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LOWER BOUND OF THE SUPREMUM OF ERGODIC AVERAGES FOR $\mathbb{Z}^d$ AND $\mathbb{R}^d$-ACTIONS

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Abstract. For ergodic $\mathbb{Z}^d$ and $\mathbb{R}^d$-actions, we obtain a pointwise lower bound for the supremum of ergodic averages. For $\mathbb{Z}^d$-actions in the case when the supremum is taken over multi-indices exceeding $\vec{n}$ located in a certain sector, the resulting inequality is not improvable over $\vec{n}$ in the class of all averaging integrable functions.

Keywords: rates of convergence in ergodic theorems, individual ergodic theorem, Wiener–Wintner ergodic theorem

1. Introduction

Let $(\Omega, \mathfrak{F}, \lambda)$ be a probability measure space and $T : \Omega \to \Omega$ be a λ-preserving transformation. Given $f \in L_1(\Omega, \mathbb{C})$, $\omega \in \Omega$, and $n \geq 1$, we put

$$A_n^T f(\omega) = \frac{1}{n} \sum_{k=0}^{n-1} f(T^k \omega).$$

The individual Birkhoff ergodic theorem asserts that the limit $f^* = \lim_{n\to\infty} A_n^T f$ exists λ-a.e. and $\mathbb{E}f^* = \mathbb{E}f$. If the transformation T is ergodic (i.e., the only T-invariant sets are those of λ-measure 0 or 1), then $f^* \equiv \mathbb{E}f$ λ-a.e.

In what follows, we assume that the function f has spatial mean zero, i.e., $\mathbb{E}f = 0$; we denote the subspace of $L_1(\Omega, \mathbb{C})$ consisting of such functions by $L_1^0(\Omega, \mathbb{C})$.

The following theorem was proved in [1]:

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Theorem 1. *If* $f \in L_1^0(\Omega, \mathbb{C})$, $f \not\equiv 0$, *and* T *is an ergodic automorphism of* $(\Omega, \mathfrak{F}, \lambda)$, *then, for a.e.* $\omega \in \Omega$, *there exists a constant* $c = c(\omega) > 0$ *such that*

$$\mathcal{A}_n^T f(\omega) = \sup_{k \geq n} |A_k^T f(\omega)| \geq \frac{c(\omega)}{n} \tag{1}$$

for all $n \geq 1$.

This result was used as a key ingredient for the proof of the zero-one law for the rate of convergence of ergodic averages, i.e., that the $\mathfrak{F}$-measurable sets

$$E_\varphi(f, T) = \{\omega \in \Omega : \ A_n^T f(\omega) = o(\varphi(n)) \ \text{ as } \ n \to \infty\},$$

$$F_\varphi(f, T) = \{\omega \in \Omega : \ A_n^T f(\omega) = \mathcal{O}(\varphi(n)) \ \text{ as } \ n \to \infty\}$$

have full or null measure for each monotonic function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, tending to zero at infinity.

In this article, we consider several variations of inequality (1). We generalize it for $\mathbb{Z}^d$-actions considering nonergodic case and using weighted averages (Theorem 3); we also present the corresponding counterpart for an ergodic $\mathbb{R}^d$-actions (Theorem 4). The proofs of these results are based on some property of the returning nets (Theorem 2), which is of independent interest.

It is worth noting that the study of the pointwise asymptotic behavior of ergodic averages for the groups $\mathbb{Z}^d$ and $\mathbb{R}^d$ follows one of two possible approaches (see [1]) to measuring the rate of convergence of such ergodic means. The second approach, related to the large deviations theory [2] for such averages, was discussed in [3].

2. The Frequency Return Lemma

The proof of Theorem 1 relies on the following statement that, which we give in the generality of interest to us and will be used further. In particular, in what follows, a probability measure space is a Lebesgue space (see [4, 5] for example).

Lemma 1. *Let* T *be an automorphism of a Lebesgue space* $(\Omega, \mathfrak{F}, \lambda)$ *and* $\lambda(A) > 0$. *Then, for a.e.* $\omega \in \bigcup_{k \in \mathbb{Z}} T^k A$, *there exists a natural number* $L = L(T, A, \omega)$ *satisfying the following condition: for all* $n \in \mathbb{N}$ *there exists a natural number* $k_n = k_n(T, A, \omega)$ *belonging to the interval* $[n, Ln)$ *such that* $T^{k_n}\omega \in A$.

Proof. Assume first that T is an ergodic automorphism. Then $\bigcup_{k \in \mathbb{Z}} T^k A = \Omega$. The Kac lemma (see [4, chapter 1, §5] for example) and the Birkhoff ergodic theorem together imply that for a.e. $\omega \in A$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} r_A(F^j \omega) = \lambda(A)^{-1},$$

where $F = T^{r_A(\omega)} : A \to A$ and $r_A(\omega) = \inf\{k \geq 1 : \ T^k \omega \in A\}$.

For a.e. $\omega \in A$, put

$$L = L(T, A, \omega) = \left\lfloor \sup_{n \geq 1} \frac{1}{n} \sum_{j=0}^{n-1} r_A(F^j \omega) \right\rfloor + 1.$$

It is clear that $L > \lambda(A)^{-1}$ and

$$k_n(T, A, \omega) := \sum_{j=0}^{n-1} r_A(F^j \omega) < Ln$$

for $n \geq 1$. The ergodicity of an automorphism T yields that a.e. $\omega \in \Omega$ hits in A under the action of T, i.e.,

$$\tilde{r}_A(\omega) = \inf\{k \geq 0 : \ T^k\omega \in A\} < \infty.$$

Then, for such a.e. $\omega \in \Omega \setminus A$, put $\tilde{\omega} = T^{\tilde{r}_A(\omega)}\omega$, and define L and k_n by $L(T, A, \tilde{\omega})$ and $k_n(\tilde{\omega})$ consequently.

Now, consider the general nonergodic case. We will use the Rokhlin theory of measurable partitions (see [5, Part IV] for example). Recall some facts from this theory. Let ξ_T be the measurable partitions of Ω into the ergodic components of T; denote by $\xi_T(\omega)$ the element of the partition ξ_T containing ω. For a.e. $\omega \in \Omega$ there exists a probability measure $\lambda_{\xi_T(\omega)}$ with support in $\xi_T(\omega)$, and the following disintegration formula holds $\lambda(A) = \int_\Omega \lambda_{\xi_T(\omega)}(A)\, d\lambda(\omega)$ for any $A \in \mathfrak{F}$. The component $\xi_T(\omega)$ is a T-invariant set, i.e., $T^{-1}(\xi_T(\omega)) = \xi_T(\omega) = \xi_T(T\omega)$, and $(\xi_T(\omega), \lambda_{\xi_T(\omega)}; T)$ is an ergodic dynamical system for a.e. $\omega \in \Omega$.

We show that $\lambda_{\xi_T(\omega)}(A) > 0$ for a.e. $\omega \in \bigcup_{k\in\mathbb{Z}} T^k A$. Put

$$E = \left\{ \omega \in \bigcup_{k\in\mathbb{Z}} T^k A : \lambda_{\xi_T(\omega)}(A) = 0 \right\}.$$

It is clear that E is a T-invariant set and

$$\lambda(E) = \int_\Omega \lambda_{\xi_T(\omega)}(E)\, d\lambda(\omega) = \int_E \lambda_{\xi_T(\omega)}(E)\, d\lambda(\omega) + \int_{\Omega\setminus E} \lambda_{\xi_T(\omega)}(E)\, d\lambda(\omega).$$

The second term equals zero since $\Omega \setminus E$ is T-invariant and the support of measure $\lambda_{\xi_T(\omega)}$ is included in $\Omega \setminus E$ for a.e. $\omega \in \Omega \setminus E$.

The first term is also zero:

$$\begin{aligned}
\int_E \lambda_{\xi_T(\omega)}(E)\, d\lambda(\omega) &= \int_E \lambda_{\xi_T(\omega)} \left(\bigcup_{k\in\mathbb{Z}} T^k A \bigcap E \right) d\lambda(\omega) \\
&= \int_E \lambda_{\xi_T(\omega)} \left(\bigcup_{k\in\mathbb{Z}} T^k (A \cap E) \right) d\lambda(\omega) \leq \sum_{k\in\mathbb{Z}} \int_E \lambda_{\xi_T(\omega)}(T^k(A\cap E))\, d\lambda(\omega) \\
&= \sum_{k\in\mathbb{Z}} \int_E \lambda_{\xi_T(\omega)}(A \cap E)\, d\lambda(\omega) \leq \sum_{k\in\mathbb{Z}} \int_E \lambda_{\xi_T(\omega)}(A)\, d\lambda(\omega) = 0.
\end{aligned}$$

Taking into account the statement proved for an ergodic dynamical system $(\xi_T(\omega), \lambda_{\xi_T(\omega)}; T)$ with the set A of positive $\lambda_{\xi_T(\omega)}$-measure, we obtain that the following constant are defined for a.e. $\omega \in \bigcup_{k\in\mathbb{Z}} T^k A$:

$$\begin{aligned}
L(T, A, \omega) &:= L(T\,|_{\xi_T(\omega)}, A \cap \xi_T(\omega), \omega) > \lambda^{-1}_{\xi_T(\omega)}(A), \\
k_n(T, A, \omega) &:= k_n(T\,|_{\xi_T(\omega)}, A \cap \xi_T(\omega), \omega),
\end{aligned}$$

which are desired natural numbers. The proof is complete. □

Remark 1. *For ergodic dynamical systems, it is well known (see* [6] *for example) that for all* $A \in \mathfrak{F}$ *with* $\lambda(A) > 0$ *the set*

$$N_A = \{n \in \mathbb{N} : \lambda(A \cap T^{-n}A) > 0\}$$

is syndetic, i.e., there exists a number $M = M(A) \geq 1$ *such that* $N_A \cap [n, n+M] \neq \emptyset$ *for all* $n \in \mathbb{N}$. *It is easy to see that the set* N_A *possesses a similar multiplicative* *property: there exists a minimal natural number* $\tilde{M} = \tilde{M}(A) > 1$ *such that for all* $n \in \mathbb{N}$ *there is a number* $r_n \in N_A \cap [n, n\tilde{M}]$*. Then*

$$L_0 := \operatorname*{ess\,inf}_{\omega \in A} L(T, A, \omega) \geq \tilde{M}. \tag{2}$$

Indeed, put $B = \{\omega \in A : L(T, A, \omega) = L_0\}$, then the Lemma 1 yields the following: for a.e. $\omega \in B$ there exists a natural number $k_n = k_n(\omega)$ such that $k_n \in [n, nL_0)$ and $T^{k_n}\omega \in A$.

Since $\lambda(B) > 0$ and B is a disjoint finite union of sets $B_k = \{\omega \in B : k_n(\omega) = k\}$, $k = 1, \ldots, (L_0 - 1)n$, there exists $r_n \in [n, nL_0)$, $n \geq 1$ such that $\lambda(B_{r_n}) > 0$. It follows that $r_n \in N_A$, since

$$\lambda(A \cap T^{-r_n}A) > \lambda(B \cap T^{-r_n}A) = \lambda(B_{r_n}) > 0.$$

Due to minimality of $\tilde{M}$, we obtain (2).

3. Returning Nets

The main result of the section, which deals with some property of returning sequences (and, more generally, for $\mathbb{Z}^d$-nets), appeared during fruitful discussions with V.V. Ryzhikov and A.G. Kachurovskii. Consider the following definition:

Definition 1. *Let* $\mathcal{T}$ *be a semigroup filtered by a partial order* $\leq$ *and let* $\mathcal{L} : \mathcal{T} \to \mathcal{T}$ *be a map such that*

(1) $t < \mathcal{L}(t)$ *for all* $t \in \mathcal{T}$*,*

(2) $\mathcal{L}(t) \leq \mathcal{L}(s)$ *as* $t \leq s$*.*

A net $\{x_t\}_{t\in\mathcal{T}}$ *of complex numbers is called* $(a, \mathcal{L})$*-returning for some* $a > 0$ *if for any* $t \in \mathcal{T}$ *there exists* $s_t \in \mathcal{T}$ *such that* $t \leq s_t < \mathcal{L}(t)$ *and* $|x_{s_t}| \geq a$*.*

As the main example of a semigroup $\mathcal{T}$, we will consider the set $\mathbb{Z}^d, d \geq 1$, (and also $\mathbb{R}^d$) with the following partial order

$$\vec{n} \leq \vec{m} \;\Leftrightarrow\; n_j \leq m_j,\ j = 1, 2, \ldots, d.$$

The strict inequality is defined in an analogous natural way. Denote the standard basis of $\mathbb{R}^d$ by $\{\vec{e}_j\}_{j=1}^d$ and put $\vec{e}_0 = \sum_{j=1}^d \vec{e}_j$, and

$$\pi(\vec{n}) = n_1 n_2 \cdots n_d$$

for $\vec{n} = (n_1, \ldots, n_d) = \sum_{j=1}^n n_j \vec{e}_j$.

The following result contains some important properties of the returning nets.

Theorem 2. *The following statements hold:*

(1) Suppose that a net $\{x_t\}_{t\in\mathcal{T}}$ *is* $(a, \mathcal{L})$*-returning and a net* $\{y_t\}_{t\in\mathcal{T}}$ *is separated from zero, i.e.,* $|y_t| \geq \delta$ *for some* $\delta > 0$ *and all* $t \in \mathcal{T}$*. Then the product net* $\{x_t y_t\}_{t\in\mathcal{T}}$ *is* $(a\delta, \mathcal{L})$*-returning.*

(2) Let the net $\{x_{\vec{n}}\}_{\vec{n}\in\mathbb{Z}^d}$ *be* $(a, \mathcal{L})$*-returning. Then*

$$\sup_{\vec{m} \geq \vec{n}} \left| \frac{1}{\pi(\vec{m})} \sum_{0 \leq \vec{k} < \vec{m}} x_{\vec{k}} \right| \geq \frac{a}{2^d \pi(\mathcal{L}(\vec{n}))} \tag{3}$$

for all $\vec{n} \geq \vec{e}_0$*.*

Proof. (1) It is evident that for any $t \in \mathcal{T}$ there exists $s_t \in \mathcal{T}$ such that $t \le s_t < \mathcal{L}(t)$ and $|x_{s_t} y_{s_t}| \ge a\delta$.

(2) Put

$$S_{\vec{n}} = \sum_{0 \le \vec{k} < \vec{n}} x_{\vec{k}},$$

and show that

$$S_{\vec{n}} = x_{\vec{n}-\vec{e}_0} + \sum_{m=1}^{d} \sum_{1 \le j_1 < j_2 < \ldots < j_m \le d} (-1)^{m+1} S_{\vec{n}-\vec{e}_{j_1}-\vec{e}_{j_2}-\ldots-\vec{e}_{j_m}} \tag{4}$$

for each multi-index $\vec{n} = (n_1, n_2, \ldots, n_d) \ge 2\vec{e}_0, d \ge 1$.

To this end, we consider the finitely additive function τ defined on the algebra of finite subsets of $\mathbb{Z}^d$ by

$$\tau(J) = \sum_{\vec{j} \in J} x_{\vec{j}}.$$

It is easy to see that $S_{\vec{n}} = \tau([0, \vec{n}))$, and

$$[0, \vec{n}) = \{\vec{n} - \vec{e}_0\} \bigcup \left(\bigcup_{k=1}^{d} [0, \vec{n} - \vec{e}_k) \right).$$

Using the well-known formula for the measure of a finite union of sets (see [7, 1.12.51] for example), we obtain

$$\begin{aligned}
\tau([0, \vec{n})) &= \tau(\{\vec{n} - \vec{e}_0\}) + \tau\left(\bigcup_{k=1}^{d} [0, \vec{n} - \vec{e}_k) \right) \\
&= \tau(\{\vec{n} - \vec{e}_0\}) + \sum_{k=1}^{d} \tau([0, \vec{n} - \vec{e}_k)) - \sum_{0 \le k < l \le d} \tau([0, \vec{n} - \vec{e}_k) \cap [0, \vec{n} - \vec{e}_l)) \\
&+ \sum_{0 \le k < l < r \le d} \tau([0, \vec{n} - \vec{e}_k) \cap [0, \vec{n} - \vec{e}_l) \cap [0, \vec{n} - \vec{e}_r)) - \ldots + (-1)^{d+1} \tau\left(\bigcap_{k=1}^{d} [0, \vec{n} - \vec{e}_k) \right) \\
&= \tau(\{\vec{n} - \vec{e}_0\}) + \sum_{k=1}^{d} \tau([0, \vec{n} - \vec{e}_k)) - \sum_{0 \le k < l \le d} \tau([0, \vec{n} - \vec{e}_k - \vec{e}_l)) \\
&\quad + \sum_{0 \le k < l < r \le d} \tau([0, \vec{n} - \vec{e}_k - \vec{e}_l - \vec{e}_r)) - \ldots + (-1)^{d+1} \tau([0, \vec{n} - \vec{e}_0)).
\end{aligned}$$

Renaming indices and combining quantities, we get

$$\tau([0, \vec{n})) = \tau(\{\vec{n} - \vec{e}_0\}) + \sum_{m=1}^{d} \sum_{1 \le j_1 < j_2 < \ldots < j_m \le d} (-1)^{m+1} \tau([0, \vec{n} - \vec{e}_{j_1} - \vec{e}_{j_2} - \ldots - \vec{e}_{j_m})).$$

This is the formula (4).

Now, suppose to the contrary, that (3) fails, i.e. there exists $\vec{n} \ge \vec{e}_0$ such that

$$\sup_{\vec{m} \ge \vec{n}} \left| \frac{1}{\pi(\vec{m})} \sum_{0 \le \vec{k} < \vec{m}} x_{\vec{k}} \right| < \frac{a}{2^d \pi(\mathcal{L}(\vec{n}))}.$$

Take $\vec{v}$ from the definition of an $(a, \mathcal{L})$-returning net (i.e. $\vec{n} \leq \vec{v} < \mathcal{L}(\vec{n})$ and $|x_{\vec{v}}| \geq a$) and put $\mathcal{A}_{\vec{m}} := \sup\limits_{\vec{p} \geq \vec{m}} \frac{1}{\pi(\vec{p})} \left| \sum\limits_{0 \leq \vec{k} < \vec{p}} x_{\vec{k}} \right|$. Then we have

$$\mathcal{A}_{\vec{v}+\vec{e}_0} \leq \mathcal{A}_{\vec{v}+\vec{e}_0-\vec{e}_{j_1}-\vec{e}_{j_2}-\ldots-\vec{e}_{j_m}} \leq \mathcal{A}_{\vec{n}} < \frac{a}{2^d \pi(\mathcal{L}(\vec{n}))}, \tag{5}$$

since $\vec{v} + \vec{e}_0 \geq \vec{v} + \vec{e}_0 - \vec{e}_{j_1} - \vec{e}_{j_2} - \ldots - \vec{e}_{j_m} \geq \vec{n}$. On the other hand, equality (4) yields (we may use it since $\vec{v} + \vec{e}_0 \geq 2\vec{e}_0$)

$$|S_{\vec{v}+\vec{e}_0}| \geq |x_{\vec{v}}| - \sum_{m=1}^{d} \sum_{1 \leq j_1 < j_2 < \ldots < j_m \leq d} |S_{\vec{v}+\vec{e}_0-\vec{e}_{j_1}-\vec{e}_{j_2}-\ldots-\vec{e}_{j_m}}|.$$

Taking into account (5) and the inequality $|x_{\vec{v}}| \geq a$, we get

$$\begin{aligned} \mathcal{A}_{\vec{v}+\vec{e}_0} &\geq \frac{|x_{\vec{v}}|}{\pi(\vec{v}+\vec{e}_0)} \\ &- \sum_{m=1}^{d} \sum_{1 \leq j_1 < j_2 < \ldots < j_m \leq d} \frac{\pi(\vec{v}+\vec{e}_0-\vec{e}_{j_1}-\vec{e}_{j_2}-\ldots-\vec{e}_{j_m})}{\pi(\vec{v}+\vec{e}_0)} \mathcal{A}_{\vec{v}+\vec{e}_0-\vec{e}_{j_1}-\vec{e}_{j_2}-\ldots-\vec{e}_{j_m}} \\ &\geq \frac{a}{\pi(\vec{v}+\vec{e}_0)} - \sum_{m=1}^{d} \sum_{1 \leq j_1 < j_2 < \ldots < j_m \leq d} \frac{a}{2^d \pi(\mathcal{L}(\vec{n}))} = \frac{a}{\pi(\vec{v}+\vec{e}_0)} - \frac{a(2^d-1)}{2^d \pi(\mathcal{L}(\vec{n}))}. \end{aligned}$$

Since $\pi(\vec{v}+\vec{e}_0) \leq \pi(\mathcal{L}(\vec{n}))$, it follows that

$$\mathcal{A}_{\vec{v}+\vec{e}_0} \geq \frac{a}{\pi(\mathcal{L}(\vec{n}))} - \frac{a(2^d-1)}{2^d \pi(\mathcal{L}(\vec{n}))} = \frac{a}{2^d \pi(\mathcal{L}(\vec{n}))}.$$

This contradicts (5). Therefore, the assumption was false, and (3) holds. □

4. An Inequality for $\mathbb{Z}^d$-actions

The consideration of the d-dimensional analog of inequality (1) raises the question of what exactly is necessary to require for a $\mathbb{Z}^d$-action: its ergodicity, ergodicity of at least one element, or of all elements? Or is an analogue of such an inequality possible in the nonergodic case? In the 1-dimensional case, the ergodicity of a $\mathbb{Z}$-action is equivalent to the ergodicity of the automorphism T; in the multidimensional case, examples of ergodic actions are known that do not have a single ergodic element (see [8] for example). On the other hand, all elements of the generic $\mathbb{Z}^d$-action are ergodic (moreover, they are even weakly mixing) [9, 10].

The ergodic averages for the group of automorphisms $T^{\vec{n}} : \Omega \to \Omega, \vec{n} \in \mathbb{Z}^d$ are defined in a natural way. Given $f \in L_1(\Omega, \mathbb{C})$, $\omega \in \Omega$, and a multi-index $\vec{n} = (n_1, \ldots, n_d)$, we put

$$A_{\vec{n}} f(\omega) = \frac{1}{\pi(\vec{n})} \sum_{\vec{j} \in [0, \vec{n})} f(T^{\vec{j}} \omega).$$

The individual ergodic theorem for $\mathbb{Z}^d$-actions also hold, but with its own constrains: either on the function f, or on the process of tending to infinity for multi-indices $\vec{n}$; a detailed discussion can be found in the monograph [11, Chapter 6].

Consider the sets

$$\mathcal{M}_f = \bigcup_{\substack{a \in (0, \operatorname{ess\,sup}|f|) \\ a \in \mathbb{Q}}} \bigcup_{\vec{n} \geq 0} T^{\vec{n}} \{\omega \in \Omega : |f(\omega)| \geq a\},$$

$$\mathcal{N}_f = \Omega \setminus \mathcal{M}_f = \bigcap_{\vec{n} \geq 0} T^{\vec{n}} \{\omega \in \Omega : f(\omega) = 0\}.$$

It is clear that $A_{\vec{n}} f(\omega) = 0$ for all $\vec{n} \geq \vec{e}_0$ and $\omega \in \mathcal{N}_f$.

The following theorem is a generalization of Theorem 1 for $\mathbb{Z}^d$-actions. Here we do not assume ergodicity and consider nonconventional averages.

Theorem 3. *Let $f \in L_1^0(\Omega, \mathbb{C})$, $f \not\equiv 0$ and $\{T^{\vec{n}}\}_{\vec{n} \in \mathbb{Z}^d}$ be a group of automorphisms of a Lebesgue space $(\Omega, \mathfrak{F}, \lambda)$. Let $g_{t,\vec{n}}$ be a family of functions uniformly separated from zero. Then, for a.e. $\omega \in \mathcal{M}_f$, there exists a positive number $c = c(\omega)$ such that*

$$\sup_t \sup_{\vec{m} \geq \vec{n}} \left| \frac{1}{\pi(\vec{m})} \sum_{0 \leq \vec{k} < \vec{m}} f(T^{\vec{k}} \omega) g_{t,\vec{k}}(\omega) \right| \geq \frac{c(\omega)}{\left(\max\limits_{1 \leq i \leq d} n_i \right)^d} \tag{6}$$

for all $\vec{n} \geq \vec{e}_0$.

Proof. Take $\omega \in \mathcal{M}_f$, then there are $a > 0$ and $\vec{\ell} \geq 0$ such that $\omega \in A := T^{\vec{\ell}} \{|f| \geq a\}$. It is clear that $\omega \in \bigcup_{k \in \mathbb{Z}} T^{k(\vec{e}_0 + \vec{\ell})} A$. Lemma 1 defines a number $L(T^{\vec{e}_0 + \vec{\ell}}, A, \omega)$ such that for all $N \in \mathbb{N}$ there exists a natural number k in the interval $[N, LN)$ with $T^{k(\vec{e}_0 + \vec{\ell})} \omega \in A$. Hence, $|f(T^{k(\vec{e}_0 + \vec{\ell}) - \vec{\ell}} \omega)| \geq a$.

For an arbitrary multi-index $\vec{n} \geq \vec{e}_0$, put

$$N = \max_{1 \leq i \leq d} \frac{n_i + \ell_i}{\ell_i + 1}.$$

Defining $\vec{v} = k(\vec{e}_0 + \vec{\ell}) - \vec{\ell}$, we obtain

$$\vec{n} \leq N(\vec{e}_0 + \vec{\ell}) - \vec{\ell} \leq \vec{v} < LN(\vec{e}_0 + \vec{\ell}) - \vec{\ell}.$$

It follows that the net $x_{\vec{n}} := f(T^{\vec{n}} \omega)$ is $(a, \mathcal{L})$-returning, where

$$\mathcal{L}(\vec{n}) = LN(\vec{e}_0 + \vec{\ell}) - \vec{\ell}.$$

Since there is such $\delta > 0$, that $|g_{t,\vec{n}}| \geq \delta$, we obtain due to Theorem 2 that the net $f(T^{\vec{n}} \omega) g_{t,\vec{n}}(\omega)$ is $(a\delta, \mathcal{L})$-returning for all t. Therefore, inequality (3) holds for this net.

It remains to note that

$$\begin{aligned} \pi(\mathcal{L}(\vec{n})) &= \pi(LN(\vec{e}_0 + \vec{\ell}) - \vec{\ell}) \leq \pi(LN(\vec{e}_0 + \vec{\ell})) \\ &\leq L^d \left(\max_{1 \leq i \leq d} \frac{n_i + \ell_i}{\ell_i + 1} \right)^d \pi(\vec{e}_0 + \vec{\ell}) \leq L^d (\max_{1 \leq i \leq d} n_i)^d \pi(\vec{e}_0 + \vec{\ell}). \end{aligned}$$

Taking into account the obtained calculations, the constant $c(\omega)$ in (6) can be taken equal to $\frac{a\delta}{2^d L^d \pi(\vec{e}_0 + \vec{\ell})}$. □

Remark 2. *The set of functions $f \in L_1^0(\Omega, \mathbb{C})$, for which $\lambda(\mathcal{N}_f) = 0$, is dense in $L_1^0(\Omega, \mathbb{C})$ since it contains the functions separated from zero. In the ergodic case, $\lambda(\mathcal{N}_f) = 0$ for every integrable function f.*

Remark 3. *Considering the ergodic case in Theorem 3 when $d = 1$ and taking functions $g_{t,k} = e^{2\pi itk}$, we obtain a lower pointwise bound for the supremum of averages appearing in Bourgain's uniform Wiener–Wintner ergodic theorem. It states (see* [12, 2.4.2] *for example) that for ergodic dynamical system and any $f \in L_2(\Omega)$ with continuous spectral measure σ_f*

$$\lim_{n\to\infty} \sup_t \left| \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) e^{2\pi ikt} \right| = 0 \quad a.e.$$

Remark 4. *Consider the multi-indices $\vec{n}$ which remain in a sector (of $\mathbb{N}^d$)* [11, §6.2]*, i.e., there exists a constant $\mathcal{C} > 0$ such that*

$$n_i \le \mathcal{C} n_j,$$

for all $1 \le i, j \le d$. Then, in (6)*, one can apply the estimate*

$$\left(\max_{1\le i\le d} n_i \right)^d \le \mathcal{C}^{d-1} \pi(\vec{n}).$$

Remark 5. *Consider the multi-indices $\vec{n}$ which remain in a sector (of $\mathbb{N}^d$). Then the inequality* (6) *(without weights) cannot be improvable in the class of all averaging integrable functions.*

Indeed, given $g \in L_\infty(\Omega)$, we define inductively the functions $f_k, k = 1, ..., d$:

$$f_1(\omega) = g(T^{\vec{e}_1}\omega) - g(\omega), \quad f_k(\omega) = f_{k-1}(T^{\vec{e_k}}\omega) - f_{k-1}(\omega).$$

We show by induction on d that

$$|A_{\vec{n}} f_d(\omega)| \le \frac{2^d \|g\|_\infty}{\pi(\vec{n})}.$$

For $d = 1$ this is a well-known and easy result. Assuming that the inequality is true for $d - 1$, we make the induction step. We have

$$\begin{aligned} |A_{\vec{n}} f_d(\omega)| &= |A_{n_1}^{T^{\vec{e}_1}} A_{n_2}^{T^{\vec{e}_2}} \cdots A_{n_d}^{T^{\vec{e}_d}} f_d(\omega)| \\ &= \frac{1}{n_d} |A_{n_1}^{T^{\vec{e}_1}} A_{n_2}^{T^{\vec{e}_2}} \cdots A_{n_{d-1}}^{T^{\vec{e}_{d-1}}} (f_{d-1}(T^{n_d \vec{e}_d}\omega) - f_{d-1}(\omega))| \\ &\le \frac{1}{n_d} \left(\frac{2^{d-1}\|g\|_\infty}{n_1 \cdots n_{d-1}} + \frac{2^{d-1}\|g\|_\infty}{n_1 \cdots n_{d-1}} \right) = \frac{2^d \|g\|_\infty}{\pi(\vec{n})}. \end{aligned}$$

Taking into account Remark 4, we see that

$$\frac{2^d \|g\|_\infty}{\pi(\vec{n})} \ge \sup_{\vec{m}\ge\vec{n}} \left| \frac{1}{\pi(\vec{m})} \sum_{0\le\vec{k}<\vec{m}} f_d(T^{\vec{k}}\omega) \right| \ge \frac{c(\omega)}{\mathcal{C}^{d-1}\pi(\vec{n})}$$

for all multi-indices $\vec{n}$ remaining in a sector.

5. An Inequality for the Ergodic $\mathbb{R}^d$-flow

To get a counterpart of the inequality (1) for ergodic $\mathbb{R}^d$-actions one can use the standard approach for transition from discrete to continuous time (as regards transition to continuous time in ergodic theorems, the reader is also referred to [13]).

The ergodic averages for $\mathbb{R}^d$-action are defined as

$$A_{\vec{t}} f(\omega) = \frac{1}{\pi(\vec{t})} \int_{[0,\vec{t})} f(T^{\vec{\tau}}\omega) d\vec{\tau}.$$

Here $[0,\vec{t}) := \{\vec{s} \in \mathbb{R}^d :\ 0 \le \vec{s} < \vec{t}\}$ and $d\vec{\tau} = d\tau_1 \dots d\tau_d$. Consider also the multiplication in $\mathbb{R}^d$ defined by

$$\vec{t} \odot \vec{s} = (t_1 s_1, t_2 s_2, \dots, t_d s_d).$$

We need the following technical lemma:

Lemma 2. *Let $f \in L_1(\Omega, \mathbb{C})$, $f \not\equiv 0$ and $\{T^{\vec{t}}\}_{\vec{t}\in\mathbb{R}^d}$ be an $\mathbb{R}^d$-flow on a probability measure space $(\Omega, \mathfrak{F}, \lambda)$. Suppose that $\mathcal{V} \subset (0,1)$ and zero is a limit point of the set $\mathcal{V}$. Then there exists a vector $\vec{p} \in \mathcal{V}^d$ such that*

$$\int_{[0,\vec{p})} f(T^{\vec{t}}\omega)\, d\vec{t} \ne 0$$

for a.e. $\omega \in \Omega$.

Proof. We again apply induction on d. Check the induction base $d = 1$. Assume to the contrary that there is no such a point $p \in \mathcal{V}$. Then, for any sequence $\{p_n\}_{n\in\mathbb{N}} \subseteq \mathcal{V}$ with $p_n \to 0$ as $n \to \infty$, we have $\int_0^{p_n} f(T^t\omega)\, dt = 0$ a.e. It follows due to Wiener's local ergodic theorem [11, §1.2] that

$$f(\omega) = \lim_{n\to\infty} \frac{1}{p_n} \int_0^{p_n} f(T^\tau \omega)\, d\tau = 0 \quad \text{a.e.}$$

This equality contradicts the condition $f \not\equiv 0$. (For comparison, observe in brackets that, for real-valued function f and for all ω from a set of full measure, there exists (see [15, 14] an infinite sequence $t_n = t_n(\omega)$ such that $\int_0^{t_n} f(T^t\omega)dt = 0$).

Applying the induction steps to f, we find a vector $(p_1, \dots, p_{d-1}) \in \mathcal{V}^{d-1}$ such that a.e.

$$0 \ne \int_0^{p_1}\int_0^{p_2} \dots \int_0^{p_{d-1}} f(T^{t_1\vec{e}_1 + t_2\vec{e}_2 + \dots t_{d-1}\vec{e}_{d-1}}\omega) dt_1 \dots dt_{d-1} := F(\omega).$$

It remains to considerthe one dimensional case again with the function F and the flow $\{T^{t_d \vec{e}_d}\}_{t_d\in\mathbb{R}}$. □

We formulate the counterpart of Theorem 3 only for ergodic flows with the usual Cesaro mean. The general case apparently does not contain any arguments different from those applied in the discrete case but entails technical problems related to the measurability of sets of type $\mathcal{M}_f$. It requires additional research.

Theorem 4. *Let $f \in L_1^0(\Omega, \mathbb{C})$, $f \not\equiv 0$ and $\{T^{\vec{t}}\}_{\vec{t}\in\mathbb{R}^d}$ be an ergodic $\mathbb{R}^d$-flow on a Lebesgue space $(\Omega, \mathfrak{F}, \lambda)$. Then there exists a vector $0 < \vec{p} < \vec{e}_0$ satisfying the following condition: for a.e. $\omega \in \Omega$ there exists a positive number $c = c(\omega)$ such that*

$$\sup_{\vec{t}\ge\vec{s}} \left| \frac{1}{\pi(\vec{t})} \int_{[0,\vec{t})} f(T^{\vec{\tau}}\omega)\, d\vec{\tau} \right| \ge \frac{c(\omega)\left(\min\limits_{1\le j\le d} p_j\right)^d}{\pi(\vec{p})\left(\max\limits_{1\le j\le d} s_j\right)^d} \tag{7}$$

for all $\vec{s} \ge \vec{p}$; and also

$$\sup_{\vec{t}\ge\vec{s}} \left| \frac{1}{\pi(\vec{t})} \int_{[0,\vec{t})} f(T^{\vec{\tau}}\omega)\, d\vec{\tau} \right| \ge \frac{c(\omega)\left(\min\limits_{1\le j\le d} p_j\right)^d}{\mathcal{C}^{d-1}\pi(\vec{p})\pi(\vec{s})} \tag{8}$$

for all $\vec{s} \ge \vec{p}$ remaining in a sector, and for some constant $\mathcal{C} > 0$.

Proof. Since the flow $\{T^{\vec t}\}_{\vec t\in\mathbb{R}^d}$ is ergodic, all the automorphisms $T^{\vec t}$ outside a set of vectors $\vec t$ containing in a countable family of hyperplanes are ergodic [16]. Then there is a set $\mathcal{V}\subset(0,1)$ with zero as a limit point such that each automorphism $T^{\vec p}, \vec p\in\mathcal{V}^d$ is ergodic. Applying Lemma 2 to $\mathcal{V}$, we conclude that $T^{\vec p}$ is ergodic for some $\vec p\in\mathcal{V}^d$ and still the function $g(\omega)=\int_{[0,\vec p]} f(T^{\vec t}\omega)\,d\vec t$ does not vanish a.e.

Now, we can use Theorem 3 with the function g and the $\mathbb{Z}^d$-action generated by the automorphisms $S_j = T^{p_j\vec e_j}$. Taking into account the inequality (6), we get

$$\begin{aligned}\sup_{\vec t\geq\vec s}\left|\frac{1}{\pi(\vec t)}\int_{[0,\vec t)} f(T^{\vec\tau}\omega)\,d\vec\tau\right| &\geq \sup_{\vec t\geq\vec s,\vec t=\vec n\odot\vec p}\left|\frac{1}{\pi(\vec t)}\int_{[0,\vec t)} f(T^{\vec\tau}\omega)\,d\vec\tau\right|\\ &= \sup_{\vec n\in\mathbb{N}^d, n_j\geq\lfloor s_j/p_j\rfloor}\left|\frac{1}{\pi(\vec n)\pi(\vec p)}\int_{[0,\vec n\odot\vec p)} f(T^{\vec\tau}\omega)\,d\vec\tau\right|\\ &= \frac{1}{\pi(\vec p)}\sup_{n_j\geq\lfloor s_j/p_j\rfloor}\left|\frac{1}{\pi(\vec n)}\sum_{\vec k\in[0,\vec n)} g(S^{\vec k}\omega)\right|\\ &\geq \frac{c(\omega)}{\pi(\vec p)\left(\max\limits_{1\leq j\leq d}\lfloor s_j/p_j\rfloor\right)^d} \geq \frac{c(\omega)\left(\min\limits_{1\leq j\leq d} p_j\right)^d}{\pi(\vec p)\left(\max\limits_{1\leq j\leq d} s_j\right)^d}\end{aligned}$$

for all $\vec s\geq\vec p$ and a.e. $\omega\in\Omega$. This completes the proof of (7). Estimate (8) is obtained from the just proved inequality (7) with the use of Remark 4. ☐

Remark 6. *In the statement of Theorem 4 for $d=1$, we can weaken the Lebesgue condition for a measure space to the condition of separability of that space (with respect to the pseudometric $d(A,B)=\lambda(A\triangle B)$ for $A,B\in\mathfrak{F}$).*

Indeed, in the proof of the case $d=1$ of Theorem 4, we can use Theorem 1 instead of Theorem 3, where the Lebesgue condition for the measure space is not required. And applying the Pugh—Shub theorem [16] requires only separability of $L_2(\Omega,\mathfrak{F},\lambda)$, which is equivalent to separability of the measure space [17, 13.4].

References

[1] A. G. Kachurovskii, I. V. Podvigin, *Measuring the rate of convergence in the Birkhoff ergodic theorem*, Mat. Zametki, **106**:1 (2019), 40–52; English transl., Math. Notes, **106**:1 (2019), 52–62. MR3981324

[2] A. G. Kachurovskii, I. V. Podvigin, *Large deviations and the rate of convergence in the Birkhoff ergodic theorem*, Mat. Zametki, **94**:4 (2013), 569–577. English transl., Math. Notes, **94**:4 (2013), 524–531. MR3423283

[3] I. V. Podvigin, *On the rate of convergence in the individual ergodic theorem for the actions of a semigroup*, Mat. Tr., **18**:2 (2015), 93–111; English transl., Siberian Adv. Math., **26**:2 (2016), 139–151. MR3588293

[4] I. P. Kornfeld, Ya. G. Sinai, S. V. Fomin, *Ergodic theory*, Nauka, Moscow, 1980; English transl., Springer, New York, 1982. MR0610981

[5] Y. Coudene, *Ergodic theory and dynamical systems*, Springer-Verlag, London, 2016. MR3586310

[6] R. Kuang, X. D. Ye, *The return times set and mixing for measure preserving transformations*, Discrete Contin. Dyn. Syst., **18**:4 (2007), 817—827. MR2318270

[7] V. I. Bogachev, *Measure theory*, Vol. I, Springer-Verlag, Berlin, 2007. MR2267655

[8] C.R.E. Raja, *On the existence of ergodic automorphisms in ergodic $\mathbb{Z}^d$-actions on compact groups*, Ergodic Theory Dyn. Syst., **30**:6 (2010), 1803–1816. MR2736896

[9] V. V. Ryzhikov, S.V̇. Tikhonov. *Generic $\mathbb{Z}^n$-action can be embedded only in injective $\mathbb{R}^n$-actions*, Mat. Zametki, **79**:6 (2006), 925—930; English transl., Math. Notes, **79**:6 (2006), 864-–868. MR2261246

[10] V. V. Ryzhikov, *Factors, rank, and embedding of a generic $\mathbb{Z}^n$-action in an $\mathbb{R}^n$-flow*, Uspekhi Mat. Nauk, **61**:4 (2006), 197–198; English transl., Russian Math. Surveys, **61**:4 (2006), 786—787. MR2278846

[11] U. Krengel, *Ergodic theorems*, Walter de Gruyter, Berlin—New York, 1985. MR0797411

[12] I. Assani, *Wiener Wintner ergodic theorems*, World Scientific, Singapore, 2003. MR1995517

[13] V. Bergelson, A. Leibman, C. G. Moreira, *From discrete-to continuous-time ergodic theorems*, Ergodic Theory Dyn. Syst., **32**:2 (2012), 383–426. MR2901353

[14] B. Marcus, K. Petersen, *Balancing ergodic averages*, (Ergodic theory, Procedengs Oberwolfach, Germany, 1978, Lectures notes in Maths., **729**, eds. M. Denker, K. Jacobs), Springer-Verlag, Berlin, 1979, 126–143. MR0550416

[15] I. Ya. Shneiberg, *Zeros of the integrals along trajectories of ergodic systems*, Funktsional. Anal. i Prilozhen., **19**:2 (1985), 92—93; English transl., Funct. Anal. Appl., **19**:2 (1985), 160–161. MR0800934

[16] C. Pugh, M. Shub, *Ergodic elements of ergodic actions*, Compos. Math., **23**:1 (1971), 115–122. MR0283174

[17] B.S. Thompson, J.B. Bruckner, A.M. Bruckner, *Elementary Real Analysis*, Second Edition, ClassicalRealAnalysis.com, 2008.

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