SOBOLEV-TYPE FUNCTIONS ON NONHOMOGENEOUS METRIC SPACES

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Abstract. We consider analogs of classical embedding theorems for function classes of Sobolev type on nonhomogeneous metric measure spaces.

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Analysis on metric structures has been actively developing since the 1990s. On metric measure spaces, various classes of functions with generalized “smoothness”, which are in a sense a generalization of the Sobolev spaces, were considered. The definitions of such spaces are based on the existence of alternative descriptions of Sobolev spaces not using the linear structure of the Euclidean space and admitting a statement in terms of the measure and metric.

The article deals with different types of an embedding theorem for Sobolev-type function classes $M^1_p(X,d,\mu)$, introduced by P. Hajlasz in [1]. The greatest analogy of the properties of the functions of the classical Sobolev spaces $W^1_p(G)$ and the functions of the spaces $M^1_p(X,d,\mu)$ is observed on homogeneous metric spaces $(X,d)$ with the measure satisfying the estimate

$$C_1 r^s \leq \mu(B(x,r)) \leq C_2 r^s,$$

for $x \in X$ and $0 < r \leq \text{diam} X$.

A quite informative theory of the spaces $M^1_p(X,d,\mu)$ is obtained in the more general situation, when the measure $\mu$ satisfies the “doubling condition”, which is weaker than (1), which implies only a bound for the measure of a ball from below. In this case, it is possible to obtain analogs of various classical results including embedding theorems, which play an exceptional role in the theory of Sobolev spaces [1, 2, 3, 4]. Moreover, the metric results and the methods used for obtaining them turn out to be useful also in studying the properties of Sobolev functions in a Euclidean space.
We will be interested in the properties of function classes of Sobolev type on nonhomogeneous metric measure spaces whose local properties depend on a point $x \in X$. In this case, it is possible to obtain embedding theorems with variable integrability exponent.

**Sobolev-Type Spaces**

In an arbitrary metric space $(X, d)$ with measure $\mu$, P. Hajlasz (see [1]) introduced the classes $M^1_p(X, d, \mu)$ of Sobolev type, the definition of which is based on a Lipschitz estimate of a special kind.

For an arbitrary $\mu$-measurable function $u : X \to \overline{\mathbb{R}}$, call a function $g : X \to [0, \infty)$ admissible if there exists a set $E \subset X$ such that $\mu(E) = 0$ and the inequality

$$|u(x) - u(y)| \leq d(x, y)(g(x) + g(y))$$

holds for all $x, y \in X \setminus E$.

Denote the set of all admissible functions for a function $u$ by $D(u)$ and, for $p \geq 1$, put $D_p(u) = D(u) \cap L_p(X, \mu)$.

The function classes $S^1_p(X, d, \mu)$ and $M^1_p(X, d, \mu)$ are defined by the conditions:

$$S^1_p(X, d, \mu) = \{u : X \to \overline{\mathbb{R}} \mid D_p(u) \neq \emptyset\},$$

$$M^1_p(X, d, \mu) = \{u \in L_p(X, \mu) \mid u \in S^1_p(X, p, \mu)\}.$$

The seminorm in $S^1_p(X, d, \mu)$ and the norm in $M^1_p(X, d, \mu)$ are defined by the equalities

$$||u||_{S^1_p} = \inf_{g \in D_p(u)} ||g||_{L_p}, \quad ||u||_{M^1_p} = ||u||_{L_p} + ||u||_{S^1_p}.$$

In Euclidean domains $G \subset \mathbb{R}^n$ with a sufficiently smooth (for instance, smooth or Lipschitz) boundary, the classical Sobolev space $W^1_p(G)$ and the space $M^1_p(G, \mu)$, considered with the standard Euclidean metric and the Lebesgue measure, coincide as sets of functions, and their norms are equivalent (see [1]).

Informative results for the spaces $M^1_p(X, d, \mu)$ can be obtained under rather natural assumptions on the relationship between the metric $d$ and the measure $\mu$. We will assume that the metric space $(X, d)$ and the finite Borel measure $\mu$ satisfies the doubling condition

$$\mu(B(x, 2r)) \leq C d\mu(B(x, r)), \quad (3)$$

i.e., the measure of the ball of doubled radius is estimated in terms of the measure of the initial ball. This simple geometric condition guarantees the fulfillment of the Vitali covering lemma and its standard consequences for the measure $\mu$. For a locally integrable function, Lebesgue's theorem on the differentiation of an integral remains valid, and, as a consequence, almost all points of the set $X$ are Lebesgue points of $u$. The property important to us is the boundedness of the Hardy–Littlewood maximal operator in the Lebesgue spaces $L_p(X, \mu)$ for $p > 1$. The doubling condition for $r \leq \text{diam} X$ implies the estimate

$$\mu(B(x, r)) \geq C_1 r^s, \quad (4)$$

where $s = \log_2 C_d$. The degree $s$ is called the regularity exponent of the measure $\mu$ with respect to the measure $d$ and plays the role of the “dimension” of the metric space $(X, d)$ in the embedding theorems.

Henceforth, we assume that the measure $\mu$ satisfies the doubling condition and is regular with exponent $s$.

Denote by $u_E$ the mean value of a function $u$ on the set $E$

$$u_E = \frac{1}{\mu(E)} \int_E u \, d\mu.$$
Define the maximal Hardy–Littlewood operator in a standard manner, by setting
\[ Mf(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |u| \, dp. \]

As was already observed,
\[ \|Mf \|_{L^p(X)} \leq C \|f \|_{L^p(X,\mu)} \]
for \( p > 1 \).

If \( 0 < \gamma \leq 1 \) then \( d_\gamma(x,y) = (d(x,y))^{\gamma} \) is again a metric. This makes it possible to introduce the Hölder function classes \( M^\gamma_p \) by replacing the estimate in the initial definition of the spaces \( M^1_p \) with
\[ |u(x) - u(y)| \leq (d(x,y))^{\gamma}(g(x) + g(y)). \]

It is easy to notice here that \( M^\gamma_p(X,d,\mu) = M^1_p(X,d_\gamma,\mu) \), i.e., the Hölder classes with respect to the initial metric can be regarded as the space with “unit smoothness” but with respect to the Hölder metric. This is often convenient since, in obtaining results for the spaces \( M^\gamma_p \), it suffices to recalculate the regularity exponent of the measure \( \mu \) with respect to the Hölder metric and use the assertion for the spaces \( M^1_p \).

In accordance with [2], the inequality
\[ |u(x) - u(y)| \leq C(d(x,y))^{\gamma}(u^\#(x) + u^\#(y)) \] (5)
holds for all the Lebesgue points of a locally integrable function \( u \), where \( 0 < \gamma \leq 1 \) and \( u^\# \) is the sharp maximal function of order \( \gamma \), defined uniquely at all points of \( X \) by the equality
\[ u^\#(x) = \sup_{r>0} r^{-\gamma} \int_{B(x,r)} |u - u_{B(x,r)}| \, d\mu. \]

For measure with doubling condition, function classes of Sobolev type admit different equivalent descriptions (see [2]).

**Lemma 1.** Let \( 1 < p \leq \infty \). The following three conditions are equivalent:

1. \( u \in S^p(X,d,\mu) \);
2. there exists a function \( h \in L^p(X,\mu) \) such that the Poincaré inequality
   \[ \int_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq r \int_{B(x,r)} h \, d\mu \] (6)
holds for arbitrary \( x \in X \) and \( r > 0 \);
3. \( u^\# \in L^p(X,\mu) \).

Moreover,
\[ |u|_{S^p(X,d,\mu)} \sim \inf_{h} \|h \|_{L^p(X,\mu)} \sim \|u^\# \|_{L^p(X,\mu)}. \]

If \( u^\# \in L^p(X,\mu) \) then, by (5), \( u \in S^p(X,d_\gamma,\mu) \).

Among all the numerous results related to function classes of Sobolev type, we will first of all be interested in analogs of the Sobolev embedding theorems. We will now confine ourselves to formulating two assertions concerning the compactness of the embedding operators.

**Lemma 2.** Let \( 1 < p < \infty \). Then the embedding operator
\[ I : M^p(X,d,\mu) \to L^q(X,\mu) \]
is compact for
1. \(1 \leq q < \frac{ps}{s-p}\) if \(p < s\);
2. \(1 \leq q < \infty\) if \(p = s\);
3. \(1 \leq q \leq \infty\) if \(p > s\).

The first two items are consequences of the results of [1, 3], and the proof of the third is given in [4].

The following assertion is an internal embedding theorem in the scale of the spaces \(M^p_g\) and shows the relationship between the classes of functions defined by various metrics (see [4]).

**Lemma 3.** Let \(1 < p < \infty\), \(0 < \gamma < 1\). The embedding operator

\[ I : M^1_g(X, d, \mu) \to M^1_g(X, d^\gamma, \mu) \]

is compact for
1. \(1 \leq \omega < \frac{ps}{s-(1-\gamma)p}\) if \((1-\gamma)p < s\);
2. \(1 \leq \omega < \infty\) if \((1-\gamma)p = s\);
3. \(1 \leq \omega \leq \infty\) if \((1-\gamma)p > s\).

Note that item 3 implies the Hölder continuity of the function understood in the usual sense \(|u(x) - u(y)| \leq C (d(x, y))^{\gamma}\).

**Measures with Variable Integrability Exponent**

Lemmas 2 and 3 are very universal but they do not depend on the specific nature of the metric space and are completely defined by the regularity exponent \(s\) characterizing the relationship between the measure and metric.

In the case of homogeneous metric spaces, when the measure of an arbitrary ball \(B(x, r)\) admits a two-sided estimate (1) via \(r^s\), the claims of lemmas 2 and 3 concerning the integrability exponents \(q\) and \(\omega\) are accurate and unimprovable.

In accordance with [5], every compact set in \(\mathbb{R}^n\) can be endowed with a metric satisfying the doubling condition (3). The doubling condition implies a bound for the measure of a ball of the form (4) but there is no upper bound in the general case.

It is not hard to give examples when the homogeneity condition (1) fails.

1. The simplest situation is when the set \(X \subset \mathbb{R}^m\) is the union of sets \(E_k\) having different Euclidean dimensions \(n_k\).

2. If the interval \([0, 1] \subset \mathbb{R}\) is endowed with the weight metric \(dm = 2x \, dx\) then
\[ \mu(B(0, r)) = r^2, \quad r \leq 1. \]
Moreover, \(\mu(B(1, r)) = 1 - (1-r)^2 = 2r - r^2 = r(2-r)\).

Consequently,
\[ r \leq \mu(B(1, r)) \leq 2r. \]

For arbitrary \(x \in (0,1)\) and \(0 < r < 1\), we have an inequality with variable regularity exponent \(\mu(B(x, r)) \geq C r^{s(x)}\), where \(s(x) \searrow 2\) as \(x \to 0\) and \(s(x) \downarrow 1\) as \(x \to 1\). Rough estimates show that, as \(x \to 0\), the inequality \(\mu(B(x, r)) \geq C r^{s(x)}\) holds for the function \(\omega(x) = 2 (1 - \log_2(1+2x))\).

Even for Euclidean domain and the Lebesgue measure, the exponent in the lower bound (4) can be different for different points of the domain, i.e., be a function of a point.

Standard examples of such a kind are peaks with Hölder singularities at the vertex.

3. Let \(1 < \lambda < \infty\); define the “zero peak” \(G_\lambda \subset \mathbb{R}^2\) by the condition:
\[ G_\lambda = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, \ 0 \leq y \leq x^\lambda\}. \]

As the measure \(\mu\), consider the restriction of the Lebesgue measure in \(\mathbb{R}^2\) to the peak \(G_\lambda\). If \(a\) is the vertex of the peak \(G_\lambda\), then \(\mu(B(a, r)) = \Lambda^{-1} r^\Lambda\), where \(\Lambda = 1 + \lambda > 2\).
If a point \( b = (x, y) \) different from the vertex belongs to \( G_1 \) then the balls of a sufficiently small radius satisfy \( \mu(B(b, r)) > C r^2 \). For a suitable choice of the constant \( C^* \), for an arbitrary point \( c = (x, y) \in G_1 \) and \( 0 < r < 1 \), we have the estimate \( \mu(B(c, r)) \geq C^* r^{s(x, y)} \), where \( s(x, y) \not< \Lambda \geq 2 \) as \( x \to 0 \) and \( s(x, y) \not< 2 \) as \( x \to 1 \).

The estimate \( \mu(B(a, r)) > C r^\Lambda \) holds for all points \( a \in G_\Lambda \); moreover, the uniform regularity exponent of the measure \( \mu \) cannot be less than \( \Lambda \). By Lemma 2, for \( p < \Lambda \), the embedding operator

\[
I : M^p(L_\Lambda, | \cdot |, \mu) \to L_q(X, \mu)
\]

is compact then

\[
q < \frac{p\Lambda}{\Lambda - p}.
\]

Moreover, local embedding into the Lebesgue space holds with a greater integrability exponent.

Put

\[
s_{a,r} = \sup_{B(a, r)} s(x, y).
\]

If a point \( a \) is different from the vertex of the peak then, for \( 1 < p < 2 \), on the ball \( B(a, r) \), the space \( M^p(L_\Lambda, | \cdot |, \mu) \) is compactly embedded in the Lebesgue space \( L_{q(a, r)}(X, \mu) \), where

\[
q(a, r) < \frac{2s_{a,r}}{s_{a,r} - p} > \frac{p\Lambda}{\Lambda - p}.
\]

This enables us to assume that, in this situation, for refining the result, it is appropriate to consider the embedding of Sobolev-type spaces into spaces with variable integrability exponents.

**Spaces with Variable Integrability Exponents**

Consider a metric space \((X, d)\) with finite diameter, a finite Borel measure \( \mu \) with support in \( X \), and fix a positive measurable function \( p : X \to [1, \infty) \). On the set of measurable functions \( f : X \to \mathbb{R} \), introduce the functional

\[
\rho_p(f) = \int_X |f(x)|^p(x) d\mu.
\]

Define the Lebesgue space with variable integrability exponent \( L_{p(\cdot)}(X, \mu) \) as the class of all such functions \( f \) such that \( \rho_{p(\cdot)}(\lambda f) < \infty \) for some \( \lambda > 0 \).

If \( p(x) \geq 1 \) then the norm in \( L_{p(\cdot)}(X, \mu) \) is introduced by the equality

\[
\|f\|_{p(\cdot)} = \|f|_{L_{p(\cdot)}(X, \mu)}\| = \inf \left\{ \alpha > 0 \mid \rho_{p(\cdot)}(\alpha f^\alpha) \leq 1 \right\}.
\]

For \( p(x) = p = \text{const} \), the above-introduced norm coincides with the standard norm in \( L_p \). The norm in \( L_{p(\cdot)}(X, \mu) \) is monotone, i.e., the condition \( |f| \leq |g| \) almost everywhere implies the inequality \( \|f\|_{p(\cdot)} \leq \|g\|_{p(\cdot)} \).

Reckoning with the finiteness of the measure \( \mu \), it is easy to show that, for \( 1 \leq p(x) \leq q(x) < \infty \), the space \( L_{p(\cdot)}(X, \mu) \) is continuously embedded in \( L_{q(\cdot)}(X, \mu) \).

For \( 1 < p(x) < \infty \), define the conjugate exponent \( p'(x) \) by the standard equality

\[
\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.
\]

It is rather easy to prove an analog of the classical Hölder inequality

\[
\int_X |f(x)g(x)| d\mu \leq C \|f\|_{p(\cdot)} \cdot \|g\|_{p'(\cdot)}.
\]

We will need the following assertion, proved in [7].

**Lemma 4.** Let \( 1 < p_- \leq p(x) \leq p^+ < \infty \). Then

(1) the conditions \( \|f\|_{p(\cdot)} \leq 1 \) and \( \rho_{p(\cdot)}(f) \leq 1 \) are equivalent;
(2) \(|f_n|_{V_0^1} \to 0\) if and only if \(\rho_{V_0^1}(f_n) \to 0\);

(3) if \(\rho_{V_0^1}(f) \leq L < \infty\) then \(|f|_{V_0^1} \leq K = K(p(x),L) < \infty\).

Replacing in the definition of the spaces \(S_{V_0^1}(X,d,\mu)\) and \(M_{V_0^1}^1(X,d,\mu)\) the usual Lebesgue space \(L_p(X,\mu)\) by the Lebesgue space with variable integrability exponent \(L_{p(q)}(X,\mu)\), we obtain the classes of functions with variable integrability exponent \(-S_{p(q)}^1(X,d,\mu)\) and \(M_{p(q)}^1(X,d,\mu)\).

Embedding Theorems for Nonhomogeneous Measures

Henceforth, we will consider the properties of functions of the Sobolev-type spaces \(M_{V_0^1}^1(X,d,\mu)\) in the case when the measure \(\mu\) satisfies:

(i) the doubling condition;

(ii) the estimate \(\mu(B(x,r)) \geq C_0 r^s(x)\) for \(x \in X, r \leq \text{diam} X\).

For \(p < s(x)\), put \(q(x) = \frac{ps(x)}{s(x) - p}\).

For proving the boundedness of the embedding operator into the Lebesgue space with limit exponent \(q(x)\), make use of the description of Sobolev-type spaces in terms of maximal order functions.

**Theorem 1.** Suppose that a measure \(\mu\) on a metric space \((X,d)\) satisfies conditions (i), (ii). If \(1 < p < s_\ast = \inf_{x \in X} s(x)\) and \(q(x) = \frac{ps(x)}{s(x) - p}\) then the embedding operator

\[ I : M_{p(q)}^{1}(X,d,\mu) \to L_{p(q)}(X,\mu) \]

is bounded. Moreover,

\[ \|u - u_X| L_{p(q)}(X,\mu)\| \leq C\|u| S_{p(q)}^{1}(X,d,\mu)\|. \]

**Proof.** By the finiteness of the measure and the boundedness of the exponent \(q(x)\), it suffices to prove the last inequality. If \(u \equiv \text{const}\) then the theorem is obvious. If the function \(u\) is nonconstant then \(u_0^p > 0\) everywhere and \(u\) is finite almost everywhere.

Consider a Lebesgue point \(x \in X\) of the function \(u\) at which \(u_0^p(x) = \lambda < \infty\) and estimate the deviation of the value of \(u\) at the point \(x\) from the mean value \(u_X\).

Put \(r_k = 2^{-k} \cdot \text{diam} X\) and consider the sequence of balls \(\{B_k = B(x,r_k)\}\). Since \(u_X = u_{B_0}\) and \(u_{B_k} \to u(x)\) as \(k \to \infty\), we get

\[ |u(x) - u_X| = |(u_{B_0} - u_{B_1}) + (u_{B_1} - u_{B_2}) + \cdots + (u_{B_k} - u_{B_{k+1}}) + \cdots| \leq \]

\[ \sum_{k=0}^{\infty} \frac{\mu(B_k)}{\mu(B_{k+1})} \int_{B_k} \|u - u_{B_k}\|d\mu \leq C_d \sum_{k=0}^{\infty} \int_{B_k} \|u - u_{B_k}\|d\mu. \] (7)

Consider two cases.

I. Let \(\text{diam} X \leq \left(\|h| L_p||/\lambda\right)^{p/(s(x))}\), where \(h\) is the function of the Poincaré inequality (6).

Using (7), we infer

\[ |u(x) - u_X| \leq C_d \sum_{k=0}^{\infty} r_k \left( \int_{B_k} \|u - u_{B_k}\|d\mu \right) \leq C_1 \lambda \text{diam} X \leq \]

\[ C_1 \|h| L_p||^{p/(s(x))}\lambda^{p/(q(x))} \leq C_1 \|h| L_p|| \left(\|h| L_p\right)^{(1/q(x))}

or

\[ \left( \frac{|u(x) - u_X|}{\|h| L_p\} \right)^{q(x)} \leq C_2 \left( u_0^p(x) \right)^p \|h| L_p||^{-p}. \] (8)

II. If \(\|h| L_p||/\lambda^{p/(s(x))} \leq \text{diam} X\) then fix \(m\) such that

\[ r_m \leq \left(\|h| L_p||/\lambda\right)^{p/(s(x))} < 2r_m. \]
By (7),

\[ |u_X - u(x)|C_d \left( \sum_{k=0}^{m} \int_{B_k} |u - u_{B_k}| \, d\mu + \sum_{k=m+1}^{\infty} \int_{B_k} |u - u_{B_k}| \, d\mu \right). \]

Estimate the first sum as in item I:

\[ \sum_{k=0}^{m} \int_{B_k} |u - u_{B_k}| \, d\mu \leq \lambda \sum_{k=0}^{m} r_k \leq C_\lambda \|h \cdot L_p\|^{p/s(x)} \lambda^{p/q(x)} = C_\lambda \|h \cdot L_p\|^{p/s(x)} \lambda^{p/q(x)}. \]  

Of the different norms observed in Lemma 1, we obtain

The constant $M$ holds for almost all $x$ in $X$. Integrating this inequality and taking into account the equivalence of the different norms observed in Lemma 1, we obtain

\[ \rho_q(x) \left( \frac{|u(x) - u_X|}{\|h \cdot L_p\|} \right) \leq \tilde{C} \|u^q \cdot L_p\| \|h \cdot L_p\|^{-p} \leq M < \infty, \]

where the constant $M$ does not depend on the function $u \in M^p_\nu(X, d, \mu)$.

By Lemma 4, inequality (11) implies

\[ \|u - u_X\|_{q(x)} \leq K \|h \cdot L_p\| \leq C \|u \cdot S^1_p(X, d, \mu)\|. \]

Like for a constant integrability exponent, in this case, we can consider the embedding into spaces of Sobolev type defined by the Hölder metrics $d_{\gamma}$. For $0 < \gamma < 1$ and $(1 - \gamma)p < s_\gamma$, we put

\[ q_{\gamma}(x) = \frac{ps(x)}{s(x) - (1 - \gamma)p}. \]

The proof of the boundedness of the embedding operator into the space $M^p_\nu(X, d, \mu)$ is based on the estimates of the sharp maximal functions of fractional order $u^q_{\gamma}$. Theorem 2. Suppose that a measure $\mu$ on a metric space $(X, d)$ satisfies conditions (i), (ii), $0 < \gamma < 1$, and $(1 - \gamma)p < s_\gamma$. Then the embedding operator

\[ I : S^1_p(X, d, \mu) \rightarrow S^1_{p,\gamma}(X, d, \mu) \]

is continuous.

Proof. By Lemma 1 and inequality (5), it suffices to demonstrate that the condition

\[ u^q_{\gamma} \in L_p(X, \mu) \text{ implies } u^q_{\gamma} \in L_{q_{\gamma}(\cdot)}(X, \mu). \]

Since

\[ u^q_{\gamma}(x) = \sup_{r > 0} r^{1-\gamma} \int_{B(x, r)} |u - u_{B(x, r)}| \, d\mu \leq (\text{diam}X)^{1-\gamma} u^q_{\gamma}(x), \]

we have $u^q_{\gamma}(x) < \infty$ for almost all $x \in X$.
Let \( u^\#(x) = \lambda < \infty \). Choose \( r_0 \) so that
\[
\lambda \leq 2 \mathbf{r}_0^{-\gamma} \int_{B(x, r_0)} |u - u_{B(x, r_0)}| \, d\mu;
\]
then
\[
\frac{\lambda r_0^{-\gamma - 1}}{2 \mathbf{r}_0^{-\gamma}} \leq \int_{B(x, r_0)} |u - u_{B(x, r_0)}| \, d\mu \leq 2 u^\#(x).
\] (12)

Using the Poincaré inequality (6), Hölder’s inequality, and condition (ii), we infer
\[
\lambda \leq 2 r_0^{-\gamma} \mathbf{r}_0^{-\gamma} \int_{B(x, r_0)} |u - u_{B(x, r_0)}| \, d\mu \leq 2 r_0^{-\gamma} \int_{B(x, r_0)} h \, d\mu \leq 2 r_0^{-\gamma} \|h| L_p\| \mu(B(x, r_0))^{-1/p} \leq C_1 r_0^{-s(x)/p} \|h| L_p\|.
\]
This gives an estimate for \( r_0 \):
\[
2^{-1} \geq C_2 x^{q_1(x)/s(x)} \|h| L_p\|^{-q_1(x)/s(x)}. \tag{13}
\]

Using (12), (13), and recalculation of the exponents, we obtain the inequality
\[
\left( \frac{u^\#(x)}{\|h| L_p\|} \right)^{q_1(x)} \leq \hat{C} \left( \frac{u^\#(x)}{h| L_p\|} \right)^{p},
\]
integrating which and taking into account the equivalence of the different norms, we obtain
\[
\rho_{q_1} \left( \frac{u^\#(x)}{\|h| L_p\|} \right) \leq \hat{C} \|u^\#| L_p\|^p \|h| L_p\|^{-p} \leq M < \infty, \tag{14}
\]
where the constant \( M \) does not depend on the choice of the function \( u \in M^p_1(X, d, \mu) \).

Lemma 4 and inequality (14) yield
\[
\|u^\#| H_{q_1} \| \leq K \|h| L_p\| \|u| S^p_1(X, d, \mu)\|.
\]

**Theorem 3.** Suppose that a measure \( \mu \) on a compact metric space \((X, d)\) satisfies conditions (i), (ii), where the function \( s(x) \) is continuous and \( 1 < p < s_- = \min x s(x) \). Then, for every sufficiently small \( \varepsilon > 0 \) and an arbitrary function \( r(x) \) satisfying the estimate \( 1 \leq r(x) \leq q(x) - \varepsilon \), the embedding operator
\[
I : M^p(X, d, \mu) \rightarrow L_{q_1}(X, \mu)
\]
is compact.

**Proof.** Fix \( \varepsilon > 0 \) and a function \( r(x) \) satisfying the hypotheses of the theorem. Since \( p < s_- \), there exists a constant \( L < \infty \) such that \( |q(x) - q(y)| \leq L|s(x) - s(y)| \). Put \( \varepsilon_1 = \varepsilon/2L \).

The function \( s(x) \), being continuous on a compact space, is uniformly continuous. Therefore, the set \( X \) can be covered by a finite family of balls \( \{B_k\} \) such that \( |s(x) - s(y)| < \varepsilon_1 \) for all \( x, y \in B_k \).

Put \( s_k = \sup_{x \in B_k} s(x) \) and \( q_k = \frac{s_k p}{s_k - p} \), then \( r(x) \leq q(x) - \varepsilon \leq q_k - \varepsilon/2 \) for all \( x \in B_k \).

Since, on the ball \( B_k \), the measure \( \mu \) is \( s_k \)-regular then the space \( M^p_1(B_k, d, \mu) \) (by Lemma 2) is compactly embedded in \( L_\omega(B_k, \mu) \) for all values of \( \omega \) satisfying the inequality \( q_k - \varepsilon/2 \leq \omega < q_k \). Moreover, the space \( L_\omega(B_k, \mu) \) is continuously embedded in \( L_{q_1}(B_k, \mu) \). Hence, the embedding operator
\[
I : M^p_1(B_k, d, \mu) \rightarrow L_{q_1}(B_k, \mu)
\]
is compact because it is the composition of a compact operator and a continuous operator.
Since the family of balls \( \{ B_k \} \) is finite, the Sobolev-type space \( M^{1,r}_p(X,d,\mu) \) is embedded in the Lebesgue space with variable integrability exponent \( L^{r(.)}(X,\mu) \). Moreover, from every bounded set of functions in \( M^{1,r}_p(X,d,\mu) \), we can choose a sequence of functions \( \{ u_n \} \) whose restrictions to each of the balls \( B_k \) constitute a Cauchy sequence in the corresponding space \( L^{r(.)}(B_k,\mu) \). By Lemma 4,

\[
\int_{B_k} |u_n(x) - u_m(x)|^{r(x)} d\mu \to 0 \quad \text{as} \quad n,m \to \infty
\]

and

\[
\rho_{r(.)}(u_n - u_m) = \int_X |u_n(x) - u_m(x)|^{r(x)} d\mu \leq \sum_k \int_{B_k} |u_n(x) - u_m(x)|^{r(x)} d\mu \to 0 \quad \text{as} \quad n,m \to \infty.
\]

Again by Lemma 4, we obtain \( \|u_n - u_m\|_{L^{r(.)}} \to 0 \), i.e., the function sequence \( \{ u_n \} \) is a Cauchy sequence in \( L^{r(.)}(X,\mu) \), which completes the proof of the compactness of the embedding operator. \( \blacksquare \)

**Theorem 4.** Suppose that a measure \( \mu \) on a compact metric space \( (X,d) \) satisfies conditions (i), (ii), where the function \( s(x) \) is continuous and \( 1 < p < s_n(1 - \gamma) \). Then, for every sufficiently small \( \varepsilon > 0 \) and an arbitrary function \( r(x) \) satisfying the estimate \( 1 \leq r(x) \leq q_\gamma(x) - \varepsilon \), the embedding operator

\[
I : M^{1,r}_p(X,d,\mu) \to M^{1,r}_{p,\gamma}(X,d,\mu)
\]

is compact.

As for reckoning with Lemma 3, the proof of Theorem 4 is practically identical to that of Theorem 3. All changes amount to formally replacing \( q(x) \) by \( q_\gamma(x) \), the space \( L^{r(.)}(B_k,\mu) \), by \( M^{1,r}_{p,\gamma}(B_k,d,\mu) \), and checking in \( L^{r(.)}(X,\mu) \) that a sequence of admissible functions \( \{ g_{\gamma,n} \} \) such that

\[
|u_n(x) - u_n(y)| \leq (d(x,y))^{\gamma} (g_{\gamma,n}(x) + g_{\gamma,n}(y))
\]

is a Cauchy sequence. \( \blacksquare \)

**Functions with Variable “Smoothness” Exponent**

In the Euclidean case, for \( p > n \), the functions of the space \( W^{1,p}_n(R^n) \) satisfy the Hölder condition with exponent \( \gamma = 1 - n/p \). In the metric case, inequality (5) and the boundedness of the sharp maximal function \( u^{s_n}_w \) imply the Hölder continuity of the function \( u \) with exponent \( n \) with respect to the metric \( d \). If a nonhomogeneous measure \( \mu \) satisfies condition (ii) then, for large integrability exponents, a function \( u \in S^{1,p}_n(X,d,\mu) \) can have different local “smoothness” characteristics in neighborhoods of different points since they are determined by the local properties of the measure and the properties of the sharp maximal function \( u^{s_n}_w \), where \( \gamma(x) = 1 - s(x)/p \).

**Theorem 5.** If a measure \( \mu \) on a metric space \( (X,d) \) satisfies conditions (i), (ii), \( p > s_n = \sup_{x \in X} s(x) \) and \( u \in S^{1,p}_n(X,d,\mu) \) then \( u^{s_n}_w \in L^{m}(X,\mu) \), where \( \gamma(x) = 1 - s(x)/p \).

**Proof.** Using the Poincaré inequality, Hölder’s inequality, and condition (ii), we obtain

\[
\frac{1}{r^{1-s(x)/p}} \int_{B(x,r)} |u - u_{B(x,r)}| d\mu \leq r^{s(x)/p} \int_{B(x,r)} h d\mu \leq r^{s(x)/p} (\mu(B(x,r)))^{-1/p} \|h\|_{L^p(X,\mu)} \leq C \|h\|_{L^p(X,\mu)} < \infty.
\]

Therefore, \( u^{s_n}_w \in L^{m}(X,\mu) \). \( \blacksquare \)
References


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