\delta = \delta(\varphi)$ of Λ_2 such that

$$A_0(\varphi)X = \beta X, XA_0(\varphi) = \gamma X, \text{ and } XA_1(\varphi)X = \delta X. \tag{47}$$

Moreover, the equation $XCX = \delta X$ with $\delta = \delta(C) \in \Lambda_2$, holds for every matrix $C \in M_2(\Lambda_2)$.

Hence we can associate with any automorphism $\varphi \in \mathrm{IAut}(M_3)$ the elements α, β, γ, and δ of Λ_2. These elements are called the *residues* as these elements are completely defined for any automorphism $\varphi \in \mathrm{IAut}(M_3)$ by the matrix X generating the the module of all $b_3^{-1}-$components over Λ_2 of dimension 1.

Theorem 5. *(V.A. Roman'kov* [7], [8]*).*
1) The map

$$\rho : \mathrm{IAut}(M_3) \to \mathrm{GL}_2(\Lambda_2) \textit{ defined by } \varphi \mapsto \begin{pmatrix} 1+\beta & \alpha \\ \delta & 1+\gamma \end{pmatrix}, \tag{48}$$

is a homomorphism.

2) A matrix C from $GL_2(\Lambda_2)$ belongs to the image of ρ if and only if its entries follows the inclusion scheme

$$\begin{pmatrix} 1+\Delta_2 & \Lambda_2 \\ \Delta_2^2 & 1+\Lambda_2 \end{pmatrix}. \tag{49}$$

Let $\mathrm{GL}_2(\Lambda_2, \Delta_2)$ denote the congruence subgroup of $\mathrm{GL}_2(\Lambda_2)$ with respect to the ideal Δ_2, and $\mathrm{GL}_2(\Lambda_2, \Delta_2, \Delta_2^2)$ the subgroup of $\mathrm{GL}_2(\Lambda_2)$ consisting of the matrices following the inclusion scheme (49). In other words,

$$\mathrm{im}(\rho) = \mathrm{GL}_2(\Lambda_2, \Delta_2, \Delta_2^2). \tag{50}$$

It is known (see [9]) that the subgroup $\mathrm{TIAut}(M_3)$ consisting of all tame IA-automorphisms of M_3 is generated by the elements $\varphi_{123}, \varphi_{132}, \varphi_{231}$, and

$$\psi_{21}, \psi_{31}, \psi_{12}, \psi_{32}, \psi_{13}, \psi_{23}.$$

These automorphisms are defined as follows.

$$\begin{aligned} &\varphi_{ijk} : x_k \mapsto [x_i, x_j]x_k, x_l \mapsto x_l \text{ for } l \neq k, \\ &\psi_{ik} : x_k \mapsto [x_i, x_k]x_k, x_l \mapsto x_l \text{ for } l \neq k. \end{aligned} \tag{51}$$

Let φ be the element of $\mathrm{IAut}(M_3)$ given by a map

$$\varphi : x_i \mapsto u_i x_i \ , u_i \in M_3', \text{ for } i = 1, 2, 3, \tag{52}$$

We shall compute the elements α, β, γ, and δ in Λ_2 corresponding to φ in terms of the values of the free Fox derivatives $d_i : \mathbb{Z}M_3 \to \mathbb{Z}A_3$ for $i = 1, 2, 3$. Following (40) we write

$$d_i(u_j) = b_3^3 u_{j3}^i + b_3^2 u_{j2}^i + b_3 u_{j1}^i + u_{j0}^i \text{ for } i, j = 1, 2, 3, \tag{53}$$

where $u^i_{jl} \in \Lambda_2$ for $l = 0, 1, 2$, and $u^i_{j3} \in \Lambda_3$ for $i, j = 1, 2, 3$.

We get by straightforward computations that

$$\alpha = -u^1_{30}b_2^{-1} = u^2_{30}b_1^{-1}, \beta = -b_1u^1_{31} - b_2u^2_{31} = u^3_{30},$$

$$\gamma = u^1_{10} - b_1b_2^{-1}u^1_{20}, \delta = -u^3_{10}b_2 + u^3_{20}b_1. \tag{54}$$

Using (54) we derive the images under ρ of the elements (51):

$$\varphi_{123} \mapsto t_{12}(1), \varphi_{132} \mapsto t_{21}(b_1^2), \varphi_{231} \mapsto t_{21}(-b_2^2), \psi_{21} \mapsto d_2(a_2),$$

$$\psi_{31} \mapsto t_{21}(b_1b_2), \psi_{12} \mapsto d_2(a_1), \psi_{13} \mapsto d_1(a_1), \psi_{23} \mapsto d_1(a_2), \tag{55}$$

where $t_{ij}(c) = E + ce_{ij}$, $(i \neq j)$ is standard denotion of transvection, and $d_i(g)$ $(i = 1, 2)$ is the diagonal matrix with g at $i-$th and 1 on the other diagonal position.

We see that the image of the subgroup $\mathrm{TIAut}(M_3)$ under ρ lies in the group $\mathrm{GE}_2(\Lambda_2)$ of elementary matrices of $\mathrm{GL}_2(\Lambda_2)$. By definition $\mathrm{GE}_2(\Lambda_2)$ is generated by all transvections $t_{ij}(\alpha)$, $\alpha \in \Lambda_2$, and all diagonal matrices $d_i(g)$ with $g \in \Lambda_2^* = \mathrm{gp}(a_i|i = 1, 2, 3)$.

It is easy to prove that since $\mathrm{GL}_2(\Lambda_2)$ contains by [13] non-elementary matrices then the image $im(\rho) = \mathrm{GL}_2(\Lambda_2, \Delta_2, \Delta_2^2)$ also contains non-trivial matrices. Hence the image under ρ of the group $\mathrm{TIAut}(M_3)$ does not equal to $im(\rho)$. It follows that every preimage of any matrix $B \in \mathrm{IAut}(M_3)\rho \setminus \mathrm{TIAut}(M_3)\rho$ is wild, and thus $\mathrm{IAut}(M_3) \neq \mathrm{TIAut}(M_3)$. Hence $\mathrm{Aut}(M_3) \neq \mathrm{TAut}(M_3)$.

Lemma 2. *Every matrix*

$$A_1 = \begin{pmatrix} 1 + \lambda_{33}b_1 & \lambda_{32} \\ \lambda_{23}b_1^2 & 1 + \lambda_{22}b_1 \end{pmatrix} \in \mathrm{GL}_2(\Lambda_2, \Delta_2, \Delta_2^2), \tag{56}$$

where $\lambda_{ij} \in \Lambda_2$ for $i, j = 2, 3$, is the image $\varphi_1\rho$ of the automorphism

$$\varphi_1 = (x_1, [x_1, x_2]^{\lambda_{22}}[x_1, x_3]^{\lambda_{23}}x_2, [x_1, x_2]^{\lambda_{32}}[x_1, x_3]^{\lambda_{33}}x_3). \tag{57}$$

A similar statement is true for any matrix

$$A_2 = \begin{pmatrix} 1 + \lambda_{33}b_2 & \lambda_{31} \\ \lambda_{13}b_2^2 & 1 + \lambda_{11}b_2 \end{pmatrix} \in \mathrm{GL}_2(\Lambda_2, \Delta_2, \Delta_2^2), \tag{58}$$

where $\lambda_{ij} \in \Lambda_2$ for $i, j = 1, 3$, which is the image $\varphi_2\rho$ of the automorphism

$$\varphi_2 = ([x_1, x_2]^{\lambda_{12}}[x_1, x_3]^{\lambda_{13}}x_1, x_2, [x_1, x_2]^{\lambda_{31}}[x_1, x_3]^{\lambda_{33}}x_3). \tag{59}$$

Proposition 5. *For every finite set S of elements of* $\mathrm{IAut}(M_3)$ *the subgroup K generated by* $\mathrm{TAut}(M_3)$ *together with S is strictly smaller than* $\mathrm{IAut}(M_3)$. *Hence the group* $\mathrm{IAut}(M_3)$ *is not finitely generated.*

Proof. Let $H = K\rho$. It is known [13] that every subgroup of $\mathrm{GL}_2(\Lambda_k)$ which is generated by $\mathrm{GE}_2(\Lambda_k)$ and any finite set of elements is strictly smaller than $\mathrm{GL}_2(\Lambda_k)$ when $k \geq 2$. Since $\mathrm{GE}_2(\mathbb{Z}) = \mathrm{GL}_2(\mathbb{Z})$ there is a matrix $C = (c_{ij}) \in \mathrm{GL}_2(\Lambda_2) \setminus \mathrm{H}$. The desired matrix is $A = t_{21}(-b_{21})C$.

Then by Theorem 5 there is an automorphism $\varphi \in \mathrm{IAut}(M_3)$ such that $\varphi\rho = A$. Hence $\varphi \notin K$. □

3. Primitive elements and systems in M_3

S. Bachmuth and H.Y. Mochizuki in [13] did not give an example of non-elementary matrix in $\mathrm{GL}_2(\Lambda_2)$. This has been done later by M.J. Evans.

Theorem 6. *(M.J. Evans [26]). Let P be a principal domain that is not a field and $P[a_1^{\pm 1}, a_2^{\pm 1}]$ be the ring of Laurent polynomials with coefficients in P. Let ζ be a non-unit in P.*
Suppose $\sigma, \tau \in P[a_1^{\pm 1}, a_2^{\pm 1}]$ be such that

- $\sigma = (a_1 - 1)(\lambda_n a_2^n + \lambda_{n-1} a_2^{n-1} + \ldots \lambda_1 a_2 + \lambda_0)$
  for some $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1, \lambda_0 \in P[a_1^{\pm 1}, a_2^{\pm 1}]$, where $\lambda_0 \neq 0$ and $n \geq 0$;
- *σ and λ_0 are coprime, and*
- $\tau \notin a_2 P[a_1^{\pm 1}, a_2]$.

Then

$$A = \begin{pmatrix} 1 - \zeta\sigma\tau a_2^{-1} & \zeta^2 \tau a_2^{-1} \\ -\sigma^2 \tau a_2^{-1} & 1 + \zeta\sigma\tau a_2^{-1} \end{pmatrix} \tag{60}$$

is an element of $\mathrm{GL}_2(\mathrm{P}[\mathrm{a}_1^{\pm 1}, \mathrm{a}_2^{\pm 1}]) \setminus \mathrm{GE}_2(\mathrm{P}[\mathrm{a}_1^{\pm 1}, \mathrm{a}_2^{\pm 1}])$.

Let $P = \mathbb{Z}$, so $P[a_1^{\pm 1}, a_2^{\pm 1}] = \Lambda_2$. Let $\zeta = 2, \tau = 1$ and $\sigma = b_1$. Then

$$A_1 = \begin{pmatrix} 1 - 2b_1 b_2^{-1} & 4a_2^{-1} \\ -b_1^2 a_2^{-1} & 1 + 2b_1 a_2^{-1} \end{pmatrix} \in \mathrm{GL}_2(\Lambda_2) \setminus \mathrm{GE}_2(\Lambda_2). \tag{61}$$

Since $A_1 \in \mathrm{GL}_2(\Lambda_2, \Delta_2, \Delta_2^2)$ we construct by Lemma 2 the corresponding automorphism φ_1 of M_3:

$$\varphi_1 = (x_1, [x_1, x_2]^{2a_2^{-1}} [x_1, x_3]^{-a_2^{-1}} x_2, [x_1, x_2]^{4a_2^{-1}} [x_1, x_3]^{-2a_2^{-1}} x_3) \tag{62}$$

which by Theorem 6 and Lemma 2 is non-tame.

A well-known theorem of S. Bachmuth and H.Y. Mochizuki [6] and the author [7], [8] implies that, for $n \neq 3$ every primitive element of M_n is tame. In contrast, we have the following result in the 3−case.

Theorem 7. *(V.A. Roman'kov [4]). Let φ be the element of* $\mathrm{IAut}(M_3)$ *given by the map (52). Suppose that the image $A = \varphi\rho$ under the homomorphism ρ from (48) does not belong to $GE_2(\Lambda_2)$. Then the primitive element $x = u_3 x_3$ is not tame.*

Proof. Note that the first row of the matrix $A = \varphi\rho$ depends of a single element $x = u_3 x_3$. Suppose that x can be included in a tame basis $\{y, z, x\}$ of M_3. Easily to prove that we can choose this basis in such way that the corresponding automorphism $\psi = (y, z, x)$ belongs to $\mathrm{TIAut}(M_3)$. But then the image $\bar{A} = \psi\rho$ lies in $\mathrm{GE}_2(\Lambda_2)$. The first rows of A and $\bar{A}$ coincide. Thus matrix $A\bar{A}^{-1}$ has trivial first row and so belongs to $\mathrm{GE}_2(\Lambda_2)$. It follows that A also belongs to $\mathrm{GE}_2(\Lambda_2)$, that contradicts our assumption. □

Now we can give a concrete non-tame primitive element of M_3.

Example 1. *The element*

$$x = [x_1, x_2]^{4a_2^{-1}} [x_1, x_3]^{2a_2^{-1}} x_3 \tag{63}$$

presented in (62) is non-tame primitive element of M_3.

Recall that the first example of non-tame automorphism of M_3 has been established by O. Chein [2]. This automorphism ξ is defined as follows:

$$\xi = ([x_2, x_3]^{b_1^2} x_1, x_2, x_3). \tag{64}$$

In [27] an element of M_3 of the form $c = [x_1, x_2]^p x_1$, where $p \in \Lambda_3$, is called a *Chein's element.* Every Chein's element defines a one-row *Chein's automorphism* $\xi_c = (c, x_2, x_3)$. An automorphism φ of a relatively free group G is called *one row* w.r.t. a fixed basis if φ fixes all except may be one elements of the basis.

It would be quite natural to expect that the Chein's elements are non-tame primitive elements of M_3. This is not true.

Proposition 6. *([27]). Every Chein's element $c = [x_2, x_3]^p x_1$, $p \in \Lambda_3$, is tame primitive element in M_3.*

Proof. The proof consists in exhibiting a tame IA-automorphism φ_p of M_3 which maps x_1 to c. It is sufficient to find an automorphism $\varphi_p = \varphi_{ijk}$ for every monomial $p = a_1^i a_2^j a_3^k$ $(i, j, k \in \mathbb{Z})$ with additional property that $x_l \varphi_p = w x_l w^{-1}$ $(l = 2, 3)$ for some $w = w(p) \in M_3'$. Note, that a product of two automorphisms φ_p, φ_q with this additional property is automorphism φ_{p+q} satisfying the property. Also note, that for every automorphism φ_p with this property we have $x_1 \varphi_{-p} = x_1 \varphi_p^{-1}$, and so we can get for an arbitrary $p \in \Lambda_3$ an automorphism of the form φ_p from φ_{ijk}'s.

It remains to construct for each $i, j, k \in \mathbb{Z}$ an automorphism φ_{ijk}. Consider the tame automorphisms α_{jk} and β_i of M_3 given by

$$\alpha_{jk} = ([x_2, x_3]^{a_2^j a_3^k} x_1, x_2, x_3), \beta_i = (x_1, x_1^i x_2 x_1^{-i}, x_1^i x_3 x_1^{-i}), \tag{65}$$

and define $\varphi_{ijk} = \beta_i \alpha_{jk} \beta_i^{-1}$. □

Notice, that each Cheins automorphism $\eta = (x_1, x_2, c)$, where c is a Chein's primitive element of the form $c = [x_2, x_3]^p x_1$, is defined by three tame primitive elements, although it is not always tame itself. The homomorphism $\xi = \theta\rho$ maps η to matrix with the second row $(0, 1)$ which is obviously tame. We'll call an element $z = u_3 x_3 \in M_3, u_3 \in M_3'$, *strongly wild* if the image $(x, y, z)\xi$ belongs to $W = \mathrm{GL}_2(\Lambda_2) \setminus \mathrm{GE}_2(\Lambda)$.

We will call a matrix in $\mathrm{GL}_2(\Lambda_2)$ *tame* if it belongs to $\mathrm{GE}_2(\Lambda_2)$, and *wild* otherwise. A row $(a, b) \in \Lambda^2$ is called *tame* if it belongs to some tame matrix, and *wild* otherwise. This definition is correct, since any row cannot belong to the tame and wild matrices at the same time. Indeed, if this row belongs to a tame matrix, one can multiply other non-tame matrix with it to the inverse of the first matrix, and obtain a matrix with one trivial row which is obviously tame, what is impossible.

Now we slightly change the homomorphism ρ to $\tilde{\rho}$. Let σ is a conjugation by matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & b_1 \end{pmatrix}. \tag{66}$$

Then we define $\tilde{\rho} = \rho\sigma$. By definition a matrix in $\mathrm{im}(\tilde{\rho})$ is *tame* if the corresponding matrix in $\mathrm{im}(\rho)$ is tame.

Theorem 8. *Let $z = u_3 x_3 \in M_3, u_3 \in M_3'$, be a strongly wild primitive element. Then every basis (x, y, z) of M_3 contains at least two wild elements.*

Proof. We assume that (x, y, z) contains at least one tame primitive element, say x. We can also assume that $x = x_1$. Let $\varphi \in \mathrm{IAut}(M_3)$ is defined by (52). Then

$$\varphi\tilde{\rho} = \begin{pmatrix} 1 + u_{30}^3 & u_{30}^2 \\ u_{20}^3 & 1 + u_{20}^2 \end{pmatrix}. \tag{67}$$

We know that the first row in (67) is wild. The second row is symmetric in indices with the first, hence it is wild too. Hence $y = u_2x_2$ is strongly wild primitive element. □

Following [28] we say that the exact presentation

$$1 \to N \to F_n \to^{\theta} G \to 1 \tag{68}$$

of a group G is *essentially* $(n-1)-$*generator* if N contains a primitive element of F_n.

Let $1 \to N \to F_n \to^{\theta} M \to 1$ be the exact presentation of a metabelian group M. Since $F_n'' \leq N$, there is an induced epimorphism $\gamma = \gamma(\theta) : M_n \to M$ such that $\theta = \pi''\gamma$.

Let in (68) $G = M_{n-1}$. Let $\ker(\gamma) = \mathrm{ncl}(g)$, $\gamma = \gamma(\theta)$, where g is a primitive element of M_n. Now if $n \neq 3$, g is tame primitive element of M_n and consequently $\ker(\theta)$ contains a primitive element of F_n, and corresponding presentation (68) for $G = M_{n-1}$ is essentially $(n-1)-$generated.

In contrast, there are non-(essentially $2-$generator) presentations (68) for $G = M_2$.

Theorem 9. (*M.J. Evans* [29]). *Let g be a non-tame primitive element of M_3 (existing by Theorem 7). Let $\gamma : M_3 \to M_2$ be an epimorphism that has* $\ker N = \mathrm{ncl}(g)$. *Define $\theta : F_3 \to M_2$ by $\theta = \pi''\gamma$. Then $1 \to N \to F_3 \to^{\theta} M_2 \to 1$ is not essentially $2-$generator, i.e.* $\ker(\theta)$ *does not contain a primitive element of F_3.*

The proof depends on the following result from [29] which is of independent interest. Let q be a primitive element of M_n and suppose that q is contained in the normal closure $\mathrm{ncl}(f)$ of some element $f \in M_n$. Then q is conjugate to f or f^{-1}.

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