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DEFINABLE FAMILIES OF THEORIES, RELATED CALCULI AND RANKS

N.D. MARKHABATOV, S.V. SUDOPLATOV

Abstract. We consider sentence-definable and diagram-definable subfamilies of given families of theories, calculi for these subfamilies, as well dynamics and characteristics of these subfamilies with respect to rank and degree.

Keywords: family of theories, definable subfamily, calculus, rank, degree.

The rank for families of theories was introduced and studied in general context in [1]. All possible values of the ranks and degrees for families of all theories in given languages were described in [2]. In the present paper we consider sentence-definable and diagram-definable subfamilies of given families of theories, calculi for these subfamilies, as well as dynamics and characteristics of these subfamilies with respect to rank and degree.

The paper is organized as follows. In Section 1, preliminary notions, notations and results are collected. In Section 2, we consider calculi subfamilies of families of theories as well as links for sentence-definable and diagram-definable subfamilies. Compactness and E-closeness for definable subfamilies are studied in Section 3. In Section 4 we consider dynamics of ranks with respect to definable subfamilies of theories and prove the existence of subfamilies of given rank.

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1. Preliminaries

Throughout the paper we consider complete first-order theories T in a predicate language $\Sigma = \Sigma(T)$. The considerations below can be naturally spread for an arbitrary language replacing operations and constants by their graphs. We use the following terminology in [1, 3, 4, 5, 6, 7, 8, 9].

Let $P = (P_i)_{i\in I}$, be a family of nonempty unary predicates, $(\mathcal{A}_i)_{i\in I}$ be a family of structures such that P_i is the universe of $\mathcal{A}_i$, $i \in I$, and the symbols P_i are disjoint with languages for the structures $\mathcal{A}_j$, $j \in I$. Consider the structure $\bigcup\limits_{i\in I} \mathcal{A}_i$ of the language $\Sigma = \bigcup\limits_{i\in I} \Sigma(\mathcal{A}_i)$, where the predicates R in $\Sigma \setminus \Sigma(\mathcal{A}_i)$ are assumed to be empty for $\mathcal{A}_i$. The expansion of $\bigcup\limits_{i\in I} \mathcal{A}_i$ by the predicates P_i is denoted by $\mathcal{A}_P$. It is called the *P-union* of the structures $\mathcal{A}_i$, and the operator mapping $(\mathcal{A}_i)_{i\in I}$ to $\mathcal{A}_P$ is the *P-operator*. The structure $\mathcal{A}_P$ is called the *P-combination* of the structures $\mathcal{A}_i$ and denoted by $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ if $\mathcal{A}_i = (\mathcal{A}_P \upharpoonright A_i) \upharpoonright \Sigma(\mathcal{A}_i)$, $i \in I$. Structures $\mathcal{A}'$, which are elementary equivalent to $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$, will be also considered as P-combinations.

Clearly, all structures $\mathcal{A}' \equiv \mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ are represented as unions of their restrictions $\mathcal{A}'_i = (\mathcal{A}' \upharpoonright P_i) \upharpoonright \Sigma(\mathcal{A}_i)$ if and only if the set $p_\infty(x) = \{\neg P_i(x) \mid i \in I\}$ is inconsistent. If $\mathcal{A}' \neq \mathrm{Comb}_P(\mathcal{A}'_i)_{i\in I}$, we write $\mathcal{A}' = \mathrm{Comb}_P(\mathcal{A}'_i)_{i\in I\cup\{\infty\}}$, where $\mathcal{A}'_\infty = \mathcal{A}' \upharpoonright \bigcap\limits_{i\in I} \overline{P_i}$, maybe applying Morleyzation. Moreover, we write

$$\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I\cup\{\infty\}}$$

for $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ with the empty structure $\mathcal{A}_\infty$.

Note that if all predicates P_i are disjoint, a structure $\mathcal{A}_P$ is a P-combination and a disjoint union of structures $\mathcal{A}_i$. In this case the P-combination $\mathcal{A}_P$ is called *disjoint*. Clearly, for any disjoint P-combination $\mathcal{A}_P$, $\mathrm{Th}(\mathcal{A}_P) = \mathrm{Th}(\mathcal{A}'_P)$, where $\mathcal{A}'_P$ is obtained from $\mathcal{A}_P$ replacing $\mathcal{A}_i$ by pairwise disjoint $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. Thus, in this case, similar to structures the P-operator works for the theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ producing the theory $T_P = \mathrm{Th}(\mathcal{A}_P)$, being *$P$-combination* of T_i, which is denoted by $\mathrm{Comb}_P(T_i)_{i\in I}$.

Notice that P-combinations are represented by generalized products of structures [10].

For an equivalence relation E replacing disjoint predicates P_i by E-classes we get the structure $\mathcal{A}_E$ of the language $\Sigma \cup \{E^{(2)}\}$, which is the *E-union* of the structures $\mathcal{A}_i$. In this case the operator mapping $(\mathcal{A}_i)_{i\in I}$ to $\mathcal{A}_E$ is the *E-operator*. The structure $\mathcal{A}_E$ is also called the *E-combination* of the structures $\mathcal{A}_i$ and denoted by $\mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$; here $\mathcal{A}_i = (\mathcal{A}_E \upharpoonright A_i) \upharpoonright \Sigma(\mathcal{A}_i)$, $i \in I$. Similar above, structures $\mathcal{A}'$, which are elementary equivalent to $\mathcal{A}_E$, are denoted by $\mathrm{Comb}_E(\mathcal{A}'_j)_{j\in J}$, where $\mathcal{A}'_j$ are restrictions of $\mathcal{A}'$ to its E-classes. The E-operator works for the theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ producing the theory $T_E = \mathrm{Th}(\mathcal{A}_E)$, being *$E$-combination* of T_i, which is denoted by $\mathrm{Comb}_E(T_i)_{i\in I}$ or by $\mathrm{Comb}_E(\mathcal{T})$, where $\mathcal{T} = \{T_i \mid i \in I\}$.

Clearly, $\mathcal{A}' \equiv \mathcal{A}_P$ realizing $p_\infty(x)$ is not elementary embeddable into $\mathcal{A}_P$ and can not be represented as a disjoint P-combination of $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. At the same time, there are E-combinations such that all $\mathcal{A}' \equiv \mathcal{A}_E$ can be represented as E-combinations of some $\mathcal{A}'_j \equiv \mathcal{A}_i$. We call this representability of $\mathcal{A}'$ to be the *E-representability*.

If there is $\mathcal{A}'\equiv\mathcal{A}_E$ which is not E-representable, we have the E'-representability replacing E by E' such that E' is obtained from E adding equivalence classes with models for all theories T, where T is a theory of a restriction $\mathcal{B}$ of a structure $\mathcal{A}'\equiv\mathcal{A}_E$ to some E-class and $\mathcal{B}$ is not elementary equivalent to the structures $\mathcal{A}_i$. The resulting structure $\mathcal{A}_{E'}$ (with the E'-representability) is a *e-completion*, or a *e-saturation*, of $\mathcal{A}_E$. The structure $\mathcal{A}_{E'}$ itself is called *e-complete*, or *e-saturated*, or *e-universal*, or *e-largest*.

For a structure $\mathcal{A}_E$ the number of *new* structures with respect to the structures $\mathcal{A}_i$, i. e., of the structures $\mathcal{B}$ which are pairwise elementary non-equivalent and elementary non-equivalent to the structures $\mathcal{A}_i$, is called the *e-spectrum* of $\mathcal{A}_E$ and denoted by e-$\mathrm{Sp}(\mathcal{A}_E)$. The value $\sup\{e\text{-}\mathrm{Sp}(\mathcal{A}')\mid\mathcal{A}'\equiv\mathcal{A}_E\}$ is called the *e-spectrum* of the theory $\mathrm{Th}(\mathcal{A}_E)$ and denoted by e-$\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_E))$. If structures $\mathcal{A}_i$ represent theories T_i of a family $\mathcal{T}$, consisting of T_i, $i\in I$, then the e-spectrum e-$\mathrm{Sp}(\mathcal{A}_E)$ is denoted by e-$\mathrm{Sp}(\mathcal{T})$.

If $\mathcal{A}_E$ does not have E-classes $\mathcal{A}_i$, which can be removed, with all E-classes $\mathcal{A}_j\equiv\mathcal{A}_i$, preserving the theory $\mathrm{Th}(\mathcal{A}_E)$, then $\mathcal{A}_E$ is called *e-prime*, or *e-minimal*.

For a structure $\mathcal{A}'\equiv\mathcal{A}_E$ we denote by $\mathrm{TH}(\mathcal{A}')$ the set of all theories $\mathrm{Th}(\mathcal{A}_i)$ of E-classes $\mathcal{A}_i$ in $\mathcal{A}'$.

By the definition, an e-minimal structure $\mathcal{A}'$ consists of E-classes with a minimal set $\mathrm{TH}(\mathcal{A}')$. If $\mathrm{TH}(\mathcal{A}')$ is the least for models of $\mathrm{Th}(\mathcal{A}')$ then $\mathcal{A}'$ is called *e-least*.

Definition [4]. Let $\overline{\mathcal{T}}_\Sigma$ be the set of all complete elementary theories of a relational language Σ. For a set $\mathcal{T}\subset\overline{\mathcal{T}}_\Sigma$ we denote by $\mathrm{Cl}_E(\mathcal{T})$ the set of all theories $\mathrm{Th}(\mathcal{A})$, where $\mathcal{A}$ is a structure of some E-class in $\mathcal{A}'\equiv\mathcal{A}_E$, $\mathcal{A}_E=\mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$, $\mathrm{Th}(\mathcal{A}_i)\in\mathcal{T}$. As usual, if $\mathcal{T}=\mathrm{Cl}_E(\mathcal{T})$ then $\mathcal{T}$ is said to be *E-closed*.

The operator Cl_E of E-closure can be naturally extended to the classes $\mathcal{T}\subset\overline{\mathcal{T}}$, where $\overline{\mathcal{T}}$ is the union of all $\overline{\mathcal{T}}_\Sigma$ as follows: $\mathrm{Cl}_E(\mathcal{T})$ is the union of all $\mathrm{Cl}_E(\mathcal{T}_0)$ for subsets $\mathcal{T}_0\subseteq\mathcal{T}$, where new language symbols with respect to the theories in $\mathcal{T}_0$ are empty.

For a set $\mathcal{T}\subset\overline{\mathcal{T}}$ of theories in a language Σ and for a sentence φ with $\Sigma(\varphi)\subseteq\Sigma$ we denote by $\mathcal{T}_\varphi$ the set $\{T\in\mathcal{T}\mid\varphi\in T\}$. Any set $\mathcal{T}_\varphi$ is called the *φ-neighbourhood*, or simply a *neighbourhood*, for $\mathcal{T}$, or the (φ-)*definable* subset of $\mathcal{T}$. The set $\mathcal{T}_\varphi$ is also called (*formula-* or *sentence-*)*definable* (by the sentence φ) with respect to $\mathcal{T}$, or (*sentence-*)*$\mathcal{T}$-definable*, or simply *s-definable*.

Proposition 1.1 [4]. *If $\mathcal{T}\subset\overline{\mathcal{T}}$ is an infinite set and $T\in\overline{\mathcal{T}}\setminus\mathcal{T}$ then $T\in\mathrm{Cl}_E(\mathcal{T})$ (i.e., T is an accumulation point for $\mathcal{T}$ with respect to E-closure Cl_E) if and only if for any formula $\varphi\in T$ the set $\mathcal{T}_\varphi$ is infinite.*

If T is an accumulation point for $\mathcal{T}$ then we also say that T is an *accumulation point* for $\mathrm{Cl}_E(\mathcal{T})$.

Theorem 1.2 [4]. *For any sets $\mathcal{T}_0,\mathcal{T}_1\subset\overline{\mathcal{T}}$, $\mathrm{Cl}_E(\mathcal{T}_0\cup\mathcal{T}_1)=\mathrm{Cl}_E(\mathcal{T}_0)\cup\mathrm{Cl}_E(\mathcal{T}_1)$.*

Definition [4]. Let $\mathcal{T}_0$ be a closed set in a topological space $(\mathcal{T},\mathcal{O}_E(\mathcal{T}))$, where $\mathcal{O}_E(\mathcal{T})=\{\mathcal{T}\setminus\mathrm{Cl}_E(\mathcal{T}')\mid\mathcal{T}'\subseteq\mathcal{T}\}$. A subset $\mathcal{T}_0'\subseteq\mathcal{T}_0$ is said to be *generating* if $\mathcal{T}_0=\mathrm{Cl}_E(\mathcal{T}_0')$. The generating set $\mathcal{T}_0'$ (for $\mathcal{T}_0$) is *minimal* if $\mathcal{T}_0'$ does not contain proper generating subsets. A minimal generating set $\mathcal{T}_0'$ is *least* if $\mathcal{T}_0'$ is contained in each generating set for $\mathcal{T}_0$.

Theorem 1.3 [4]. *If $\mathcal{T}_0'$ is a generating set for a E-closed set $\mathcal{T}_0$ then the following conditions are equivalent:*

(1) $\mathcal{T}_0'$ *is the least generating set for* $\mathcal{T}_0$*;*
(2) $\mathcal{T}_0'$ *is a minimal generating set for* $\mathcal{T}_0$*;*
(3) *any theory in* $\mathcal{T}_0'$ *is isolated by some set* $(\mathcal{T}_0')_\varphi$*, i.e., for any* $T \in \mathcal{T}_0'$ *there is* $\varphi \in T$ *such that* $(\mathcal{T}_0')_\varphi = \{T\}$*;*
(4) *any theory in* $\mathcal{T}_0'$ *is isolated by some set* $(\mathcal{T}_0)_\varphi$*, i.e., for any* $T \in \mathcal{T}_0'$ *there is* $\varphi \in T$ *such that* $(\mathcal{T}_0)_\varphi = \{T\}$*.*

Definition [9]. Let $\mathcal{T}$ be a family of theories and T be a theory, $T \notin \mathcal{T}$. The theory T is called $\mathcal{T}$*-approximated*, or *approximated by* $\mathcal{T}$, or $\mathcal{T}$*-approximable*, or a *pseudo-*$\mathcal{T}$*-theory*, if for any formula $\varphi \in T$ there is $T' \in \mathcal{T}$ such that $\varphi \in T'$.

If T is $\mathcal{T}$-approximated then $\mathcal{T}$ is called an *approximating family* for T, theories $T' \in \mathcal{T}$ are *approximations* for T, and T is an *accumulation point* for $\mathcal{T}$.

An approximating family $\mathcal{T}$ is called *e-minimal* if for any sentence $\varphi \in \Sigma(\mathcal{T})$, $\mathcal{T}_\varphi$ is finite or $\mathcal{T}_{\neg\varphi}$ is finite.

It was shown in [9] that any e-minimal family $\mathcal{T}$ has unique accumulation point T with respect to neighbourhoods $\mathcal{T}_\varphi$, and $\mathcal{T} \cup \{T\}$ is also called *e-minimal*.

Following [1] we define the *rank* $\mathrm{RS}(\cdot)$ for the families of theories, similar to Morley rank [11], and a hierarchy with respect to these ranks in the following way.

For the empty family $\mathcal{T}$ we put the rank $\mathrm{RS}(\mathcal{T}) = -1$, for finite nonempty families $\mathcal{T}$ we put $\mathrm{RS}(\mathcal{T}) = 0$, and for infinite families $\mathcal{T}$ — $\mathrm{RS}(\mathcal{T}) \geq 1$.

For a family $\mathcal{T}$ and an ordinal $\alpha = \beta + 1$ we put $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if there are pairwise inconsistent $\Sigma(\mathcal{T})$-sentences φ_n, $n \in \omega$, such that $\mathrm{RS}(\mathcal{T}_{\varphi_n}) \geq \beta$, $n \in \omega$.

If α is a limit ordinal then $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \beta$ for any $\beta < \alpha$.

We set $\mathrm{RS}(\mathcal{T}) = \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \alpha$ and $\mathrm{RS}(\mathcal{T}) \not\geq \alpha + 1$.

If $\mathrm{RS}(\mathcal{T}) \geq \alpha$ for any α, we put $\mathrm{RS}(\mathcal{T}) = \infty$.

A family $\mathcal{T}$ is called *e-totally transcendental*, or *totally transcendental*, if $\mathrm{RS}(\mathcal{T})$ is an ordinal.

Similarly [11], for a nonempty family $\mathcal{T}$, we denote by $\mathcal{B}(\mathcal{T})$ the Boolean algebra consisting of all subfamilies $\mathcal{T}_\varphi$, where φ are sentences in the language $\Sigma(\mathcal{T})$.

Theorem 1.4 [1, 11]. *A nonempty family* $\mathcal{T}$ *is e-totally transcendental if and only if the Boolean algebra* $\mathcal{B}(\mathcal{T})$ *is superatomic.*

Proposition 1.5 [1]. *If an infinite family* $\mathcal{T}$ *does not have e-minimal subfamilies* $\mathcal{T}_\varphi$ *then* $\mathcal{T}$ *is not e-totally transcendental.*

If $\mathcal{T}$ is e-totally transcendental, with $\mathrm{RS}(\mathcal{T}) = \alpha \geq 0$, we define the *degree* $\mathrm{ds}(\mathcal{T})$ of $\mathcal{T}$ as the maximal number of pairwise inconsistent sentences φ_i such that $\mathrm{RS}(\mathcal{T}_{\varphi_i}) = \alpha$.

Proposition 1.6 [1]. *A family* $\mathcal{T}$ *is e-minimal if and only if* $\mathrm{RS}(\mathcal{T}) = 1$ *and* $\mathrm{ds}(\mathcal{T}) = 1$.

Proposition 1.7 [1]. *For any family* $\mathcal{T}$*,* $\mathrm{RS}(\mathcal{T}) = \mathrm{RS}(\mathrm{Cl}_E(\mathcal{T}))$*, and if* $\mathcal{T}$ *is nonempty and e-totally transcendental then* $\mathrm{ds}(\mathcal{T}) = \mathrm{ds}(\mathrm{Cl}_E(\mathcal{T}))$.

Recall the definition of the Cantor–Bendixson rank. It is defined on the elements of a topological space X by induction: $\mathrm{CB}_X(p) \geq 0$ for all $p \in X$; $\mathrm{CB}_X(p) \geq \alpha$ if and only if for any $\beta < \alpha$, p is an accumulation point of the points of CB_X-rank at least β. $\mathrm{CB}_X(p) = \alpha$ if and only if both $\mathrm{CB}_X(p) \geq \alpha$ and $\mathrm{CB}_X(p) \not\geq \alpha + 1$ hold; if such an ordinal α does not exist then $\mathrm{CB}_X(p) = \infty$. Isolated points of X are precisely those having rank 0, points of rank 1 are those which are isolated in the subspace of all non-isolated points, and so on. For a non-empty $C \subseteq X$ we define $\mathrm{CB}_X(C) = \sup\{\mathrm{CB}_X(p) \mid p \in C\}$; in this way $\mathrm{CB}_X(X)$ is defined and $\mathrm{CB}_X(\{p\}) = \mathrm{CB}_X(p)$ holds. If X is compact and C is closed in X then the sup is achieved: $\mathrm{CB}_X(C)$ is the maximum value of $\mathrm{CB}_X(p)$ for $p \in C$; there are finitely many points of maximum rank in C and the number of such points is the CB_X-*degree* of C, denoted by $n_X(C)$.

If X is countable and compact then $\mathrm{CB}_X(X)$ is a countable ordinal and every closed subset has ordinal-valued rank and finite CB_X-degree $n_X(X) \in \omega \setminus \{0\}$.

For any ordinal α the set $\{p \in X \mid \mathrm{CB}_X(p) \geq \alpha\}$ is called the α-th CB-*derivative* X_α of X.

Elements $p \in X$ with $\mathrm{CB}_X(p) = \infty$ form the *perfect kernel* X_∞ of X.

Clearly, $X_\alpha \supseteq X_{\alpha+1}$, $\alpha \in \mathrm{Ord}$, and $X_\infty = \bigcap\limits_{\alpha \in \mathrm{Ord}} X_\alpha$.

It is noticed in [1] that any e-totally transcendental family $\mathcal{T}$ defines a superatomic Boolean algebra $\mathcal{B}(\mathcal{T})$ with $\mathrm{RS}(\mathcal{T}) = \mathrm{CB}_{\mathcal{B}(\mathcal{T})}(B(\mathcal{T}))$, $\mathrm{ds}(\mathcal{T}) = n_{\mathcal{B}(\mathcal{T})}(B(\mathcal{T}))$, i.e., the pair $(\mathrm{RS}(\mathcal{T}), \mathrm{ds}(\mathcal{T}))$ consists of Cantor–Bendixson invariants for $\mathcal{B}(\mathcal{T})$ [12]. The algebra $\mathcal{B}(\mathcal{T})$ is the sentence algebra, i.e., the Lindenbaum–Tarski algebra, and the invariants $\mathrm{CB}_{\mathcal{B}(\mathcal{T})}(B(\mathcal{T}))$, $n_{\mathcal{B}(\mathcal{T})}(B(\mathcal{T}))$ can be obtained on a base of classification for sentence algebras [13].

By the definition for any e-totally transcendental family $\mathcal{T}$ each theory $T \in \mathcal{T}$ obtains the CB-rank $\mathrm{CB}_{\mathcal{T}}(T)$ starting with $\mathcal{T}$-isolated points T_0, of $\mathrm{CB}_{\mathcal{T}}(T_0) = 0$. We will denote the values $\mathrm{CB}_{\mathcal{T}}(T)$ by $\mathrm{RS}_{\mathcal{T}}(T)$ as the rank for the point T in the topological space on the E-closure $\mathrm{Cl}_E(\mathcal{T})$ of $\mathcal{T}$ which is defined with respect to $\Sigma(\mathcal{T})$-sentences.

Definition [1]. Let α be an ordinal. A family $\mathcal{T}$ of rank α is called *α-minimal* if for any sentence $\varphi \in \Sigma(\mathcal{T})$, $\mathrm{RS}(\mathcal{T}_\varphi) < \alpha$ or $\mathrm{RS}(\mathcal{T}_{\neg\varphi}) < \alpha$.

Proposition 1.8 [1]. (1) *A family $\mathcal{T}$ is 0-minimal if and only if $\mathcal{T}$ is a singleton.*

(2) *A family $\mathcal{T}$ is 1-minimal if and only if $\mathcal{T}$ is e-minimal.*

(3) *For any ordinal α a family $\mathcal{T}$ is α-minimal if and only if* $\mathrm{RS}(\mathcal{T}) = \alpha$ *and* $\mathrm{ds}(\mathcal{T}) = 1$.

Proposition 1.9 [1]. *For any family $\mathcal{T}$,* $\mathrm{RS}(\mathcal{T}) = \alpha$, *with* $\mathrm{ds}(\mathcal{T}) = n$, *if and only if $\mathcal{T}$ is represented as a disjoint union of subfamilies $\mathcal{T}_{\varphi_1}, \ldots, \mathcal{T}_{\varphi_n}$, for some pairwise inconsistent sentences $\varphi_1, \ldots, \varphi_n$, such that each $\mathcal{T}_{\varphi_i}$ is α-minimal.*

2. Calculi for families of theories. Links for sentence-definable and diagram-definable families

In this section we define calculi for families of theories, similar to first-order calculi for sentences, as well as discuss properties and links for these calculi.

For a family $\mathcal{T}$ and sentences φ and ψ we say that φ *$\mathcal{T}$-forces* ψ, written $\varphi \vdash_{\mathcal{T}} \psi$ if $\mathcal{T}_\varphi \subseteq \mathcal{T}_\psi$.

We put $\vdash_{\mathcal{T}} \psi$ if $\mathcal{T}_\psi = \mathcal{T}$, and $\varphi \vdash_{\mathcal{T}}$ if $\mathcal{T}_\varphi = \emptyset$. For $\vdash_{\mathcal{T}} \psi$ we say that ψ is *$\mathcal{T}$-provable*, and if $\varphi \vdash_{\mathcal{T}}$ then we say that φ is *$\mathcal{T}$-contradictory* or *$\mathcal{T}$-inconsistent*.

By the definition the relation $\vdash_{\mathcal{T}} \psi$ is equivalent to $\chi \vdash_{\mathcal{T}} \psi$ for any identically true sentence χ, and $\varphi \vdash_{\mathcal{T}}$ is equivalent to $\varphi \vdash_{\mathcal{T}} \theta$ for any identically false sentence θ. So below we consider only relations of form $\varphi \vdash_{\mathcal{T}} \psi$ and their natural modifications.

Ordinary axioms and rules for calculi of sentences can be naturally transformed for the relations $\varphi \vdash_{\mathcal{T}} \psi$ obtaining *$\mathcal{T}$-calculi*, i.e., calculi with respect to families $\mathcal{T}$.

Clearly, $\varphi \vdash_{\emptyset} \psi$ for any sentences φ and ψ. Therefore there are sentences φ and ψ such that $\varphi \vdash_{\mathcal{T}} \psi$ but $\varphi \not\vdash \psi$. Indeed, if φ and ψ are sentences in a language Σ satisfying $\vdash \varphi$ and $\not\vdash \psi$ then we have $\varphi \not\vdash \psi$ whereas $\varphi \vdash_{\emptyset} \psi$. Besides, for the set $\mathcal{T}_\Sigma$ of all theories in the language Σ and for $\mathcal{T} = (\mathcal{T}_\Sigma)_\psi$ we have $\varphi \vdash_{\mathcal{T}} \psi$. Additionally, for any sentence φ which does not belong to theories in a family $\mathcal{T}$, i.e., $\mathcal{T}_\varphi = \emptyset$, and for any sentence ψ we have $\varphi \vdash_{\mathcal{T}} \psi$.

The following obvious proposition asserts that the relation $\varphi \vdash_{\mathcal{T}} \psi$ is monotone under $\vdash$ and inclusion:

Proposition 2.1. *For any sentences $\varphi, \varphi', \psi, \psi'$ and families $\mathcal{T}, \mathcal{T}'$, if $\varphi' \vdash \varphi$, $\psi \vdash \psi'$, and $\mathcal{T}' \subseteq \mathcal{T}$ then $\varphi \vdash_{\mathcal{T}} \psi$ implies $\varphi' \vdash_{\mathcal{T}'} \psi'$.*

The following proposition asserts the finite character for the relations $\varphi \vdash_{\mathcal{T}} \psi$.

Proposition 2.2. *For any sentences φ, ψ and a family $\mathcal{T}$ of theories the following conditions are equivalent:*
(1) *$\varphi \vdash_{\mathcal{T}} \psi$;*
(2) *$\varphi \vdash_{\mathcal{T}_0} \psi$ for any finite $\mathcal{T}_0 \subseteq \mathcal{T}$;*
(3) *$\varphi \vdash_{\{T\}} \psi$ for any singleton $\{T\} \subseteq \mathcal{T}$.*

Proof. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) hold by Proposition 2.1.

(3) $\Rightarrow$ (1). In view of $\varphi \vdash_{\emptyset} \psi$ it suffices to show $\varphi \vdash_{\mathcal{T}} \psi$ for nonempty $\mathcal{T}$ having $\varphi \vdash_{\{T\}} \psi$ for any singleton $\{T\} \subseteq \mathcal{T}$. But if $T \in \mathcal{T}_\varphi$ then $T \in \{T\}_\varphi$ and using $\varphi \vdash_{\{T\}} \psi$ we obtain $T \in \{T\}_\psi$ implying $T \in \mathcal{T}_\psi$. Thus, $\varphi \vdash_{\mathcal{T}} \psi$. $\Box$

Proposition 2.3. *For any sentences φ and ψ in a language Σ the following conditions are equivalent:*
(1) *$\varphi \vdash \psi$;*
(2) *$\varphi \vdash_{\mathcal{T}_\Sigma} \psi$;*
(3) *$\varphi \vdash_{\mathcal{T}} \psi$ for any (finite) family (singleton) $\mathcal{T} \subseteq \mathcal{T}_\Sigma$;*
(4) *$\varphi \vdash_{\mathcal{T}} \psi$ for any (finite) family (singleton) $\mathcal{T}$;*
(5) *$T \cup \{\varphi\} \vdash \psi$ for any $T \in \mathcal{T}_\Sigma$.*

Proof. (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2) are obvious using Proposition 2.2.

(2) $\Rightarrow$ (1). Assume on contrary that $\varphi \vdash_{\mathcal{T}_\Sigma} \psi$ and $\varphi \not\vdash \psi$. Then $\varphi \wedge \neg\psi$ is consistent. Extending $\{\varphi \wedge \neg\psi\}$ till a complete theory T in the language Σ we obtain $T \in (\mathcal{T}_\Sigma)_\varphi$ and $T \notin (\mathcal{T}_\Sigma)_\psi$ contradicting $\varphi \vdash_{\mathcal{T}_\Sigma} \psi$.

(1) $\Rightarrow$ (4). If $\varphi \vdash \psi$ then for any theory T with $\varphi \in T$ we have $\psi \in T$, hence $\mathcal{T}_\varphi \subseteq \mathcal{T}_\psi$ for any family $\mathcal{T}$, i.e., $\varphi \vdash_{\mathcal{T}} \psi$.

(3) $\Leftrightarrow$ (5). $\varphi \vdash_{\{T\}} \psi$ means that $\varphi \in T$ implies $\psi \in T$. So if $\varphi \in T$ then $T \vdash \psi$ implying $T \cup \{\varphi\} \vdash \psi$. Otherwise if $\varphi \notin T$ then $\neg\varphi \in T$. Therefore we have $\{\varphi, \neg\varphi\} \vdash \psi$ implying $T \cup \{\varphi\} \vdash \psi$. Conversely, assuming on contrary $\varphi \not\vdash_{\{T\}} \psi$ we have $\varphi \in T$ and $\psi \notin T$, so $\neg\psi \in T$. Hence $T \cup \{\varphi\} \not\vdash \psi$ since T is complete theory and containing $\neg\psi$ it can not force ψ, i.e., T can not contain ψ. $\Box$

Definition. If $\mathcal{T}$ is a family of theories and Φ is a set of sentences, then we put $\mathcal{T}_\Phi = \bigcap\limits_{\varphi \in \Phi} \mathcal{T}_\varphi$ and the set $\mathcal{T}_\Phi$ is called (*type-* or *diagram-*)*definable* (by the set Φ) with respect to $\mathcal{T}$, or (*diagram-*)*$\mathcal{T}$-definable*, or simply *d-definable*.

By the definition we have the following properties:

0. Any d-definable subfamily of E-closed family $\mathcal{T}$ is again E-closed.

1. $\mathcal{T}_{\{\varphi\}} = \mathcal{T}_\varphi$.

2. $\mathcal{T}_\Phi = \{T \in \mathcal{T} \mid \Phi \subseteq T\}$.

3. $\mathcal{T}_\Phi = \mathcal{T}$ if and only if $\Phi \subseteq \cap\mathcal{T}$. In particular, $\mathcal{T}_\emptyset = \mathcal{T}$.

4. $\mathcal{T}_{\Phi\cup\Psi} = \mathcal{T}_\Phi \cap \mathcal{T}_\Psi$.

5. $\mathcal{T}_\Phi = (\mathcal{T}_\Phi)_\Psi$ for any Ψ consisting of sentences ψ with $\Phi \vdash \psi$. In particular, the operation $(\cdot)_\Phi$ is idempotent: $(\mathcal{T}_\Phi)_\Phi = \mathcal{T}_\Phi$.

6. $\mathcal{T}_{\{\varphi_1,\dots,\varphi_n\}} = \mathcal{T}_{\varphi_1\wedge\dots\wedge\varphi_n}$, i.e., definable sets $\mathcal{T}_\Phi$ by finite Φ are sentence-definable.

7. $\mathcal{T}_\Phi = \mathcal{T}_\Psi$, where Ψ is the closure of Φ under conjunctions.

By the latter property, studying d-definable sets, we will usually consider sets Φ closed under conjunctions. Moreover, by Property 5, considering d-definable families we can additionally assume that any Φ is closed under logical conclusions with respect to $\vdash$. It means that it suffices to assume that Φ corresponds a filter with respect to the family of d-definable subsets of $\mathcal{T}$.

8. For any sets Φ and Ψ containing all their logical conclusions, $\mathcal{T}_{\Phi\cap\Psi} = \mathcal{T}_\Phi \cup \mathcal{T}_\Psi$.

Indeed, if $T \in \mathcal{T}_{\Phi\cap\Psi}$ then $\Phi \cap \Psi \subseteq T$. Assuming $T \notin \mathcal{T}_\Phi \cup \mathcal{T}_\Psi$ we have $\Phi \not\subseteq T$ and $\Psi \not\subseteq T$. So there are sentences $\varphi \in \Phi \setminus T$ and $\psi \in \Psi \setminus T$. Then $\varphi \vee \psi \notin T$. But by conjecture, $\varphi \vee \psi \in \Phi \cap \Psi$ contradicting $\Phi \cap \Psi \subseteq T$. Conversely, if $T \in \mathcal{T}_\Phi \cup \mathcal{T}_\Psi$ then $\Phi \subseteq T$ or $\Psi \subseteq T$ implying $\Phi \cap \Psi \subseteq T$ and $T \in \mathcal{T}_{\Phi\cap\Psi}$.

9. For any $T \in \mathcal{T}$ and $\Phi \subseteq T$ with $\Phi \vdash \varphi$ for all $\varphi \in T$, $\mathcal{T}_\Phi = \{T\}$. So any set of axioms for T isolates T in $\mathcal{T}$. In particular, since T is an ultrafilter and axiomatized by itself, $\mathcal{T}_T = \{T\}$.

The following proposition gives obvious criteria for d-definable sets to be s-definable.

Proposition 2.4. *For any d-definable set $\mathcal{T} = \mathcal{T}_\Phi$, where Φ is closed under conjunctions, and a sentence $\varphi \in \Phi$ the following conditions are equivalent:*
(1) *$\mathcal{T}$ is s-definable by φ: $\mathcal{T} = \mathcal{T}_\varphi$;*
(2) *$\varphi \vdash_\mathcal{T} \psi$ for any $\psi \in \Phi$;*
(3) *each $\psi \in \Phi$ with $\psi \vdash \varphi$ satisfies $\mathcal{T}_\varphi = \mathcal{T}_\psi$;*
(3) *there are no $T \in \mathcal{T}$ containing $\varphi \wedge \neg\psi$ for any $\psi \in \Phi$.*

The sentence φ with $\mathcal{T}_\varphi \neq \emptyset$ and satisfying the conditions in Proposition 2.4 is called *$\mathcal{T}$-isolating*, *$\mathcal{T}$-principal* or *$\mathcal{T}$-complete* for Φ, and Φ is called *$\mathcal{T}$-isolated* or *$\mathcal{T}$-principal.*

By Proposition 2.3, $\mathcal{T}_\Sigma$-isolating sentences are isolating for Φ, in the ordinary sense. Besides, if Φ is forced by some $\varphi \in \Phi$ then for any family $\mathcal{T}$, Φ is $\mathcal{T}$-isolated, but not vice versa.

Clearly, each d-definable set $\mathcal{T}_\Phi$ equals the set $\mathcal{T}_\theta$, where $\theta = \bigwedge\Phi$ with possibly infinite conjunction and $\mathcal{T}_\theta$ is the set of all theories $T \in \mathcal{T}$ containing conjunctive members of θ.

By Property 8 finite unions of d-definable sets are again d-definable. Considering infinite unions $\mathcal{T}'$ of d-definable sets $\mathcal{T}_{\Phi_i}$, $i \in I$, we can represent them by sets of formulas with infinite disjunctions $\bigvee\limits_{i\in I}\varphi_i$, $\varphi_i \in \Phi_i$. We call these unions $\mathcal{T}'$ as d_∞-*definable* sets.

Now the definability for subfamilies of $\mathcal{T}$ can be extended for infinite unions, intersections and their complements. Notice that since all singletons $\{T\} \subseteq \mathcal{T}$ are d-definable, each subfamily $\mathcal{T}' \subseteq \mathcal{T}$ is d_∞-definable.

The relations $\varphi \vdash_\mathcal{T} \psi$ can be naturally spread to sets Φ and Ψ of sentences producing relations $\Phi \vdash_\mathcal{T} \Psi$ meaning $\mathcal{T}_\Phi \subseteq \mathcal{T}_\Psi$.

By Proposition 2.1 the relations $\Phi \vdash_\mathcal{T} \Psi$ are again monotone:

Proposition 2.5. *For any sets Φ, Φ', Ψ, Ψ' of sentences and families $\mathcal{T}, \mathcal{T}'$, if $\Phi' \vdash \Phi$, $\Psi \vdash \Psi'$, and $\mathcal{T}' \subseteq \mathcal{T}$ then $\Phi \vdash_\mathcal{T} \Psi$ implies $\Phi' \vdash_{\mathcal{T}'} \Psi'$.*

Proposition 2.2 implies the following:

Proposition 2.6. *For any sets Φ and Ψ of sentences and a family $\mathcal{T}$ of theories the following conditions are equivalent:*
(1) $\Phi \vdash_\mathcal{T} \Psi$;
(2) $\Phi \vdash_{\mathcal{T}_0} \Psi$ *for any finite* $\mathcal{T}_0 \subseteq \mathcal{T}$;
(3) $\Phi \vdash_{\{T\}} \Psi$ *for any singleton* $\{T\} \subseteq \mathcal{T}$.

Proposition 2.3 immediately implies

Proposition 2.7. *For any sets Φ and Ψ of sentences in a language Σ the following conditions are equivalent:*
(1) $\Phi \vdash \Psi$, *i.e., each sentence in Ψ is forced by some conjunction of sentences in Φ;*
(2) $\Phi \vdash_{\mathcal{T}_\Sigma} \Psi$;
(3) $\Phi \vdash_\mathcal{T} \Psi$ *for any* (*finite*) *family* (*singleton*) $\mathcal{T} \subseteq \mathcal{T}_\Sigma$;
(4) $\Phi \vdash_\mathcal{T} \Psi$ *for any* (*finite*) *family* (*singleton*) $\mathcal{T}$.

Extending the list for criteria of $\Phi \vdash_\mathcal{T} \Psi$ we have the following:

Theorem 2.8. *For any sets Φ and Ψ of sentences and a family $\mathcal{T}$ of theories the following conditions are equivalent:*
(1) $\Phi \vdash_\mathcal{T} \Psi$;
(2) $\Phi \vdash_{\mathrm{Cl}_E(\mathcal{T})} \Psi$.

Proof. Since $\mathcal{T} \subseteq \mathrm{Cl}_E(\mathcal{T})$ we have (2) $\Rightarrow$ (1) by Proposition 2.5.

(1) $\Rightarrow$ (2). Assume that $\Phi \vdash_\mathcal{T} \Psi$. It suffices to show that if $\varphi \in \Phi$, $\psi \in \Psi$ with $\mathcal{T}_\varphi \subseteq \mathcal{T}_\psi$ then $(\mathrm{Cl}_E(\mathcal{T}))_\varphi \subseteq (\mathrm{Cl}_E(\mathcal{T}))_\psi$. Let $T \in (\mathrm{Cl}_E(\mathcal{T}))_\varphi$ By the hypothesis we can assume that $T \in \mathrm{Cl}_E(\mathcal{T}) \setminus \mathcal{T}$ and using Proposition 1.1 we have infinite $\mathcal{T}_\chi$ for any $\chi \in T$. Since $\varphi \in T$, $(\mathcal{T}_\varphi)_\chi = \mathcal{T}_{\varphi\wedge\chi}$ are also infinite for any $\chi \in T$ and therefore $\mathcal{T}_\varphi \subseteq \mathcal{T}_\psi$ implies that all $(\mathcal{T}_\psi)_\chi$ are infinite. Thus again by Proposition 1.1, $T \in \mathrm{Cl}_E(\mathcal{T}_\psi) = (\mathrm{Cl}_E(\mathcal{T}))_\psi$. □

Theorem 2.8 immediately implies the following:

Corollary 2.9. *For any sets Φ and Ψ of sentences, and families $\mathcal{T}$, $\mathcal{T}'$, $\mathcal{T}''$ of theories such that $\mathcal{T}'$ generates $\mathrm{Cl}_E(\mathcal{T})$ and $\mathcal{T}' \subseteq \mathcal{T}'' \subseteq \mathrm{Cl}_E(\mathcal{T})$, the following conditions are equivalent:*
(1) $\Phi \vdash_\mathcal{T} \Psi$;
(2) $\Phi \vdash_{\mathcal{T}'} \Psi$;
(3) $\Phi \vdash_{\mathcal{T}''} \Psi$.

Remark 2.10. Notice that in general case Corollary 2.9 can not be extended to families $\mathcal{T}'' \not\subseteq \mathrm{Cl}_E(\mathcal{T})$. Indeed, taking any theory $T \notin \mathrm{Cl}_E(\mathcal{T})$ we have, by Proposition 1.1, a sentence $\chi \in T$ such that $(\mathrm{Cl}_E(\mathcal{T}))_\chi$ is finite. Since $T \notin (\mathrm{Cl}_E(\mathcal{T}))_\chi$

and $(\mathrm{Cl}_E(\mathcal{T}))_\chi$ is finite, there is a sentence $\theta \in T$ such that $(\mathrm{Cl}_E(\mathcal{T}))_\theta = \emptyset$. Thus for any inconsistence sentence φ we have $\theta \vdash_{\mathrm{Cl}_E(\mathcal{T})} \varphi$ whereas $\theta \not\vdash_{\mathrm{Cl}_E(\mathcal{T})\cup\{T\}} \varphi$. $\square$

The assertions above show that for any family $\mathcal{T}$ there are *calculi*, connected with ordinary calculi for first-order sentences [14], both for the relations $\varphi \vdash_{\mathcal{T}} \psi$ and $\Phi \vdash_{\mathcal{T}} \Psi$, which satisfy monotone properties, are reflexive ($\Phi \vdash_{\mathcal{T}} \Phi$) and transitive (if $\Phi \vdash_{\mathcal{T}} \Psi$ and $\Psi \vdash_{\mathcal{T}} \mathrm{X}$ then $\Phi \vdash_{\mathcal{T}} \mathrm{X}$).

Definition. Sets Φ and Ψ of sentences are called *$\mathcal{T}$-equivalent*, written $\Phi \equiv_{\mathcal{T}} \Psi$, if $\Phi \vdash_{\mathcal{T}} \Psi$ and $\Psi \vdash_{\mathcal{T}} \Phi$, i.e., $\mathcal{T}_\Phi = \mathcal{T}_\Psi$.

Sentences φ and ψ are called *$\mathcal{T}$-equivalent*, written $\varphi \equiv_{\mathcal{T}} \psi$, if $\{\varphi\} \equiv_{\mathcal{T}} \{\psi\}$.

Clearly, the relations $\equiv_{\mathcal{T}}$ are equivalent relations both for sentences and for sets of sentences.

Proposition 2.7 and Theorem 2.8 immediately implies the following:

Proposition 2.11. *For any sets Φ and Ψ of sentences in a language Σ the following conditions are equivalent:*

(1) $\Phi \vdash \Psi$ *and* $\Psi \vdash \Phi$*, i.e.,* Φ *and* Ψ *force each other;*
(2) $\Phi \equiv_{\mathcal{T}_\Sigma} \Psi$*;*
(3) $\Phi \equiv_{\mathcal{T}} \Psi$ *for any* (*finite*) *family* (*singleton*) $\mathcal{T} \subseteq \mathcal{T}_\Sigma$*;*
(4) $\Phi \equiv_{\mathcal{T}} \Psi$ *for any* (*finite*) *family* (*singleton*) $\mathcal{T}$*.*

Corollary 2.12. *For any sentences φ and ψ in a language Σ the following conditions are equivalent:*

(1) $\varphi \vdash \psi$ *and* $\psi \vdash \varphi$*;*
(2) $\varphi \equiv_{\mathcal{T}_\Sigma} \psi$*;*
(3) $\varphi \equiv_{\mathcal{T}} \psi$ *for any* (*finite*) *family* (*singleton*) $\mathcal{T} \subseteq \mathcal{T}_\Sigma$*;*
(4) $\varphi \equiv_{\mathcal{T}} \psi$ *for any* (*finite*) *family* (*singleton*) $\mathcal{T}$*.*

Theorem 2.8 implies

Corollary 2.13. *For any sets Φ and Ψ of sentences, and families $\mathcal{T}$, $\mathcal{T}'$, $\mathcal{T}''$ of theories such that $\mathcal{T}'$ generates $\mathrm{Cl}_E(\mathcal{T})$ and $\mathcal{T}' \subseteq \mathcal{T}'' \subseteq \mathrm{Cl}_E(\mathcal{T})$, the following conditions are equivalent:*

(1) $\Phi \equiv_{\mathcal{T}} \Psi$*;*
(2) $\Phi \equiv_{\mathcal{T}'} \Psi$*;*
(3) $\Phi \equiv_{\mathcal{T}''} \Psi$*.*

3. Compactness and E-closed families

Definition. A d-definable set $\mathcal{T}_\Phi$ is called *$\mathcal{T}$-consistent* if $\mathcal{T}_\Phi \neq \emptyset$, and $\mathcal{T}_\Phi$ is called *locally $\mathcal{T}$-consistent* if for any finite $\Phi_0 \subseteq \Phi$, $\mathcal{T}_{\Phi_0}$ is $\mathcal{T}$-consistent.

Clearly, locally $\mathcal{T}$-consistent $\mathcal{T}$-principal sets $\mathcal{T}_\Phi$ are $\mathcal{T}$-consistent.

Notice also that there are locally $\mathcal{T}$-consistent d-definable sets $\mathcal{T}_\Phi$ which are not $\mathcal{T}$-consistent. Indeed, let, for instance, $\mathcal{T}$ be an e-minimal family which does not contain its unique accumulation point T. Then by the definition of accumulation point, $\mathcal{T}_T$ is locally $\mathcal{T}$-consistent whereas $\mathcal{T}_T = \emptyset$.

The following *Compactness Theorem* shows that this effect does not occur for E-closed families.

Theorem 3.1. *For any nonempty E-closed family $\mathcal{T}$, every locally $\mathcal{T}$-consistent d-definable set $\mathcal{T}_\Phi$ is $\mathcal{T}$-consistent.*

Proof. If all neighbourhoods $\mathcal{T}_\varphi$, $\varphi \in \Phi$, contain same theory $T \in \mathcal{T}$ then $\mathcal{T}_\Phi$ is $\mathcal{T}$-consistent. So we can assume that Φ is infinite, closed under conjunctions, non-$\mathcal{T}$-principal, and for any $\varphi \in \Phi$, $\mathcal{T}_\varphi$ contains infinitely many theories in $\mathcal{T}$. Now we extend step-by-step the set Φ till a non-principal ultrafilter T of sentences of the language $\Sigma(\mathcal{T})$ such that each $\psi \in T$ satisfies $|\mathcal{T}_\psi| \geq \omega$. Applying Proposition 1.1 we obtain $T \in \mathrm{Cl}_E(\mathcal{T}) = \mathcal{T}$, and by $T \supset \Phi$ we have $T \in \mathcal{T}_\Phi$, i.e., $\mathcal{T}_\Phi$ is $\mathcal{T}$-consistent. □

Theories $T \in \mathcal{T}$ belonging to locally $\mathcal{T}$-consistent d-definable sets $\mathcal{T}_\Phi$ are called their *realizations*.

The following proposition, along Proposition 1.1 and compactness above, clarifies the mechanism of construction of $\mathrm{Cl}_E(\mathcal{T})$ via realizations of d-definable subfamilies of $\mathcal{T}$.

Proposition 3.2. *For any family $\mathcal{T}$, $\mathrm{Cl}_E(\mathcal{T})$ consists of elements of $\mathcal{T}$ and of accumulation points realizing locally $\mathcal{T}$-consistent d-definable sets $\mathcal{T}_\Phi$.*

Proof. By monotonicity property in Proposition 2.5, implying $\mathcal{T}_\Phi \supseteq \mathcal{T}_\Psi$ for $\Phi \subseteq \Psi$, it suffices to note that for any theory T, $T \in \mathrm{Cl}_E(\mathcal{T})$ if and only if T is a (unique) realization of locally $\mathcal{T}$-consistent d-definable subfamily $\mathcal{T}_T$. □

The following theorem gives a criterion of existence of d-definable family which is not s-definable.

Theorem 3.3. *For any E-closed family $\mathcal{T}$, there is a d-definable family $\mathcal{T}_\Phi$ which is not s-definable if and only if $\mathcal{T}$ is infinite.*

Proof. If $\mathcal{T}$ is finite then each theory $T \in \mathcal{T}$ is isolated by some sentence φ. So each nonempty subfamily of $\mathcal{T}$ is s-definable by some disjunction of the sentences φ. Thus, since the empty subfamily of $\mathcal{T}$ is s-definable, by an inconsistent sentence, then each d-definable family $\mathcal{T}_\Phi$ is s-definable.

Now we assume that $\mathcal{T}$ is infinite. By compactness, since $\mathcal{T}$ is E-closed and infinite, the set Φ of all sentences φ such that $|\mathcal{T}_{\neg\varphi}| = 1$ is $\mathcal{T}$-consistent. Taking an arbitrary theory $T \in \mathcal{T}_\Phi$ we obtain a d-definable singleton $\mathcal{T}_T = \{T\}$ which can not be s-definable by choice of Φ. □

Remark 3.4. Theorem 3.3 does not hold for families $\mathcal{T}$ which are not E-closed. Indeed, take an arbitrary e-minimal family $\mathcal{T}$, which does not contain its (unique) accumulation point T. Repeating arguments for the proof of Theorem 3.3 we find the set Φ which is locally $\mathcal{T}$-consistent but $\mathcal{T}_\Phi = \emptyset$ in view of $T \notin \mathcal{T}$. Since all s-definable subfamilies of $\mathcal{T}$ are either finite or cofinite, the only possibility for new d-definable subfamily of $\mathcal{T}$ is $\mathcal{T}_\Phi$. Since $\mathcal{T}_\Phi$ is empty, $\mathcal{T}$ does not have d-definable subfamilies which are not s-definable. □

4. Dynamics of ranks with respect to definable subfamilies of theories

Let $\mathcal{T}$ be a family of theories, Φ be a set of sentences, α be an ordinal $\leq \mathrm{RS}(\mathcal{T})$ or -1. The set Φ is called *α-ranking* for $\mathcal{T}$ if $\mathrm{RS}(\mathcal{T}_\Phi) = \alpha$. A sentence φ is called *α-ranking* for $\mathcal{T}$ if $\mathrm{RS}(\mathcal{T}_{\{\varphi\}}) = \alpha$.

The set Φ (the sentence φ) is called *ranking* for $\mathcal{T}$ if it is α-ranking for $\mathcal{T}$ with some α.

Definition [9]. For a family $\mathcal{T}$, a theory T is *$\mathcal{T}$-finitely axiomatizable*, or *finitely axiomatizable with respect to $\mathcal{T}$*, or *$\mathcal{T}$-relatively finitely axiomatizable*, if $\mathcal{T}_\varphi = \{T\}$ for some $\Sigma(\mathcal{T})$-sentence φ.

For a family $\mathcal{T}$ of a language Σ, a sentence φ of the language Σ is called *$\mathcal{T}$-complete* if φ isolates a unique theory in $\mathcal{T}$, i.e., $\mathcal{T}_\varphi$ is a singleton.

Proposition 4.1. (1) *A set Φ (a sentence φ) is (-1)-ranking for $\mathcal{T}$ if and only if $\mathcal{T} = \emptyset$ or Φ (respectively φ) is inconsistent with theories in $\mathcal{T}$.*

(2) *A set Φ (a sentence φ) is 0-ranking for $\mathcal{T}$, with $\mathrm{ds}(\mathcal{T}_\Phi) = n$, if and only if Φ (respectively φ) is consistent exactly with some $n \in \omega \setminus \{0\}$ theories in $\mathcal{T}$.*

(3) *Any 0-ranking sentence φ for $\mathcal{T}$, with $\mathrm{ds}(\mathcal{T}_\varphi) = n$, is $\mathcal{T}$-equivalent to a disjunction of n (pairwise inconsistent) $\mathcal{T}$-complete sentences.*

Proof. (1) and (2) immediately follow from the definition.

(3) In view of $\mathrm{RS}(\mathcal{T}_\varphi) = 0$ and $\mathrm{ds}(\mathcal{T}_\varphi) = n$ we have $\mathcal{T}_\varphi = \{T_1, \ldots, T_n\}$ for some distinct theories $T_1, \ldots, T_n \in \mathcal{T}$. Since the theories T_i are distinct, there are sentences $\psi_i \in T_i$ such that $\neg\psi_i \in T_j$ for $j \neq i$. Thus the formulas

$$(\varphi \wedge \psi_1 \wedge \neg\psi_2 \wedge \ldots \wedge \neg\psi_n),$$

$$\ldots,$$

$$(\varphi \wedge \neg\psi_1 \wedge \neg\psi_2 \wedge \ldots \wedge \neg\psi_{n-2} \wedge \psi_{n-1} \wedge \neg\psi_n),$$

$$(\varphi \wedge \neg\psi_1 \wedge \ldots \wedge \neg\psi_{n-1})$$

are $\mathcal{T}$-complete, pairwise inconsistent and such that their disjunction is $\mathcal{T}$-equivalent to φ. $\square$

Remark 4.2. By Proposition 4.1, if $T \in \mathcal{T}$ then $\Phi = T$ is 0-ranking, with $\mathcal{T}_T = \{T\}$. More generally, for any distinct $T_1, \ldots, T_n \in \mathcal{T}$ the set $T_1 \vee \ldots \vee T_n = \{\varphi_1 \vee \ldots \vee \varphi_n \mid \varphi_i \in T_i\}$ is 0-ranking, with $\mathrm{ds}(\mathcal{T}_{T_1 \vee \ldots \vee T_n}) = n$.

As shown in Remark 4.2 each finite subset $\mathcal{T}_0 \subseteq \mathcal{T}$ is d-definable, and Proposition 4.1 gives a characterization for $\mathcal{T}_0$ to be s-definable.

The following theorem produces a characterization for a subfamily $\mathcal{T}' \subseteq \mathcal{T}$ to be d-definable.

Theorem 4.3. *A subfamily $\mathcal{T}' \subseteq \mathcal{T}$ is d-definable in $\mathcal{T}$ if and only if $\mathcal{T}'$ is E-closed in $\mathcal{T}$, i.e., $\mathcal{T}' = \mathrm{Cl}_E(\mathcal{T}') \cap \mathcal{T}$.*

Proof. In view of Remark 4.2 we can assume that $\mathcal{T}'$ is infinite. Let $\mathcal{T}'$ be d-definable, i.e., $\mathcal{T}' = \mathcal{T}_\Phi$ for some set Φ. By Proposition 1.1, all theories in $\mathrm{Cl}_E(\mathcal{T}')$ contain the set Φ, i.e., $\mathrm{Cl}_E(\mathcal{T}') \cap \mathcal{T} \subseteq \mathcal{T}_\Phi$. Indeed, if a theory $T \in \mathrm{Cl}_E(\mathcal{T}')$ does not contain a sentence $\varphi \in \Phi$ then $\neg\varphi \in T$ and $(\mathcal{T}')_{\neg\varphi} = \emptyset$ contradicting $T \in \mathrm{Cl}_E(\mathcal{T}')$. Since $\mathcal{T}' \subseteq \mathrm{Cl}_E(\mathcal{T}') \cap \mathcal{T}$, we have $\mathcal{T}' = \mathrm{Cl}_E(\mathcal{T}') \cap \mathcal{T}$, i.e., the subfamily $\mathcal{T}'$ is E-closed in $\mathcal{T}$.

Now let the subfamily $\mathcal{T}'$ be E-closed in $\mathcal{T}$. Denote by Φ the set $\bigcap \mathcal{T}'$, i.e., the set of all $\Sigma(\mathcal{T})$-sentences belonging to all theories in $\mathcal{T}'$. Clearly, $\mathcal{T}' \subseteq \mathcal{T}_\Phi$. If $\mathcal{T}' \subset \mathcal{T}_\Phi$, i.e., there is $T \in \mathcal{T}_\Phi \setminus \mathcal{T}'$ then $T \notin \mathrm{Cl}_E(\mathcal{T}')$. Applying Proposition 1.1 we find a sentence $\varphi \in T$ such that $(\mathcal{T}')_\varphi$ is finite, say, $(\mathcal{T}')_\varphi = \{T_1, \ldots, T_n\}$. Since $T_i \neq T$ there are sentences $\psi_i \in T \setminus T_i$, $i = 1, \ldots, n$. For the sentence $\chi = \varphi \wedge \psi_1 \wedge \ldots \wedge \psi_n$ we have $\chi \in T$ and $(\mathcal{T}')_\chi = \emptyset$. It implies $\neg\chi \in \Phi$, contradicting $T \in \mathcal{T}_\Phi$. $\square$

The following proposition shows that s-definable subsets of a family $\mathcal{T}$ witnessing $\mathrm{RS}(\mathcal{T}) = \beta$ produce a hierarchy of α-ranking sentences for all ordinals $\alpha \leq \beta$.

Proposition 4.4. *For any ordinals $\alpha \leq \beta$, if* $\mathrm{RS}(\mathcal{T}) = \beta$ *then* $\mathrm{RS}(\mathcal{T}_\varphi) = \alpha$ *for some (α-ranking) sentence φ. Moreover, there are* $\mathrm{ds}(\mathcal{T})$ *pairwise $\mathcal{T}$-inconsistent β-ranking sentences for $\mathcal{T}$, and if $\alpha < \beta$ then there are infinitely many pairwise $\mathcal{T}$-inconsistent α-ranking sentences for $\mathcal{T}$.*

Proof. Since the Boolean algebra $F(\mathcal{T})$ is superatomic by Theorem 1.1, each $\mathcal{T}_\varphi$ belongs to a hierarchy with respect to the rank $\mathrm{RS}(\cdot)$ starting with singletons, e-minimal subfamilies, etc. Thus, each $\mathcal{T}_\varphi$ obtains a value $\mathrm{RS}(\mathcal{T}_\varphi) = \alpha$ in this hierarchy such that all $\alpha \leq \beta$ are witnessed by some $\mathcal{T}_\varphi$. By the definition of $\mathrm{RS}(\cdot)$, $\mathcal{T}$ can be divided onto $\mathrm{ds}(\mathcal{T})$ disjoint parts $\mathcal{T}_\varphi$ having the rank β. Again by the definition, if $\alpha < \beta$ then there are infinitely many pairwise $\mathcal{T}$-inconsistent α-ranking sentences for $\mathcal{T}$. □

By Proposition 4.4, for every family $\mathcal{T}$ with $\mathrm{RS}(\mathcal{T}) = \beta \geq 0$ the possibilities for $\mathrm{RS}(\mathcal{T}')$ with $\mathcal{T}' \subseteq \mathcal{T}$ are realized by s-definable subsets $\mathcal{T}_\varphi$ with $\mathrm{RS}(\mathcal{T}_\varphi) = \alpha$ for all $\alpha \leq \beta$. Thus the following natural question arises for families $\mathcal{T}$ which are not e-totally transcendental.

Question. *Let $\mathcal{T}$ be a family with* $\mathrm{RS}(\mathcal{T}) = \infty$*. What are the* RS*-possibilities for s-definable / d-definable subfamilies of $\mathcal{T}$?*

As shown in Remark 4.2 every finite subset of $\mathcal{T}$ is d-definable and 0-ranking. So in fact the question arises for $\alpha \geq 0$ with s-definable subfamilies, and for $\alpha > 0$ with d-definable subfamilies.

Partially answering the question we notice that obtaining an s-definable / d-definable subfamily $\mathcal{T}_\beta$ of $\mathcal{T}$ with $\mathrm{RS}(\mathcal{T}_\beta) = \beta \geq 0$ we have, by Proposition 4.4, s-definable / d-definable subfamilies $\mathcal{T}_\alpha$ of $\mathcal{T}$ with $\mathrm{RS}(\mathcal{T}_\alpha) = \alpha$, for all ordinals $\alpha \leq \beta$. Thus, the required ordinals α form an initial segment.

Illustrating the question we notice that, in some more or less general cases, the possibility for $\alpha = 0$ with s-definable subfamilies can be realized:

Remark 4.5. If a family $\mathcal{T}$ has an α-ranking sentence, for $\alpha \geq 0$, it does not imply that $\mathcal{T}$ is e-totally transcendental. Indeed, any family $\mathcal{T}$, for instance, of functional language, e-totally transcendental or not, and with a theory T of an one-element algebra has a 0-ranking sentence φ saying that the universe is a singleton. Clearly, $\mathcal{T}_\varphi = \{T\}$.

At the same time there are many examples of families of theories without nonempty s-definable e-totally transcendental subfamilies. Indeed, taking, for instance, a family $\mathcal{T}_\Sigma$ of all theories in a language Σ containing infinitely many predicate symbols, we can not control, by a sentence, links between all predicates. In particular, there are at least continuum many possibilities arbitrarily varying empty/nonempty predicates. These variations produce unbounded ranks for any nonempty s-definable subfamilies $\mathcal{T}_\varphi$ implying $\mathrm{RS}(\mathcal{T}_\varphi) = \infty$.

The following theorem gives an answer to the question for d-definable subfamilies of theories in countable languages. The arguments for this answer can be naturally spread for arbitrary languages.

Theorem 4.6. *Let $\mathcal{T}$ be a family of a countable language Σ and with* $\mathrm{RS}(\mathcal{T}) = \infty$*, $\alpha \in \{0, 1\}$, $n \in \omega \setminus \{0\}$. Then there is a d-definable subfamily $\mathcal{T}_\Phi$ such that* $\mathrm{RS}(\mathcal{T}_\Phi) = \alpha$ *and* $\mathrm{ds}(\mathcal{T}_\Phi) = n$.

Proof. We fix a family $\mathcal{T}$ of countable language Σ, with $\mathrm{RS}(\mathcal{T})=\infty$, a countable ordinal α, and $n\in\omega\setminus\{0\}$. By Theorem 4.3 it suffices to find an E-closed subfamily $\mathcal{T}'$ in $\mathcal{T}$ with $\mathrm{RS}(\mathcal{T}')=1$ and $\mathrm{ds}(\mathcal{T}')=n$.

If $\alpha=0$ then $\mathcal{T}'$ exists by Remark 4.2.

If $\alpha=1$ we take n pairwise inconsistent sentences φ_i, $i=1,\dots,n$, such that $\mathrm{RS}(\mathcal{T}_{\varphi_i})=\infty$ and for each $\mathcal{T}_{\varphi_i}$ find E-closed, in $\mathcal{T}_{\varphi_i}$ (and so in $\mathcal{T}$), e-minimal subfamily $\mathcal{T}'_i$ in the following way. We enumerate the set of all Σ-sentences which force φ_i: ψ_{ik}, $k\in\omega$, and form $\mathcal{T}'_i$ step-by-step with respect to that enumeration using the following subfamilies $\mathcal{T}_{ik}$ of $\mathcal{T}_{\varphi_i}$ with $\mathcal{T}_{ik}\supseteq\mathcal{T}_{i,k+1}$.

At the initial step if $\mathcal{T}_{\psi_{i0}}$ is cofinite in $\mathcal{T}_{\varphi_i}$, we set $\mathcal{T}_{i0}=\mathcal{T}_{\psi_{i0}}$, if $\mathcal{T}_{\varphi_i}=\mathcal{T}_{\psi_{i0}}$, and $\mathcal{T}_{i0}=\mathcal{T}_{\psi_{i0}}\cup\{T_0\}$, with an arbitrary theory $T^i_0\in\mathcal{T}_i\setminus\mathcal{T}_{\psi_{i0}}$, if $\mathcal{T}_i\neq\mathcal{T}_{\psi_{i0}}$. If $\mathcal{T}_{\psi_{i0}}$ is co-infinite we repeat the process replacing ψ_{i0} by $\varphi_i\wedge\neg\psi_{i0}$: for infinite $\mathcal{T}_{\varphi_i\wedge\neg\psi_{i0}}$ instead of $\mathcal{T}_{\psi_{i0}}$.

Let at the step k a family $\mathcal{T}_{ik}$ is already formed with some theories $T^i_0,\dots,T^i_r$ added to families $\mathcal{T}_{\psi_{is}}$ or $\mathcal{T}_{\varphi_i\wedge\neg\psi_{is}}$. Now we consider the sentence $\psi_{i,k+1}$. If $(\mathcal{T}_{ik})_{\psi_{i,k+1}}$ is cofinite in $\mathcal{T}_{ik}$, we set $\mathcal{T}_{i,k+1}=\mathcal{T}_{ik}$, if $\mathcal{T}_{ik}=(\mathcal{T}_{ik})_{\psi_{i,k+1}}$ modulo $\{T^i_0,\dots,T^i_r\}$, and $\mathcal{T}_{i,k+1}=(\mathcal{T}_{ik})_{\psi_{i,k+1}}\cup\{T^i_{r+1}\}$, with an arbitrary theory $T^i_{r+1}\in\mathcal{T}_{i,k+1}\setminus((\mathcal{T}_{ik})_{\psi_{i,k+1}}\cup\{T^i_0,\dots,T^i_r\})$, if $\mathcal{T}_{ik}\neq(\mathcal{T}_{ik})_{\psi_{i,k+1}}$ modulo $\{T^i_0,\dots,T^i_r\}$. If $(\mathcal{T}_{ik})_{\psi_{i,k+1}}$ is co-infinite we repeat the process for infinite $(\mathcal{T}_{ik})_{\varphi_i\wedge\neg\psi_{i,k+1}}$.

By the construction the subfamilies $\mathcal{T}'_i$ consisting of the theories $T^i_0,\dots,T^i_r,\dots$, are infinite and can not be divided into two infinite parts by Σ-sentences. Indeed, $\mathcal{T}'_i$ is infinite because each set $\{T^i_0,\dots,T^i_r\}$ is extended in some step by some new theory since $\mathcal{T}_{ik}\neq(\mathcal{T}_{ik})_{\psi_{i,k+1}}$ modulo $\{T^i_0,\dots,T^i_r\}$ for some $\psi_{i,k+1}$ negating all theories in $\{T^i_0,\dots,T^i_r\}$ and some theory in $\mathcal{T}_{ik}$. The subfamilies $\mathcal{T}'_i$ are e-minimal since each Σ-sentence is equivalent to some ψ_{ik} modulo φ_i and each ψ_{ik} can divide only $\mathcal{T}_{i0},\dots,\mathcal{T}_{i,k-1}$ modulo $\{T^i_0,\dots,T^i_r\}$.

Thus, the subfamilies $\mathcal{T}'_i$ of $\mathcal{T}$ are e-minimal, $i=1,\dots,n$. By Proposition 1.6 we have $\mathrm{RS}(\mathcal{T}'_i)=1$ and $\mathrm{ds}(\mathcal{T}'_i)=1$, and by Proposition 1.7 we can assume that the families $\mathcal{T}'_i$ are E-closed in $\mathcal{T}$. Hence, for $\mathcal{T}'=\mathcal{T}'_1\cup\ldots\cup\mathcal{T}'_n$, which is d-definable by Theorem 4.3, we have $\mathrm{RS}(\mathcal{T}')=1$ and $\mathrm{ds}(\mathcal{T}')=n$. $\square$

Remark 4.7. Notice that the arguments in the proof of Theorem 4.6 do not work for $\alpha\geq 2$ since taking infinitely many disjoint s-definable infinite subfamilies $\mathcal{T}_{\varphi_i}$ we can not guarantee that $\varphi_i\not\vdash_{\mathcal{T}}\psi_j$ for infinitely many $\mathcal{T}$-disjoint sentences ψ_j. Thus constructing d-definable e-minimal subfamilies $\mathcal{T}_i$ of $\mathcal{T}_{\varphi_i}$ it is possible to obtain $\mathrm{RS}\left(\bigcup\limits_i\mathcal{T}_i\right)\geq 3$, not $\mathrm{RS}\left(\bigcup\limits_i\mathcal{T}_i\right)=2$.

At the same time, constructing countably many d-definable subfamilies $\mathcal{T}_i$ of $\mathcal{T}_{\varphi_i}$, $i\in\omega$, with pairwise inconsistent φ_i, we can choose some infinite $I\subseteq\omega$, such that accumulation points T_i for $\mathcal{T}_i$, $i\in I$, form an e-minimal family. Thus, possibly loosing the d-definability we obtain a d_∞-definable subfamily $\mathcal{T}'=\bigcup\limits_{i\in I}\mathcal{T}_i$ with $\mathrm{RS}(\mathcal{T}')=2$ and $\mathrm{ds}(\mathcal{T}')=1$. Taking some n disjoint $\mathcal{T}'$ we obtain a subfamily $\mathcal{T}''$, being the union of $\mathcal{T}'$, with $\mathrm{RS}(\mathcal{T}')=2$ and $\mathrm{ds}(\mathcal{T}')=n$.

Now we can continue the process for greater countable ordinals α obtaining a d_∞-definable subfamily $\mathcal{T}^*\subset\mathcal{T}$ with $\mathrm{RS}(\mathcal{T}^*)=\alpha$ and $\mathrm{ds}(\mathcal{T}^*)=n$ for given $n\in\omega\setminus\{0\}$. $\square$

Theorem 4.6 and Remark 4.7 imply the following:

Theorem 4.8. *Let $\mathcal{T}$ be a family of a countable language Σ and with* $\mathrm{RS}(\mathcal{T}) = \infty$*, α be a countable ordinal, $n \in \omega \setminus \{0\}$. Then there is a d_∞-definable subfamily $\mathcal{T}^* \subset \mathcal{T}$ such that* $\mathrm{RS}(\mathcal{T}^*) = \alpha$ *and* $\mathrm{ds}(\mathcal{T}^*) = n$.

Example 4.9. Let $\mathcal{T}_\Sigma$ be a family of all theories of a countable language Σ with $\mathrm{RS}(\mathcal{T}^*) = \infty$ [2], say of unary predicates Q_n, $n \in \omega$. Taking a countable d-definable subfamily $\mathcal{T} \subset \mathcal{T}_\Sigma$ with either empty or complete predicates Q_n such that complete predicates in $\mathcal{T}$ are linearly ordered and indexes for complete predicates form an infinite set $I \subset \omega$ with infinite $\omega \setminus I$ we can assume that $\mathcal{T}$ is e-minimal has unique accumulation point witnessing $\mathrm{RS}(\mathcal{T}) = 1$ and $\mathrm{ds}(\mathcal{T}) = 1$. Taking indexes in $\omega \setminus I$ we can define countably many disjoint e-minimal d-definable subfamilies $\mathcal{T}_k$ with unique accumulation point for the set of all accumulation points of $\mathcal{T}_k$ witnessing $\mathrm{RS} = 2$ and $\mathrm{ds} = 1$. Now applying Theorem 4.8 we can continue the process obtaining $\mathrm{RS} = \alpha$ and $\mathrm{ds} = n$ for arbitrary countable ordinal α and $n \in \omega \setminus \{0\}$. □

Definition. An α-ranking set Φ for $\mathcal{T}$, and $\mathcal{T}_\Phi$ are called *$\mathcal{T}$-irreducible* if for any $\mathcal{T}$-inconsistent $\Psi, \mathrm{X} \supseteq \Phi$, i.e., with $\mathcal{T}_\Psi \cap \mathcal{T}_\mathrm{X} \supseteq \mathcal{T}_\Phi$, $\mathrm{RS}(\mathcal{T}_\Psi) < \alpha$ or $\mathrm{RS}(\mathcal{T}_\mathrm{X}) < \alpha$. An α-ranking sentence φ for $\mathcal{T}$, and $\mathcal{T}_\varphi$ are called *$\mathcal{T}$-irreducible* if the singleton $\{\varphi\}$ is $\mathcal{T}$-irreducible.

If $\mathcal{T}$ is fixed, $\mathcal{T}$-irreducible sets are called simply *irreducible*.

By the definition each $\mathcal{T}$-inconsistent set Φ, with $\mathcal{T}_\Phi = \emptyset$, is irreducible, as well as singletons $\mathcal{T}_\Phi$.

Moreover, nonempty E-closed families $\mathcal{T}_\Phi$ of rank α are irreducible if and only if $\mathrm{ds}(\mathcal{T}_\Phi) = 1$.

Indeed, if $\mathcal{T}_\Phi$ is irreducible it can not be divided by a sentence into two parts of rank α implying $\mathrm{ds}(\mathcal{T}_\Phi) = 1$. Conversely, having $\mathcal{T}$-inconsistent $\Psi, \mathrm{X} \supseteq \Phi$, with $\mathcal{T}_\Psi \cap \mathcal{T}_\mathrm{X} \supseteq \mathcal{T}_\Phi$, $\mathrm{RS}(\mathcal{T}_\Psi) = \alpha$ and $\mathrm{RS}(\mathcal{T}_\mathrm{X}) = \alpha$, we obtain, by compactness, some $\mathcal{T}$-inconsistent $\psi \in \Psi$ and $\chi \in \mathrm{X}$ such that $\mathrm{RS}((\mathcal{T}_\Phi)_\psi) = \alpha$ and $\mathrm{RS}((\mathcal{T}_\Phi)_\chi) = \alpha$ contradicting $\mathrm{ds}(\mathcal{T}_\Phi) = 1$.

Since each family $\mathcal{T}$ with $\mathrm{RS}(\mathcal{T}) = \alpha \geq 0$ has a finite degree $\mathrm{ds}(\mathcal{T}) = n$, there are pairwise inconsistent sentences $\varphi_1, \ldots, \varphi_n$ such that $\mathcal{T} = \mathcal{T}_{\varphi_1} \dot\cup \ldots \dot\cup \mathcal{T}_{\varphi_n}$, $\mathrm{RS}(\mathcal{T}_{\varphi_i}) = \alpha$ and $\mathrm{ds}(\mathcal{T}_{\varphi_i}) = 1$, $i = 1, \ldots, n$.

Thus, all e-totally transcendental E-closed families and, in particular, d-definable α-ranking E-closed families are reduced to irreducible ones:

Proposition 4.10. *Any e-totally transcendental E-closed family $\mathcal{T}$ is represented as a finite disjoint union of s-definable irreducible subfamilies of rank* $\alpha = \mathrm{RS}(\mathcal{T})$.

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Nurlan Darkhanovich Markhabatov
Novosibirsk State Technical University,
20, K. Marx ave.,
Novosibirsk, 630073, Russia.
E-mail address: nur_24.08.93@mail.ru

Sergey Vladimirovich Sudoplatov
Sobolev Institute of Mathematics,
4, Acad. Koptyug ave.,
Novosibirsk, 630090, Russia
Novosibirsk State Technical University,
20, K. Marx ave.,
Novosibirsk, 630073, Russia
Novosibirsk State University,
1, Pirogova str.,
Novosibirsk, 630090, Russia
E-mail address: sudoplat@math.nsc.ru