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ASYMPTOTIC MODELLING OF BONDED PLATES BY A SOFT THIN ADHESIVE LAYER

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Abstract. In the present paper, a composite structure is considered. The structure is made of three homogeneous plates: two linear elastic adherents and a thin adhesive. It is assumed that elastic properties of the adhesive layer depend on its thickness ε as ε to the power of 3. Passage to the limit as ε goes to zero is justified and a limit model is found in which the influence of the thin adhesive layer is replaced by an interface condition between adherents. As a result, we have analog of the spring type condition in the plate theory. Moreover, a representation formula of the solution in the adhesive layer has been obtained.

Keywords: bonded structure, Kirchhoff-Love's plate, composite material, spring type interface condition, biharmonic equation.

1. Introduction

The characterization of interface conditions between bonded elastic media is a classical problem in solids mechanics (see, e.g., [1, 2, 3, 4]). This problem arises, when composite material should be modelled. Due to small thickness of a glue layer (or, so called, an adhesive) numerical computation of the solution of the corresponding boundary value problem can be very difficult because it requires fine meshes. In this situation, instead of the full model the approximate one with the interface condition between adherents is introduced. Many interface conditions are currently studied rather well from both mathematical and mechanical point of view for different models of solids mechanics: linear and nonlinear elasticity, piezoelectricity, magneto-electro-thermo-elasticity, delamination cracks (see, e.g., [5, 6, 7, 8, 9, 10, 11]).

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However, to our best knowledge, the problem of constructing an asymptotic expansion for bonded Kirchhoff-Love's plates has not been addressed in the existing literature.

In the present work, we consider a composite structure consisting of two plates glued together by a third one (adhesive layer) along some common interfaces. The structure is in equilibrium under the action of applied forces and equilibrium of each plates is described by biharmonic equations. The problem (case of pure bending) is formulated as a variational one. Namely, we consider a minimization problem of the energy functional over a set of admissible deflections of the composite plate in the space H^2 is considered (see, e.g., [12, 13]).

It is assumed that the elastic properties of the adhesive layer depend on its thickness ε as ε^3. ε is the small parameter of the problem. But the elastic properties of the glued plates do not depend on ε and remain constants. The main goal of the present paper is to strictly mathematically justify the passage to the limit when ε tends to zero.

The result is a model in which the adhesive layer is replaced by conditions on the interface of two adherent plates. By analogy with elasticity (see, e.g., [4, 5, 14] we obtain the so-called spring-type condition, which for plates means that the transverse forces acting on common interface are proportional to the jump of the deflections of the plates.

2. Statement of the problem

Let $\Omega_\pm \subset \mathbb{R}^2$ be two disjoint domains with boundaries $\partial\Omega_\pm$. Denote by $S = \partial\Omega_- \cap \partial\Omega_+$ the common part of boundaries of domains $\Omega_\pm$ and assume that S is an interval, laying on x_2-axis, i.e., $S = \{(0, x_2) \in \mathbb{R}^2 \mid x_2 \in (a, b)\}$, $a < b$; $\partial\Omega_\pm = \overline{\Gamma}_{D\pm} \cup \overline{\Gamma}_{N\pm}$, $\Gamma_{D\pm} \cap \Gamma_{N\pm} = \emptyset$, and $\overline{\Gamma}_{D\pm} \cap \overline{S} = \emptyset$.

To put an adhesive rectangular layer of the thickness $2\varepsilon d$ between the domains $\Omega_\pm$, we shift the domains $\Omega_\pm$ along the x_1-axis by $\pm\varepsilon d$, respectively, i.e.,

$$\Omega^\varepsilon_\pm = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 = x_1 \pm \varepsilon d,\ y_2 = x_2,\ (x_1, x_2) \in \Omega_\pm\}.$$

Here d is a global characteristic length (for example, diameter of the union $\Omega_- \cup \Omega_+$); $\varepsilon > 0$ is a small dimensionless parameter. Then an adhesive between Ω_- and Ω_+ is described by

$$\Omega^\varepsilon_m = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \in (-\varepsilon d, \varepsilon d),\ y_2 \in (a, b)\}.$$

Let $S^\varepsilon_\pm$ denote the common interfaces between the adherents $\Omega^\varepsilon_\pm$ and the adhesive Ω^ε_m, i.e., $S^\varepsilon_\pm = \partial\Omega^\varepsilon_m \cap \partial\Omega^\varepsilon_\pm$, where

$$S^\varepsilon_\pm = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 = \pm\varepsilon d, \quad y_2 \in (a, b)\}.$$

Let $\Omega^\varepsilon = \Omega^\varepsilon_- \cup \Omega^\varepsilon_+ \cup S^\varepsilon_- \cup S^\varepsilon_+ \cup \Omega^\varepsilon_m$ be a middle surface of the composite structure (a heterogeneous plate), consisting of middle surfaces $\Omega_\pm$ of two homogeneous plates, which are glued together by the thin plate with a middle surface Ω^ε_m. We assume that the thickness of all plates is equal to h (see Fig. 1).

Let $E_\pm$, E^ε_m and $k_\pm$, k_m be Young's modulus and Poisson's ratio corresponding to $\Omega_\pm$, Ω^ε_m, respectively, such that $E_\pm$, $k_\pm$, and k_m are constant, while Young's modulus E^ε_m depends on ε as follows

$$E^\varepsilon_m = \varepsilon^3 E_m,$$

where $E_m = const$. It means that we deal with a soft adhesive.

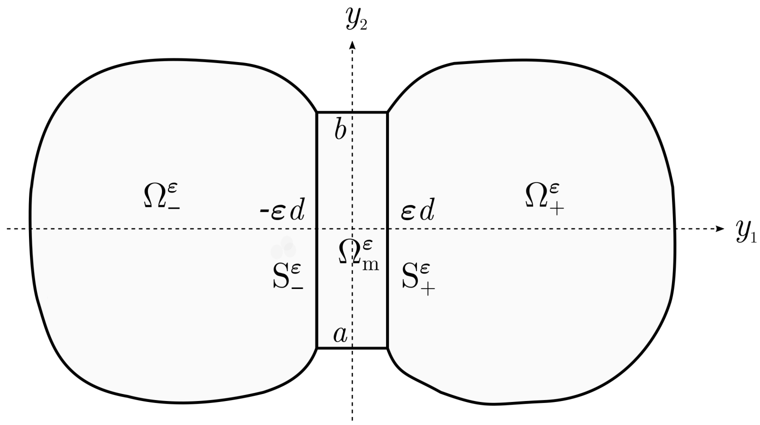

FIG. 1. Middle surface of the composite structure

We introduce the following notation for the bending stiffness

$$\mu_\pm = \frac{E_\pm h^3}{12(1-k_\pm^2)}, \quad \mu_m^\varepsilon = \varepsilon^3 \mu_m$$

with $\mu_m = \frac{E_m h^3}{12(1-k_m^2)}$.

Let u_ε be deflections of Ω^ε. We prescribe the homogeneous Dirichlet conditions on $\Gamma^\varepsilon_{D\pm} \subset \partial\Omega^\varepsilon$, where

$$\Gamma^\varepsilon_{D\pm} = \{(y_1, y_2) \in \mathbb{R}^2 \;\mid\; y_1 = x_1 \pm \varepsilon d,\; y_2 = x_2,\; (x_1, x_2) \in \Gamma_{D\pm}\}$$

. We assume that the composite plate is in equilibrium under acting of an external force $f \in L_2(\Omega^\varepsilon)$ such that $f = 0$ a.e. in Ω^ε_m.

Let us denote by μ^ε the bending stiffness and by k the Poisson ratio of the composite plate, where $\mu^\varepsilon = \mu_\pm$ and $k = k_\pm$ in $\Omega^\varepsilon_\pm$, and $\mu^\varepsilon = \mu^\varepsilon_m$ and $k = k_m$ in Ω^ε_m.

We will formulate the equilibrium problem of the composite plate with the middle surface Ω^ε as a variational problem. Let us define a Sobolev space

$$H^{2,0}(\Omega^\varepsilon) = \{v \in H^2(\Omega^\varepsilon) \;\mid\; v = \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma^\varepsilon_{D\pm}\};$$

the energy functional

$$\Pi(v) = \frac{1}{2} B^\varepsilon(v, v) - l(v)$$

with

$$B^\varepsilon(v, w) = \int\limits_{\Omega^\varepsilon} \mu^\varepsilon (v_{,11} w_{,11} + v_{,22} w_{,22} + k(v_{,11} w_{,22} + v_{,22} w_{,11}) + 2(1-k) v_{,12} w_{,12})\, dy,$$

$$l(v) = \int\limits_{\Omega^\varepsilon} f v \, dy.$$

Then the equilibrium problem is as follows: find a function $u_\varepsilon \in H^{2,0}(\Omega^\varepsilon)$ such that

$$\Pi(u_\varepsilon) = \inf_{v \in H^{2,0}(\Omega^\varepsilon)} \Pi(v), \tag{1}$$

or, equivalently,

$$B^\varepsilon(u^\varepsilon, v) = l(v) \quad \forall v \in H^{2,0}(\Omega^\varepsilon). \tag{2}$$

3. Decomposition of problem (1)

Let us reformulate problem (1) in an equivalent form. Namely, we decompose it into three subproblems defined in domains $\Omega^\varepsilon_\pm$, Ω^ε_m and connected along the common interfaces $S^\varepsilon_\pm$. Let us introduce a set

$$K_\varepsilon = \{(v_-, v_+, v_m) \in H^{2,0}(\Omega^\varepsilon_-) \times H^{2,0}(\Omega^\varepsilon_+) \times H^2(\Omega^\varepsilon_m) \ \mid \ v_\pm = v_m, \quad \frac{\partial v_\pm}{\partial n} = \frac{\partial v_m}{\partial n} \text{ on } \mathrm{S}^\varepsilon_\pm, \}$$

where

$$H^{2,0}(\Omega^\varepsilon_\pm) = \{v_\pm \in H^2(\Omega^\varepsilon_\pm) \ \mid \ v_\pm = \frac{\partial v_\pm}{\partial n} = 0 \text{ on } \Gamma^\varepsilon_{\mathrm{D}\pm}\};$$

and define bilinear forms

$$b^\varepsilon_\pm(v_\pm, u_\pm) = \mu_\pm \int\limits_{\Omega^\varepsilon_\pm} (v_{,11}w_{,11} + v_{,22}w_{,22} + k_\pm(v_{,11}w_{,22} + v_{,22}w_{,11}) + 2(1-k_\pm)v_{,12}w_{,12})\, dy,$$

$$b^\varepsilon_m(v_m, u_m) = \mu_m \int\limits_{\Omega^\varepsilon_m} (v_{,11}w_{,11} + v_{,22}w_{,22} + k_m(v_{,11}w_{,22} + v_{,22}w_{,11}) + 2(1-k_m)v_{,12}w_{,12})\, dy.$$

Then the problem (2) can be reformulate equivalently as follows: find a triplet $(u_{\varepsilon-}, u_{\varepsilon+}, v_{\varepsilon m}) \in K_\varepsilon$ such that

$$b^\varepsilon_-(u_{\varepsilon-}, v_-) + b^\varepsilon_+(u_{\varepsilon+}, v_+) + \varepsilon^3 b^\varepsilon_m(u_{\varepsilon m}, v_m) = l^\varepsilon_-(v_-) + l^\varepsilon_+(v_+) \quad \forall (v_-, v_+, v_m) \in K_\varepsilon, \tag{3}$$

where

$$l^\varepsilon_\pm(v_\pm) = \int\limits_{\Omega^\varepsilon_\pm} f_{\varepsilon\pm} v_\pm \, dy,$$

and $f_{\varepsilon\pm}$ is the restriction of the function f on domains $\Omega^\varepsilon_\pm$, respectively.

4. Rescaling and asymptotic expansions

We change coordinates in each domains $\Omega^\varepsilon_\pm$ and Ω^ε_m to obtain domains which are independent of ε. Namely, we consider the following coordinate transformations:

$$x_1 = y_1 \mp \varepsilon d, \ x_2 = y_2, \quad (x_1, x_2) \in \Omega_\pm, \ (y_1, y_2) \in \Omega^\varepsilon_\pm, \tag{4}$$

$$z_1 = \frac{y_1}{\varepsilon}, \ z_2 = y_2, \quad (z_1, z_2) \in \Omega_m, \ (y_1, y_2) \in \Omega^\varepsilon_m, \tag{5}$$

where

$$\Omega_m = \{(z_1, z_2) \in \mathbb{R}^2 \ \mid \ z_1 \in (-d, d), \ z_2 \in (a, b)\}.$$

Denote by

$$S^\pm_m = \{(z_1, z_2) \in \mathbb{R}^2 \ \mid \ z_1 = \pm d, \ z_2 \in (a, b)\},$$
$$S^a_m = \{(z_1, z_2) \in \mathbb{R}^2 \ \mid \ z_1 \in (-d, d) \ z_2 = a\},$$

$$S_m^b = \{(z_1, z_2) \in \mathbb{R}^2 \ \mid \ z_1 \in (-d, d)\ z_2 = b\}$$

parts of $\partial\Omega_m$ of the domain Ω_m, i.e., $\partial\Omega_m = \overline{S^- \cup S^+ \cup S_m^a \cup S_m^b}$.

The advantage of decomposing the problem is that we have after coordinate transformations domains which are independent of ε unlike, for example, [4], where adherents was shifted from the adhesive by the distance εd.

Note that the coordinate transformations (4) and (5) are smooth and one-to-one for all $\varepsilon > 0$. It implies one-to-one correspondence between $H^{2,0}(\Omega_\pm^\varepsilon)$, $H^2(\Omega_m^\varepsilon)$ and $H^{2,0}(\Omega_\pm)$, $H^2(\Omega_m)$, respectively. Moreover, the set of admissible deflections K_ε is one-to-one transformed into a set K^ε, where

$$K^\varepsilon = \{(v_-, v_+, v_m) \in H^{2,0}(\Omega_-) \times H^{2,0}(\Omega_+) \times H^2(\Omega_m) \ \mid \ v_\pm|_S = v_m|_{S_m^\pm}, \\ v_{\pm,1}|_S = \frac{1}{\varepsilon} v_{m,1}|_{S_m^\pm}\}.$$

Hereinafter, we assume that for any functions $v_\pm(x)$, $x \in \Omega_\pm$, and $v_m(z)$, $z \in \Omega_m$, equality $v_\pm|_S = v_m|_{S_m^\pm}$ means that

$$v_\pm(0, x_2) = v_m(\pm d, z_2), \quad x_2 = z_2 \in (a, b).$$

Apply coordinate transformations to integrals in (3). As a result we have that a triple $U^\varepsilon = (u_-^\varepsilon, u_+^\varepsilon, u_m^\varepsilon) \in K^\varepsilon$ is a solution of the following variational equality

$$(6) \quad b_-(u_-^\varepsilon, v_-) + b_+(u_+^\varepsilon, v_+) + B_m^\varepsilon(u_\varepsilon^m, v_m) = l_-(v_-) + l_+(v_+) \\ \forall\, V = (v_-, v_+, v_m) \in K^\varepsilon$$

where

$$u_\pm^\varepsilon(x_1, x_2) = u_{\varepsilon\pm}(x_1 \pm \varepsilon d, x_2), \quad u_m^\varepsilon(z_1, z_2) = u_{\varepsilon m}(\varepsilon z_1, z_2),$$

$$B_m^\varepsilon(v, w) = \mu_m \int\limits_{\Omega_m} \big(v_{,11} w_{,11} + \varepsilon^4 v_{,22} w_{,22} + \varepsilon^2 k_m (v_{,11} w_{,22} + v_{,22} w_{,11}) + \\ + 2(1 - k_m)\varepsilon^2 v_{,12} w_{,12}\big)\, dz,$$

$$l_\pm(v_\pm) = \int\limits_{\Omega_\pm} f_\pm^\varepsilon v_\pm \, dx, \quad f_\pm^\varepsilon(x_1, x_2) = f_\pm(x_1 \pm \varepsilon d, x_2).$$

We will need several auxiliary statements.

Lemma 1. *For any function $v_m \in H^1(\Omega_m)$ the following inequalities*

$$(7) \qquad \|v_m\|^2_{L_2(\Omega_m)} \le C \left(\|v_{m,1}\|^2_{L_2(\Omega_m)} + \|v_m\|^2_{L_2(S_m^\pm)} \right)$$

hold, where a constant C does not depend on v_m.

Proof. This results from some direct integrations along x_1-axis in the domain Ω_m. □

Corollary 1. *For any function $v_m \in H^2(\Omega_m)$ the followings inequalities*

$$(8) \qquad \|v_{m,1}\|^2_{L_2(\Omega_m)} \le C \left(\|v_{m,11}\|^2_{L_2(\Omega_m)} + \|v_{m,1}\|^2_{L_2(S_m^\pm)} \right),$$

$$(9) \qquad \|v_m\|^2_{L_2(\Omega_m)} \le C \left(\|v_{m,11}\|^2_{L_2(\Omega_m)} + \|v_{m,1}\|^2_{L_2(S_m^\pm)} + \|v_m\|^2_{L_2(S_m^\pm)} \right)$$

hold, where a constant C does not depend on v_m.

Proof. Inequality (8) results from inequality (7) for function $v_{m,1} \in H^1(\Omega_m)$. In turn, (9) follows from (7) and (8). $\square$

Corollary 2. *For any function $V = (v_-, v_+, v_m) \in K^\varepsilon$ and $\varepsilon \in (0,1)$ the followings inequalities*

$$\|v_{m,1}\|^2_{L_2(\Omega_m)} + \|v_m\|^2_{L_2(\Omega_m)} \le C \left(\|v_{m,11}\|^2_{L_2(\Omega_m)} + \|v_\pm\|^2_{H^{2,0}(\Omega_\pm)} \right) \tag{10}$$

hold, where a constant C does not depend on V and $\varepsilon > 0$.

Proof. Adding (8) and (9) and taking into account the continuity of the trace operator, we obtain (10) for all $\varepsilon \in (0,1)$. $\square$

Now we introduce a set

$$K = \{(v_-, v_+, v_m) \in H^{2,0}(\Omega_-) \times H^{2,0}(\Omega_+) \times H^2(\Omega_m) \ | \ v_{m,1}|_{S_m^\pm} = 0, \ \ v_m|_{S_m^\pm} = v_\pm|_S\}$$

and prove the following lemma.

Lemma 2. *For any function $V = (v_-, v_+, v_m) \in K$ there exists $V^\varepsilon = (v_-^\varepsilon, v_+^\varepsilon, v_m^\varepsilon) \in K^\varepsilon$ such that*

$$V^\varepsilon \to V \quad \textit{strongly in} \quad H^{2,0}(\Omega_-) \times H^{2,0}(\Omega_+) \times H^2(\Omega_m). \tag{11}$$

Proof. Let us construct an extension of functions defined on $S_m^\pm$ in the nonsmooth domain Ω_m. Take any bounded domain $\tilde{\Omega}_m$ with a smooth boundary $\partial\tilde{\Omega}_m$ such that $\Omega_m \subset \tilde{\Omega}_m$ and $\partial\tilde{\Omega}_m \setminus \partial(\tilde{\Omega}_m \setminus \Omega_m) = S_m^- \cup S_m^+$.

Take a function $v_\pm \in H^{2,0}(\Omega_\pm)$ and fix $\varepsilon > 0$. Further we extend restrictions of functions $\pm v_{\pm,1} \in H^{1/2}(S_m^\pm)$ on whole the boundary $\partial\tilde{\Omega}_m$ (for instance, solving the mixed boundary value problem for the Laplace operator with the Dirichlet conditions $\pm v_{\pm,1}$ on $S_m^\pm$ and homogeneous Neumann conditions on $\partial\tilde{\Omega} \setminus (S_m^- \cup S_m^+)$). Denote this extension by $q \in H^{1/2}(\partial\tilde{\Omega}_m)$.

Next, in the smooth domain $\tilde{\Omega}_m$ we consider рассмотрим the following mixed boundary value problem for bilaplacian:

$$\Delta^2 p = 0 \quad \text{in} \quad \tilde{\Omega}_m,$$

$$p = 0 \quad \text{on} \quad \partial\tilde{\Omega}_m,$$

$$\frac{\partial p}{\partial n} = q \quad \text{on} \quad \partial\tilde{\Omega}_m,$$

where n is a unit normal vector to $\partial\tilde{\Omega}_m$. Due to the definition q note that

$$\frac{\partial p}{\partial n} = \pm p_{,1} = \pm v_{\pm,1} \quad \text{on} \quad S_m^\pm.$$

At last, denote by $\varphi_m \in H^2(\Omega_m)$ a restriction of p to the domain Ω_m. Note that the function φ_m satisfies the following conditions:

$$\varphi_m = 0 \quad \text{a.e. on} \quad S_m^\pm, \tag{12}$$

$$\varphi_{m,1} = v_{\pm,1} \quad \text{a.e. on} \quad S_m^\pm. \tag{13}$$

Finally, for the function $V \in K$ we put

$$V^\varepsilon = (v_-, v_+, v_m + \varepsilon\varphi_m).$$

It is obviously that due to (12), (13) the function V^ε belongs to the set K^ε and convergence (11) holds. $\square$

Lemma 3. *Assume that for some $i, j = 1, 2$ the following convergences*

$$u_m^\varepsilon \to u_m \quad \textit{weakly in} \quad L_2(\Omega_m),$$
$$u_{m,i}^\varepsilon \to p_i \quad \textit{weakly in} \quad L_2(\Omega_m),$$
$$u_{m,ij}^\varepsilon \to q_{ij} \quad \textit{weakly in} \quad L_2(\Omega_m)$$

hold as $\varepsilon \to 0$. Then $p_i = u_{m,i}$, $q_{ij} = u_{m,ij}$ a.e. in Ω_m.

Proof. Let us consider a relation

$$\int\limits_{\Omega_m} u_m^\varepsilon \varphi_{,ij}\, dz = \int\limits_{\Omega_m} u_{m,ij}^\varepsilon \varphi\, dz$$

which is valid for all $\varphi \in C_0^\infty(\Omega_m)$. Passing to the limit in this relation as $\varepsilon \to 0$ we have the following one

$$\int\limits_{\Omega_m} u_m \varphi_{,ij}\, dz = \int\limits_{\Omega_m} q_{ij} \varphi\, dz \quad \forall \varphi \in C_0^\infty(\Omega_m).$$

It implies that $q_{ij} = u_{m,ij}$ a.e. in Ω_m. Using similar reasoning, we can show that $p_i = u_{m,i}$ a.e. in Ω_m. □

Lemma 4. *Assume that for $\alpha > 0$, $\beta > 0$, and for $i, j = 1, 2$ the following convergences*

$$u_m^\varepsilon \to u_m \quad \textit{weakly in} \quad L_2(\Omega_m),$$
$$\varepsilon^\alpha u_{m,i}^\varepsilon \to p_i \quad \textit{weakly in} \quad L_2(\Omega_m),$$
$$\varepsilon^\beta u_{m,ij}^\varepsilon \to q_{ij} \quad \textit{weakly in} \quad L_2(\Omega_m)$$

hold as $\varepsilon \to 0$. Then $p_i = 0$, $q_{ij} = 0$ a.e. in Ω_m.

Proof. Again let us consider the relation

$$\int\limits_{\Omega_m} u_m^\varepsilon \varphi_{,ij}\, dz = \int\limits_{\Omega_m} u_{m,ij}^\varepsilon \varphi\, dz$$

which is valid for all $\varphi \in C_0^\infty(\Omega_m)$. Multiplying this relation by ε^β and passing to the limit as $\varepsilon \to 0$ we have the following relation

$$0 = \int\limits_{\Omega_m} q \varphi\, dz \quad \forall \varphi \in C_0^\infty(\Omega_m).$$

It implies that $q = 0$ a.e. in Ω_m. Using similar reasoning, we can show that $p = 0$ a.e. in Ω_m. □

5. Limit problem

Now we are ready to justify the passage to the limit as $\varepsilon \to 0$. Let us substitute $U^\varepsilon = (u_-^\varepsilon, u_+^\varepsilon, u_m^\varepsilon)$ in (6) as test functions. As a result, we get the following estimate:

$$\text{(14)} \quad \|u_-^\varepsilon\|^2_{H^{2,0}(\Omega_-)} + \|u_+^\varepsilon\|^2_{H^{2,0}(\Omega_+)} + \|u_{m,11}^\varepsilon\|^2_{L_2(\Omega_m)} + \\ + \|\varepsilon u_{m,12}^\varepsilon\|^2_{L_2(\Omega_m)} + \|\varepsilon^2 u_{m,22}^\varepsilon\|^2_{L_2(\Omega_m)} \le C,$$

where by C we denote, as usual, a positive constant which is independent of ε. Moreover, due to (14) and Corollary 1 we have

$$\text{(15)} \qquad \|u_m^\varepsilon\|^2_{L_2(\Omega_m)} + \|u_{m,1}^\varepsilon\|^2_{L_2(\Omega_m)} \le C.$$

The inequalities (14), (15), and Corollary 2 imply that there exists subsequences and functions $u_\pm$, u_m, p, q such that

$$
\begin{aligned}
u_\pm^\varepsilon &\to u_\pm \quad \text{weakly in} \quad H^{2,0}(\Omega_\pm),\\
u_m^\varepsilon &\to u_m \quad \text{weakly in} \quad L_2(\Omega_m),\\
u_{m,1}^\varepsilon &\to u_{m,1} \quad \text{weakly in} \quad L_2(\Omega_m),\\
u_{m,11}^\varepsilon &\to u_{m,11} \quad \text{weakly in} \quad L_2(\Omega_m),\\
\varepsilon u_{m,12}^\varepsilon &\to p \quad \text{weakly in} \quad L_2(\Omega_m),\\
\varepsilon^2 u_{m,22}^\varepsilon &\to q \quad \text{weakly in} \quad L_2(\Omega_m).
\end{aligned}
\tag{16}
$$

Note that from definition of the set K^ε it follows

$$u_{m,1} = 0 \quad \text{a.e. on} \quad S_m^\pm, \tag{17}$$

$$u_m|_{S_m^\pm} = u_\pm|_S. \tag{18}$$

Theorem 1. *Let* $U = (u_-, u_+, u_m)$ *be the limit function from* (16)*. Then*

$$u_{m,11}(z_1, z_2) = -\frac{3(u_+(0, z_2) - u_-(0, z_2))}{2d^3} z_1 \quad \textit{a.e. in} \quad \Omega_m.$$

Proof. Take $\varphi \in C_0^\infty(\Omega_m)$ and substitute $(0, 0, \varphi) \in K^\varepsilon$ in (6) as a test function. As a result, we have

$$\int\limits_{\Omega_m} u_{m,11}\varphi_{,11} dz = 0 \ \ \forall \varphi \in C_0^\infty(\Omega_m).$$

It means that there exist two function $\alpha(z_2)$ and $\beta(z_2)$ such that

$$u_{m,11}(z_1, z_2) = \alpha(z_2) z_1 + \beta(z_2) \quad \text{a.e. in} \quad \Omega_m. \tag{19}$$

Find functions α and β. On the one hand, due to (17) we have

$$\int\limits_{-d}^{d} u_{m,11}(z_1, z_2)\, dz_1 = 0 \quad \text{a.e. on} \quad (a, b).$$

On the other hand, from (19) we get

$$\int\limits_{-d}^{d} (\alpha(z_2) z_1 + \beta(z_2))\, dz_1 = 2d\beta(z_2) \quad \text{a.e. on} \quad (a, b).$$

Thus, we get

$$\beta(z_2) = 0 \quad \text{a.e. on} \quad (a, b).$$

In virtue of (17) we have

$$\int\limits_{-d}^{d} u_{m,11}^2(z_1, z_2)\, dz_1 = \int\limits_{-d}^{d} \alpha(z_2) z_1 u_{m,11}(z_1, z_2)\, dz_1 = -\alpha(z_2)(u_m(d, z_2) - u_m(-d, z_2))$$

a.e. on (a, b). In the same time

$$\int\limits_{-d}^{d} u_{m,11}^2(z_1, z_2)\, dz_1 = \int\limits_{-d}^{d} (\alpha(z_2) z_1)^2\, dz_1 = \frac{2}{3}\alpha^2(z_2) d^3$$

a.e. on (a, b).

Thus, the following relation

$$\frac{2}{3}\alpha^2(z_2)d^3 = -\alpha(z_2)(u_m(d, z_2) - u_m(-d, z_2))$$

holds.

Let us assume that for some point $\bar{z}_2 \in (a, b)$ the function $\alpha(\bar{z}_2)$ is not equal to zero; then due to (18)

$$\alpha(\bar{z}_2) = -\frac{3(u_+(0, \bar{z}_2) - u_-(0, \bar{z}_2))}{2d^3}$$

Now suppose that there exists a set $Z_2 \subset (a, b)$ of nonzero measure such that $\alpha(z_2) = 0$ for all $z_2 \in Z_2$. Then $u_{m,11}(z_1, z_2) = 0$ a.e. in $(-d, d) \times Z_2$. It means that there exist functions $\gamma(z_2)$ and $\delta(z_2)$ such that $u_m(z_1, z_2) = \gamma(z_2)z_1 + \delta(z_2)$ a.e. in $(-d, d) \times Z_2$. Due to (17), $\gamma(z_2) = 0$ a.e. in Z_2. In virtue of (18), $0 = u_m(d, z_2) - u_m(-d, z_2) = u_+(0, x_2) - u_-(0, x_2)$. Thus, we have the following condition: if $\alpha(z_2) = 0$ then $u_+(0, x_2) - u_-(0, x_2) = 0$. □

Introduce a space K_0, where

$$K_0 = \{V = (v_-, v_+, v_m) \in H^{2,0}(\Omega_-) \times H^{2,0}(\Omega_+) \times L_2(\Omega_m) \ | \ v_{m,1} \in L_2(\Omega_m),\ v_{m,11} \in L_2(\Omega_m),\ v_{m,1} = 0 \text{ a.e. on } S_m^\pm,\ \ v_m|_{S_m^\pm} = v_\pm|_S\},$$

endowed with a norm

$$\|V\|^2_{K_0} = \|v_-\|^2_{H^{2,0}(\Omega_-)} + \|v_+\|^2_{H^{2,0}(\Omega_+)} + \|v_m\|^2_{L_2(\Omega_m)} + \|v_{m,1}\|^2_{L_2(\Omega_m)} + \|v_{m,11}\|^2_{L_2(\Omega_m)}.$$

Note that in virtue of Corollary 2 the norm $\|\cdot\|_{K_0}$ is equivalent to $|\cdot|_{K_0}$, where

$$|V|^2_{K_0} = \|v_-\|^2_{H^{2,0}(\Omega_-)} + \|v_+\|^2_{H^{2,0}(\Omega_+)} + \|v_{m,11}\|^2_{L_2(\Omega_m)}.$$

Theorem 2. *Let $U^\varepsilon = (u_-^\varepsilon, u_+^\varepsilon, u_m^\varepsilon)$ be the solution of* (6). *Let $U = (u_-, u_+, u_m) \in K_0$ be a solution the following variational equality:*

$$(20)\quad b_-(u_-, v_-) + b_+(u_+, v_+) + \mu_m \int_{\Omega_m} u_{m,11}v_{m,11}\,dz = \int_{\Omega_-} f_-v_-\,dz + \int_{\Omega_+} f_+v_+\,dz$$

for all $V \in K_0$. Then

$$U^\varepsilon \to U \quad \textit{weakly in} \quad K_0.$$

Proof. Let $V \in K$ be an arbitrary function. Due to the Lemma 2 for $\varepsilon > 0$ there exists $V^\varepsilon \in K^\varepsilon$ strongly convergences to V in $H^{2,0}(\Omega_-) \times H^{2,0}(\Omega_+) \times H^2(\Omega_m)$. Substitute V^ε in the variational equality (6) and pass to the limit as $\varepsilon \to 0$. Taking into account (16) and the fact that K is dense in K_0, we get (20). □

Now we reformulate problem (20) in an equivalent form, which does not contain the function u_m and find the representation formula for u_m. Namely, the following theorem holds.

Theorem 3. *Let $U = (u_-, u_+, u_m)$ be a solution of (20). Then (u_-, u_+) is a solution to the following variational problem:*

$$(21)\quad b_-(u_-, v_-) + b_+(u_+, v_+) + \frac{3\mu_m}{2d^3}\int\limits_S (u_+ - u_-)(v_+ - v_-)\, ds = \int\limits_{\Omega_-} f_- v_-\, dz + \int\limits_{\Omega_+} f_+ v_+\, dz$$

for all $(u_-, u_+) \in H^{2,0}(\Omega_-) \times H^{2,0}(\Omega_+)$. Moreover, the following formula

$$(22)\quad u_m(z_1, z_2) = -\frac{u_+(0, z_2) - u_-(0, z_2)}{4d^3} z_1^3 + \frac{3(u_+(0, z_2) - u_-(0, z_2))}{4d} z_1 + \frac{u_+(0, z_2) + u_-(0, z_2)}{2} \quad a.e. \text{ in } \Omega_m$$

holds.

Proof. Since $U \in K_0$, due to the Theorem 1 for any $v \in K_0$ we have

$$\int\limits_{\Omega_m} u_{m,11} v_{m,11}\, dz = \frac{3}{2d^3}\int\limits_S (u_+(0, z_2) - u_-(0, z_2))(v_+(0, z_2) - v_-(0, z_2))\, dz_2.$$

The formula (22) it follows from Theorem 1 and properties (17) and (18). □

Assuming that the solution (u_-, u_+) of (21) has additional regularity, by applying the generalized Green formula, we are about to deduce the following differential equations and boundary conditions for the functions u_- and u_+:

$$(23)\quad \begin{aligned} \mu_\pm \Delta^2 u_\pm = f_\pm \quad &\text{in} \quad \Omega_\pm, \\ u_\pm = \frac{\partial u_\pm}{\partial n} = 0 \quad &\text{on} \quad \Gamma_{D\pm}, \\ m(u_\pm) = t(u_\pm) = 0 \quad &\text{on} \quad \Gamma_{N\pm}, \\ m(u_\pm) = 0, \quad t(u_\pm) + \frac{3\mu_m}{2d^3}(u_+ - u_-) = 0 \quad &\text{on} \quad S, \end{aligned}$$

where

$$m(u_\pm) = \mu_\pm\Big(k_\pm \Delta u_\pm + (1 - k_\pm)\frac{\partial^2 u_\pm}{\partial \nu^2}\Big) \text{ and } t(u_\pm) = \mu_\pm \frac{\partial}{\partial \nu}\Big(\Delta u_\pm + (1 - k_\pm)\frac{\partial^2 u_\pm}{\partial \tau^2}\Big).$$

are bending moment and transverse force, respectively.

Note that the last condition in (23) is an analog of spring type interface condition which is widely used in elasticity and means that the transverse forces on the common interface of adherents are proportional to the jump of deflections. This explains the choice of the dependence of the elastic properties of the adhesive layer on ε as ε^3.

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