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APPROXIMATIONS OF THEORIES

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Abstract. We study approximations of theories both in general context and with respect to some natural classes of theories. Some kinds of approximations are considered, connections with finitely axiomatizable theories and minimal generating sets of theories as well as their e-spectra are found. e-categorical approximating families are introduced and characterized.

Keywords: approximation of theory, combination of structures, closure, finitely axiomatizable theory, e-spectrum.

Approximations of structures and theories as well as closures with respect to these approximations were studied in a series of papers, both implicitly [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] and explicitly [11, 12, 13, 14, 15, 16]. They are connected with the technique for finitely axiomatizable theories [17, 18, 19, 20, 21, 22, 23, 24]. Some kinds of model-theoretic approximations are considered in [25].

The aim of the paper is to introduce and investigate approximations of theories both in general context and with respect to some natural classes of theories.

The paper is organized as follows. In Section 1 we collect preliminary definitions and assertions. In Section 2 we define approximations relative given family $\mathcal{T}$ of theories and characterize the property "to be $\mathcal{T}$-approximated". In Section 3 we connect approximable theories with finite axiomatizable ones, introduce the notion of $\mathcal{T}$-relatively finite axiomatizability and characterize this notion. In Section 4 we consider λ-approximable theories, i.e., theories generated by families of theories

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such that these families have the cardinality λ, and characterize the property of λ-approximation. Approximations by almost language uniform theories are considered in Section 5. A characterization for approximating subfamilies and lower bounds for e-spectra are proved in Section 6. In Section 7, e-categorical approximating families are introduced and characterized.

1. Preliminaries

Throughout the paper we consider complete first-order theories T in predicate languages $\Sigma(T)$ and use the following terminology in [11, 12, 13, 14, 15, 16].

Let $P = (P_i)_{i\in I}$, be a family of nonempty unary predicates, $(\mathcal{A}_i)_{i\in I}$ be a family of structures such that P_i is the universe of $\mathcal{A}_i$, $i \in I$, and the symbols P_i are disjoint with languages for the structures $\mathcal{A}_j$, $j \in I$. The structure $\mathcal{A}_P \rightleftharpoons \bigcup\limits_{i\in I} \mathcal{A}_i$ expanded by the predicates P_i is the P-*union* of the structures $\mathcal{A}_i$, and the operator mapping $(\mathcal{A}_i)_{i\in I}$ to $\mathcal{A}_P$ is the P-*operator*. The structure $\mathcal{A}_P$ is called the P-*combination* of the structures $\mathcal{A}_i$ and denoted by $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ if $\mathcal{A}_i = (\mathcal{A}_P \upharpoonright A_i) \upharpoonright \Sigma(\mathcal{A}_i)$, $i \in I$. Structures $\mathcal{A}'$, which are elementary equivalent to $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$, will be also considered as P-combinations.

Clearly, all structures $\mathcal{A}' \equiv \mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ are represented as unions of their restrictions $\mathcal{A}'_i = (\mathcal{A}' \upharpoonright P_i) \upharpoonright \Sigma(\mathcal{A}_i)$ if and only if the set $p_\infty(x) = \{\neg P_i(x) \mid i \in I\}$ is inconsistent. If $\mathcal{A}' \neq \mathrm{Comb}_P(\mathcal{A}'_i)_{i\in I}$, we write $\mathcal{A}' = \mathrm{Comb}_P(\mathcal{A}'_i)_{i\in I\cup\{\infty\}}$, where $\mathcal{A}'_\infty = \mathcal{A}' \upharpoonright \bigcap\limits_{i\in I} \overline{P_i}$, maybe applying Morleyzation. Moreover, we write

$$\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I\cup\{\infty\}}$$

for $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ with the empty structure $\mathcal{A}_\infty$.

Note that if all predicates P_i are disjoint, a structure $\mathcal{A}_P$ is a P-combination and a disjoint union of structures $\mathcal{A}_i$. In this case the P-combination $\mathcal{A}_P$ is called *disjoint*. Clearly, for any disjoint P-combination $\mathcal{A}_P$, $\mathrm{Th}(\mathcal{A}_P) = \mathrm{Th}(\mathcal{A}'_P)$, where $\mathcal{A}'_P$ is obtained from $\mathcal{A}_P$ replacing $\mathcal{A}_i$ by pairwise disjoint $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. Thus, in this case, similar to structures the P-operator works for the theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ producing the theory $T_P = \mathrm{Th}(\mathcal{A}_P)$, being P-*combination* of T_i, which is denoted by $\mathrm{Comb}_P(T_i)_{i\in I}$.

For an equivalence relation E replacing disjoint predicates P_i by E-classes we get the structure $\mathcal{A}_E$ being the E-*union* of the structures $\mathcal{A}_i$. In this case the operator mapping $(\mathcal{A}_i)_{i\in I}$ to $\mathcal{A}_E$ is the E-*operator*. The structure $\mathcal{A}_E$ is also called the E-*combination* of the structures $\mathcal{A}_i$ and denoted by $\mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$; here $\mathcal{A}_i = (\mathcal{A}_E \upharpoonright A_i) \upharpoonright \Sigma(\mathcal{A}_i)$, $i \in I$. Similar above, structures $\mathcal{A}'$, which are elementary equivalent to $\mathcal{A}_E$, are denoted by $\mathrm{Comb}_E(\mathcal{A}'_j)_{j\in J}$, where $\mathcal{A}'_j$ are restrictions of $\mathcal{A}'$ to its E-classes. The E-operator works for the theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ producing the theory $T_E = \mathrm{Th}(\mathcal{A}_E)$, being E-*combination* of T_i, which is denoted by $\mathrm{Comb}_E(T_i)_{i\in I}$ or by $\mathrm{Comb}_E(\mathcal{T})$, where $\mathcal{T} = \{T_i \mid i \in I\}$.

Clearly, $\mathcal{A}' \equiv \mathcal{A}_P$ realizing $p_\infty(x)$ is not elementary embeddable into $\mathcal{A}_P$ and can not be represented as a disjoint P-combination of $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. At the same time, there are E-combinations such that all $\mathcal{A}' \equiv \mathcal{A}_E$ can be represented as E-combinations of some $\mathcal{A}'_j \equiv \mathcal{A}_i$. We call this representability of $\mathcal{A}'$ to be the E-*representability*.

If there is $\mathcal{A}' \equiv \mathcal{A}_E$ which is not E-representable, we have the E'-representability replacing E by E' such that E' is obtained from E adding equivalence classes with models for all theories T, where T is a theory of a restriction $\mathcal{B}$ of a structure $\mathcal{A}' \equiv \mathcal{A}_E$ to some E-class and $\mathcal{B}$ is not elementary equivalent to the structures $\mathcal{A}_i$. The resulting structure $\mathcal{A}_{E'}$ (with the E'-representability) is a *e-completion*, or a *e-saturation*, of $\mathcal{A}_E$. The structure $\mathcal{A}_{E'}$ itself is called *e-complete*, or *e-saturated*, or *e-universal*, or *e-largest*.

For a structure $\mathcal{A}_E$ the number of *new* structures with respect to the structures $\mathcal{A}_i$, i. e., of the structures $\mathcal{B}$ which are pairwise elementary non-equivalent and elementary non-equivalent to the structures $\mathcal{A}_i$, is called the *e-spectrum* of $\mathcal{A}_E$ and denoted by e-$\mathrm{Sp}(\mathcal{A}_E)$. The value $\sup\{e\text{-}\mathrm{Sp}(\mathcal{A}') \mid \mathcal{A}' \equiv \mathcal{A}_E\}$ is called the *e-spectrum* of the theory $\mathrm{Th}(\mathcal{A}_E)$ and denoted by e-$\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_E))$. If structures $\mathcal{A}_i$ represent theories T_i of a family $\mathcal{T}$, consisting of T_i, $i \in I$, then the e-spectrum e-$\mathrm{Sp}(\mathcal{A}_E)$ is denoted by e-$\mathrm{Sp}(\mathcal{T})$.

If $\mathcal{A}_E$ does not have E-classes $\mathcal{A}_i$, which can be removed, with all E-classes $\mathcal{A}_j \equiv \mathcal{A}_i$, preserving the theory $\mathrm{Th}(\mathcal{A}_E)$, then $\mathcal{A}_E$ is called *e-prime*, or *e-minimal*.

For a structure $\mathcal{A}' \equiv \mathcal{A}_E$ we denote by $\mathrm{TH}(\mathcal{A}')$ the set of all theories $\mathrm{Th}(\mathcal{A}_i)$ of E-classes $\mathcal{A}_i$ in $\mathcal{A}'$.

By the definition, an e-minimal structure $\mathcal{A}'$ consists of E-classes with a minimal set $\mathrm{TH}(\mathcal{A}')$. If $\mathrm{TH}(\mathcal{A}')$ is the least for models of $\mathrm{Th}(\mathcal{A}')$ then $\mathcal{A}'$ is called *e-least*.

Definition [12]. Let $\overline{\mathcal{T}}_\Sigma$ be the set of all complete elementary theories of a relational language Σ. For a set $\mathcal{T} \subset \overline{\mathcal{T}}_\Sigma$ we denote by $\mathrm{Cl}_E(\mathcal{T})$ the set of all theories $\mathrm{Th}(\mathcal{A})$, where $\mathcal{A}$ is a structure of some E-class in $\mathcal{A}' \equiv \mathcal{A}_E$, $\mathcal{A}_E = \mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$, $\mathrm{Th}(\mathcal{A}_i) \in \mathcal{T}$. As usual, if $\mathcal{T} = \mathrm{Cl}_E(\mathcal{T})$ then $\mathcal{T}$ is said to be *E-closed*.

The operator Cl_E of E-closure can be naturally extended to the classes $\mathcal{T} \subset \overline{\mathcal{T}}$, where $\overline{\mathcal{T}}$ is the union of all $\overline{\mathcal{T}}_\Sigma$ as follows: $\mathrm{Cl}_E(\mathcal{T})$ is the union of all $\mathrm{Cl}_E(\mathcal{T}_0)$ for subsets $\mathcal{T}_0 \subseteq \mathcal{T}$, where new language symbols with respect to the theories in $\mathcal{T}_0$ are empty.

For a set $\mathcal{T} \subset \overline{\mathcal{T}}$ of theories in a language Σ and for a sentence φ with $\Sigma(\varphi) \subseteq \Sigma$ we denote by $\mathcal{T}_\varphi$ the set $\{T \in \mathcal{T} \mid \varphi \in T\}$. Any set $\mathcal{T}_\varphi$ is called a *φ-neighbourhood*, or simply a *neighbourhood*, for $\mathcal{T}$.

Proposition 1.1 [12]. *If $\mathcal{T} \subset \overline{\mathcal{T}}$ is an infinite set and $T \in \overline{\mathcal{T}} \setminus \mathcal{T}$ then $T \in \mathrm{Cl}_E(\mathcal{T})$ (i.e., T is an accumulation point for $\mathcal{T}$ with respect to E-closure Cl_E) if and only if for any formula $\varphi \in T$ the set $\mathcal{T}_\varphi$ is infinite.*

If T is an accumulation point for $\mathcal{T}$ then we also say that T is an *accumulation point* for $\mathrm{Cl}_E(\mathcal{T})$.

Theorem 1.2 [12]. *For any sets $\mathcal{T}_0, \mathcal{T}_1 \subset \overline{\mathcal{T}}$, $\mathrm{Cl}_E(\mathcal{T}_0 \cup \mathcal{T}_1) = \mathrm{Cl}_E(\mathcal{T}_0) \cup \mathrm{Cl}_E(\mathcal{T}_1)$.*

Definition [12]. Let $\mathcal{T}_0$ be a closed set in a topological space $(\mathcal{T}, \mathcal{O}_E(\mathcal{T}))$, where $\mathcal{O}_E(\mathcal{T}) = \{\mathcal{T} \setminus \mathrm{Cl}_E(\mathcal{T}') \mid \mathcal{T}' \subseteq \mathcal{T}\}$. A subset $\mathcal{T}'_0 \subseteq \mathcal{T}_0$ is said to be *generating* if $\mathcal{T}_0 = \mathrm{Cl}_E(\mathcal{T}'_0)$. The generating set $\mathcal{T}'_0$ (for $\mathcal{T}_0$) is *minimal* if $\mathcal{T}'_0$ does not contain proper generating subsets. A minimal generating set $\mathcal{T}'_0$ is *least* if $\mathcal{T}'_0$ is contained in each generating set for $\mathcal{T}_0$.

Theorem 1.3 [12]. *If $\mathcal{T}'_0$ is a generating set for a E-closed set $\mathcal{T}_0$ then the following conditions are equivalent:*

(1) *$\mathcal{T}'_0$ is the least generating set for $\mathcal{T}_0$;*

(2) *$\mathcal{T}'_0$ is a minimal generating set for $\mathcal{T}_0$;*

(3) *any theory in $\mathcal{T}'_0$ is isolated by some set $(\mathcal{T}'_0)_\varphi$, i.e., for any $T \in \mathcal{T}'_0$ there is $\varphi \in T$ such that $(\mathcal{T}'_0)_\varphi = \{T\}$;*

(4) *any theory in* $\mathcal{T}_0'$ *is isolated by some set* $(\mathcal{T}_0)_\varphi$, *i.e., for any* $T\in\mathcal{T}_0'$ *there is* $\varphi\in T$ *such that* $(\mathcal{T}_0)_\varphi=\{T\}$.

Proposition 1.4. *Any family* $\mathcal{T}$ *of theories can be expanded till a family* $\mathcal{T}'$ *with the least generating set.*

Proof. By Theorem 1.3 it suffices to introduce, for each theory $T\in\mathcal{T}$, a new unary predicate P_T such that P_T is complete for T and empty for all $T'\in\mathcal{T}\setminus\{T\}$. Clearly, that the formula witnessing that P_T is complete separates T from $\mathcal{T}\setminus\{T\}$. Thus, the family $\mathcal{T}'$ itself is the least generating set. □

2. $\mathcal{T}$-APPROXIMATIONS

Definition. Let $\mathcal{T}$ be a class of theories and T be a theory, $T\notin\mathcal{T}$. The theory T is called $\mathcal{T}$*-approximated*, or *approximated by* $\mathcal{T}$, or $\mathcal{T}$*-approximable*, or a *pseudo-*$\mathcal{T}$*-theory*, if for any formula $\varphi\in T$ there is $T'\in\mathcal{T}$ such that $\varphi\in T'$.

If T is $\mathcal{T}$-approximated then $\mathcal{T}$ is called an *approximating family* for T, and theories $T'\in\mathcal{T}$ are *approximations* for T.

Remark 2.1. If T is $\mathcal{T}$-approximated then T is $\mathcal{T}'$-approximated for any $\mathcal{T}'\supseteq\mathcal{T}$. At the same time, if T is $\mathcal{T}$-approximated then T is $\mathcal{T}\setminus\{T'\}$-approximated for any $T'\in\mathcal{T}$. Indeed, since $T'\neq T$, there is $\psi\in T$ such that $\neg\psi\in T'$, and for any formula $\varphi\in T$ the formula $\varphi\wedge\psi$ belongs both to T and to some $T''\in\mathcal{T}\setminus\{T'\}$, so $\varphi\in T''$.

Besides, an approximation family $\mathcal{T}$ for T can be extended by an arbitrary theory $T'\neq T$, assuming the possibility to extend the language $\Sigma(T)$. Thus, if there an approximating family $\mathcal{T}$ for T then $\mathcal{T}$ can not be chosen minimal or maximal by inclusion, and if the language $\Sigma(T)$ is fixed then the maximal one exists containing all $\Sigma(T)$-theories $T'\neq T$.

Remark 2.1 implies the following proposition, but we will give slightly different arguments.

Proposition 2.2. *If there is a* $\mathcal{T}$*-approximated theory then* $\mathcal{T}$ *is infinite.*

Proof. If T is $\mathcal{T}$-approximated and $\mathcal{T}$ is finite consisting of $T_1,\ldots,T_n$ then having $T\notin\mathcal{T}$ there are formulas $\varphi_i\in T_i$ such that $\psi\rightleftharpoons\neg\varphi_1\wedge\ldots\wedge\neg\varphi_n\in T$. Since $\psi\notin T_1\cup\ldots\cup T_n$, T can not be $\mathcal{T}$-approximated, implying a contradiction. □

Proposition 2.3. *A theory* $T\notin\mathcal{T}$ *is* $\mathcal{T}$*-approximated if and only if* $T\in\mathrm{Cl}_E(\mathcal{T})$.

Proof. Let T be $\mathcal{T}$-approximated. By Proposition 1.1 it suffices to show that for any $\varphi\in T$ there are infinitely many $T'\in\mathcal{T}$ such that $\varphi\in T'$. Assuming on contrary that there are $\varphi\in T$ and only finitely many $T'\in\mathcal{T}$ with $\varphi\in T'$, say $T_1,\ldots,T_n$, then there are $\varphi_i\in T_i$ such that $\psi\rightleftharpoons\varphi\wedge\neg\varphi_1\wedge\ldots\wedge\neg\varphi_n\in T$. Since ψ does not belong to $\cup\mathcal{T}$, then T is not $\mathcal{T}$-approximated.

If $T\in\mathrm{Cl}_E(\mathcal{T})$ then, by Proposition 1.1, for any $\varphi\in T$ there are infinitely many $T'\in\mathcal{T}$ such that $\varphi\in T'$. □

Recall that $\mathcal{T}$ is *E-closed* if $\mathcal{T}=\mathrm{Cl}_E(\mathcal{T})$. By Proposition 2.3 we have:

Corollary 2.4. *For any family* $\mathcal{T}$ *there is a* $\mathcal{T}$*-approximated theory if and only if* $\mathcal{T}$ *is not E-closed.*

Definition [6]. An infinite structure $\mathcal{M}$ is *pseudo-finite* if every sentence true in $\mathcal{M}$ has a finite model.

If $T = \mathrm{Th}(\mathcal{M})$ for pseudo-finite $\mathcal{M}$ then T is called *pseudo-finite* as well.

Following [14] we denote by $\overline{\mathcal{T}}$ the class of all complete elementary theories of relational languages, by $\overline{\mathcal{T}}_{\rm fin}$ the subclass of $\overline{\mathcal{T}}$ consisting of all theories with finite models, and by $\overline{\mathcal{T}}_{\rm inf}$ the class $\overline{\mathcal{T}} \setminus \overline{\mathcal{T}}_{\rm fin}$.

Proposition 2.5. *For any theory T the following conditions are equivalent:*
(1) *T is pseudo-finite;*
(2) *T is $\overline{\mathcal{T}}_{\rm fin}$-approximated;*
(3) $T \in \mathrm{Cl}_E(\overline{\mathcal{T}}_{\rm fin}) \setminus \overline{\mathcal{T}}_{\rm fin}$.

Proof. (1) $\Leftrightarrow$ (2) holds by the definition. (2) $\Leftrightarrow$ (3) is satisfied by Proposition 2.3. □

3. Approximable and finitely axiomatizable theories

Definition. A theory T is called *approximable* if T is $\mathcal{T}$-approximable for some $\mathcal{T}$.

Recall [23] that a theory T is *finitely axiomatizable* if T is forced by some formula $\varphi \in T$.

Notice that by the definition finitely axiomatizable theories have finite languages.

Proposition 3.1. *For any theory T the following conditions are equivalent:*
(1) *T is approximable;*
(2) *T is $\overline{\mathcal{T}} \setminus \{T\}$-approximated;*
(3) *T is not finitely axiomatizable.*

Proof. (1) $\Leftrightarrow$ (2) holds by the definition.

(2) $\Rightarrow$ (3). Assume that T is finitely axiomatizable witnessed by a formula $\varphi \in T$. Then $\varphi \notin T'$ for any $T' \in \overline{\mathcal{T}} \setminus \{T\}$. Hence, T is not $\mathcal{T}$-approximated for any $\mathcal{T} \subseteq \overline{\mathcal{T}}$. In particular, T in not $\overline{\mathcal{T}} \setminus \{T\}$-approximated.

(3) $\Rightarrow$ (2). Let T be not finitely axiomatizable. Then for any $\varphi \in T$ there is $T' \in \overline{\mathcal{T}} \setminus \{T\}$ with $\varphi \in T'$, since otherwise T is axiomatizable by φ. Therefore, T is $\overline{\mathcal{T}} \setminus \{T\}$-approximated. □

Illustrating Proposition 3.1 we note that any theory T, in an infinite relational language Σ, is approximable by theories $T \upharpoonright \Sigma_0$, for finite Σ_0, expanded by empty or complete predicates for symbols $P \in \Sigma \setminus \Sigma_0$ such that P is empty for these expansions if P is not empty for T, and P is complete for these expansions if P is empty for T.

We denote by $\overline{\mathcal{T}}_{\rm fa}$ the class of all finitely axiomatizable theories, which coincide, by Proposition 3.1, with the class of all non-approximable theories. By the definition the class $\overline{\mathcal{T}}_{\rm fa}$ consists exactly of theories T having singletons $\overline{\mathcal{T}}_\varphi = \{T\}$, where $\varphi \vdash T$. Thus, by Propositions 1.1, the class $\overline{\mathcal{T}} \setminus \overline{\mathcal{T}}_{\rm fa}$ is E-closed, whereas $\overline{\mathcal{T}}_{\rm fa}$ is not E-closed, whose E-closure contains at least all pseudo-finite theories.

In the connection with this property it is natural to pose the following

Problem 1. *Describe $\mathcal{T}$-approximable theories and $\mathrm{Cl}_E(\mathcal{T})$ for natural classes $\mathcal{T} \subseteq \overline{\mathcal{T}}_{\rm fa}$.*

This problem can be considered in the following context.

Definition. For a family $\mathcal{T}$, a theory T is *$\mathcal{T}$-finitely axiomatizable*, or *finitely axiomatizable with respect to $\mathcal{T}$*, or *$\mathcal{T}$-relatively finitely axiomatizable*, if $\mathcal{T}_\varphi = \{T\}$ for some $\Sigma(\mathcal{T})$-sentence φ.

Remark 3.2. 1. A theory T is finitely axiomatizable if and only if T is $\mathcal{T}$-finitely axiomatizable for any $\mathcal{T}$ in the language $\Sigma(T)$.

2. A theory T is $\mathcal{T}$-finitely axiomatizable if and only if there is finite $\mathcal{T}_\varphi$ containing T, for some $\Sigma(\mathcal{T})$-sentence φ.

In this context Theorem 1.3 can be reformulated in the following way.

Theorem 3.3. *If $\mathcal{T}'_0$ is a generating set for a E-closed set $\mathcal{T}_0$ then the following conditions are equivalent:*
(1) *$\mathcal{T}'_0$ is the least generating set for $\mathcal{T}_0$;*
(2) *$\mathcal{T}'_0$ is a minimal generating set for $\mathcal{T}_0$;*
(3) *any theory in $\mathcal{T}'_0$ is $\mathcal{T}'_0$-finitely axiomatizable;*
(4) *any theory in $\mathcal{T}'_0$ is $\mathcal{T}_0$-finitely axiomatizable.*

Problem 1 can be divide into two possibilities with respect to $\mathrm{Cl}_E(\mathcal{T})$: with or without the least generating sets. In particular, it admits the following refinement.

Problem 1′. *Describe $\mathcal{T}$-approximable theories and $\mathrm{Cl}_E(\mathcal{T})$ for natural sets $\mathcal{T}$ containing $\mathcal{T}$-finitely axiomatizable generating sets.*

Definition. For a family $\mathcal{T}$ of a language Σ, a sentence φ of the language Σ is called *$\mathcal{T}$-complete* if φ isolates a unique theory in $\mathcal{T}$, i.e., $\mathcal{T}_\varphi$ is a singleton.

Clearly, a sentence φ is complete if and only if φ is $\mathcal{T}$-complete for any family $\mathcal{T}$ with a theory $T \in \mathcal{T}$ containing φ.

Obviously, if $|\mathcal{T}_\varphi| \in \omega \setminus \{0\}$ then each theory in $\mathcal{T}_\varphi$ contains a $\mathcal{T}$-complete sentence, but not vice versa. Indeed, $\mathcal{T}$ can consist of infinitely many finitely axiomatizable theories, so each theory in $\mathcal{T}_{\forall x(x\approx x)}$ contains a $\mathcal{T}$-complete sentence whereas $|\mathcal{T}_{\forall x(x\approx x)}| \geq \omega$.

Since $\mathcal{T}$-complete sentences confirm the $\mathcal{T}$-finite axiomatizability of theories in $\mathcal{T}$, and a theory T contains a $\mathcal{T}$-complete sentence if and only if T contains a disjunction of $\mathcal{T}$-complete sentences, Theorem 3.3 admits the following reformulation, with a slight extension.

Theorem 3.4. *If $\mathcal{T}'_0$ is a generating set for a E-closed set $\mathcal{T}_0$ then the following conditions are equivalent:*
(1) *$\mathcal{T}'_0$ is the least generating set for $\mathcal{T}_0$;*
(2) *$\mathcal{T}'_0$ is a minimal generating set for $\mathcal{T}_0$;*
(3) *any theory in $\mathcal{T}'_0$ contains a $\mathcal{T}'_0$-complete sentence;*
(4) *any theory in $\mathcal{T}'_0$ contains a $\mathcal{T}_0$-complete sentence;*
(5) *any theory in $\mathcal{T}'_0$ contains a disjunction of $\mathcal{T}'_0$-complete sentences;*
(6) *any theory in $\mathcal{T}'_0$ contains a disjunction of $\mathcal{T}_0$-complete sentences.*

4. λ-approximable theories

Below we consider a series of notions related to cardinalities for approximations of theories.

Definition. Let λ be a cardinality, $\mathcal{T}$ be a family of theories. A theory T is called *$(\lambda, \mathcal{T})$-approximable*, or *λ-approximable* (*λ-approximated*) by $\mathcal{T}$, if T is $\mathcal{T}'$-approximable for some $\mathcal{T}' \subseteq \mathcal{T}$ with $|\mathcal{T}'| = \lambda$. A theory T is called *somewhere* (*almost everywhere*) *$(\lambda, \mathcal{T})$-approximable* if $T \upharpoonright \Sigma$ is $(\lambda, \mathcal{T} \upharpoonright \Sigma)$-approximable for some (any) $\Sigma \subseteq \Sigma(T)$, where $|\Sigma| + \omega = \lambda$ and $\mathcal{T} \upharpoonright \Sigma$ is the restriction of theories in $\mathcal{T}$ till the language Σ.

A theory T is called *exactly* $(\lambda,\mathcal{T})$*-approximable* (respectively, *exactly somewhere* $(\lambda,\mathcal{T})$*-approximable*, or *exactly almost everywhere* $(\lambda,\mathcal{T})$*-approximable*) if T is (somewhere, or almost everywhere) $(\lambda,\mathcal{T})$-approximable and T is not (somewhere, or almost everywhere) $(\mu,\mathcal{T})$-approximable for $\mu<\lambda$.

A theory T is called λ*-approximable* (respectively, *somewhere* λ*-approximable*, *almost everywhere* λ*-approximable*, *exactly* λ*-approximable*, *exactly somewhere* λ*-approximable*, *exactly almost everywhere* λ*-approximable*) if T is $(\lambda,\mathcal{T})$-approximable (somewhere $(\lambda,\mathcal{T})$-approximable, almost everywhere $(\lambda,\mathcal{T})$-approximable, exactly $(\lambda,\mathcal{T})$-approximable, exactly somewhere $(\lambda,\mathcal{T})$-approximable, exactly almost everywhere $(\lambda,\mathcal{T})$-approximable) for some $\mathcal{T}$.

Remark 4.1. By the definition, if T is (exactly / somewhere) λ-approximable then $\lambda\geq\omega$. Besides, if T is almost everywhere $(\omega,\mathcal{T})$-approximable then $\mathcal{T}$ has infinitely many theories of structures having distinct finite cardinalities, since restrictions to the empty language can be approximated only by theories of finite structures. Moreover, by Proposition 3.1, T does not have finitely axiomatizable restrictions.

Again by the definition, if T is almost everywhere $(\lambda,\mathcal{T})$-approximable then T is somewhere $(\lambda,\mathcal{T})$-approximable. But not vice versa, since T can contain finitely axiomatizable restrictions.

If $\lambda=\omega$ we also say about *countably approximable theories* instead of ω-approximable and $(\omega,\mathcal{T})$-approximable ones.

Clearly by the definition that countably approximable theories are exactly countably approximable.

The following problem is related to the series of notions above.

Problem 2. *Describe cardinalities λ and forms of approximations for natural classes of theories.*

Illustrating the notions above and partially answering the problem we consider the following:

Theorem 4.2. (1) *Any theory T_0 of unary predicates P_i, $i\in I$, and with infinite models, is exactly $|T_0|$-approximable by the class $\mathcal{T}_0$ of theories of unary predicates, in finite languages and with finite models.*

(2) *Any theory T_0 of unary predicates P_i, $i\in I$, and with infinite models, is countably approximable, by an appropriate class $\mathcal{T}_0$ of theories of unary predicates.*

(3) *A theory T_1 of unary predicates and with finite models is* (*countably*) *approximable* (*by an appropriate class*) *if and only if T_1 has an infinite language.*

Proof. (1) Since the theory T_0 is based by the formulas describing cardinality estimations for intersections of unary predicates P_i, $i\in I$, and their complements, i.e., $P_{i_1}^{\delta_1}\wedge\ldots\wedge P_{i_k}^{\delta_k}$, $\delta_j\in\{0,1\}$, then each formula $\varphi\in T_0$ belongs to some theory in $\mathcal{T}_0$ whose models satisfy these cardinality estimations. Since $|T_0|=|I|+\omega$, T_0 is $|T_0|$-approximable by $\mathcal{T}_0$. Since the theories in $\mathcal{T}_0$ have finite languages T_0 can not be μ-approximable for $\mu<|T_0|$, i.e., T_0 is exactly $|T_0|$-approximable.

(2) If the language for T_0 is at most countable then we apply the item 1. The same approach is valid if T_0 has at most countably many independent predicates, i.e., all predicates are boolean combinations of some at most countable family of them. So below we assume that the language $\Sigma(T_0)$ is infinite and, moreover, there are uncountably many independent predicates.

If T_0 has an infinite and co-infinite predicate P_i then we approximate links of P_i with $\Sigma_i = \{P_j \mid j \in I \setminus \{i\}\}$ by a countable family $\mathcal{T}_i$ of theories $T'_k \neq T_0$, $k \in \omega$, in the language P_i, $i \in I$, such that $T'_k \upharpoonright \Sigma_i = T_0 \upharpoonright \Sigma_i$ and P_i for T'_k corresponds to T_0 step-by-step, with respect to some countable strictly increasing chain $(\Sigma'_k)_{k\in\omega}$, where $\Sigma_i = \bigcup\limits_{k\in\omega} \Sigma'_k$.

If T_0 contains just (co-)finite predicates, we put, for T'_k, predicates in $\Sigma'_k \cup \{P_i\}$ as required, and $P \in \Sigma_i \setminus \Sigma'_k$ is either empty or complete such that P is empty for T'_k if and only if P is not empty for T_0.

(3) If T_1 has a finite language then T_1 is isolated by a sentence describing all connections between elements in a model of T_1. Otherwise we can approximate T_1 by a countable family, as in the item 2. □

Definition [19]. A consistent sentence φ of a language Σ is called Σ-*complete*, or simply *complete*, if φ forces a complete theory of the language Σ.

Clearly, a theory T of a finite language Σ contains a Σ-complete sentence if and only if T is finitely axiomatizable (by that sentence).

Theorem 4.3. *For any theory T the following conditions are equivalent:*

(1) *T is λ-approximable for some λ;*

(2) *T is ω-approximable;*

(3) *the language $\Sigma(T)$ is finite and T does not contain a $\Sigma(T)$-complete sentence, or $\Sigma(T)$ is infinite.*

Proof. (2) $\Rightarrow$ (1) is obvious.

(1) $\Rightarrow$ (3). If $\Sigma(T)$ is finite then the conclusion follows by Proposition 3.1.

(3) $\Rightarrow$ (2). Let $\Sigma(T)$ is finite, and by assumption T is not finitely axiomatizable. Then by Proposition 3.1 the theory T is approximable, and being countable T is ω-approximable.

If $\Sigma(T)$ is infinite we approximate T by a countable family $\{T_n \mid n \in \omega\}$ as shown in the proof of Theorem 4.2: again considering a strictly increasing family of languages Σ_n, $n \in \omega$, $\Sigma(T) = \bigcup\limits_{n\in\omega} \Sigma_n$, and $\Sigma(T)$-theories T_n such that $T_n \upharpoonright \Sigma_n = T \upharpoonright \Sigma_n$ and $P \in \Sigma(T) \setminus \Sigma_n$ is either empty or complete, where P is empty/complete for T_n if and only if P is not empty/complete for T. □

Any approximating family $\{T_n \mid n \in \omega\}$ for T in the proof of Theorem 4.3 is called *trivial*, or *standard*.

Thus by Theorem 4.3 each theory, in an infinite language, has a standard approximating family.

5. Approximations by almost language uniform theories

Definition [14]. A theory T in a predicate language Σ is called *almost language uniform*, or a ALU-*theory* if for each arity n with n-ary predicates for Σ there is a partition for all n-ary predicates, corresponding to the symbols in Σ, with finitely many classes K such that any substitution preserving these classes preserves T, too.

Below we consider approximations of theories by families of ALU-theories. Theories with these approximations are called ALU-*approximable.*

Since theories in $\overline{\mathcal{T}}_{\rm fin}$, being theories of finite structures, are ALU-theories ([14, Proposition 5.1]) then theories T, which are approximable by families $\mathcal{T} \subset \overline{\mathcal{T}}_{\rm fin}$ are

ALU-approximable. In particular, by Theorem 4.2 (1), theories of unary predicates, with infinite models, are ALU-approximable.

By the definition each theory in a finite language is an ALU-theory. Since theories in a family for an approximation satisfy the required approximable theory step-by-step, some standard approximating family $\{T_n \mid n \in \omega\}$ for a countable theory T, in an infinite language, can consist of ALU-theories such that for each T_n only finitely many predicates can differ from empty or complete ones. Thus we have the following:

Proposition 5.1. *Any countable theory is an* ALU*-theory or it is* ALU*-approximable.*

6. Approximating subfamilies

Theorem 6.1. *A family $\mathcal{T}$ of theories contains an approximating subfamily if and only if $\mathcal{T}$ is infinite.*

Proof. Since any approximating family is infinite then, having an approximating subfamily, $\mathcal{T}$ is infinite.

Conversely, let $\mathcal{T}$ be infinite. Firstly, we assume that the language $\Sigma = \Sigma(\mathcal{T})$ of $\mathcal{T}$ is at most countable. We enumerate all Σ-sentences: φ_n, $n \in \omega$, and construct an accumulation point for $\mathcal{T}$ by induction. Since $\mathcal{T}_{\varphi_0}$ or $\mathcal{T}_{\neg\varphi_0}$ is infinite we can choose $\psi_0 = \varphi_0^\delta$ with infinite $\mathcal{T}_{\varphi_0^\delta}$, $\delta \in \{0, 1\}$. If ψ_n is already defined, with infinite $\mathcal{T}_{\psi_n}$, then we choose $\psi_{n+1} = \psi_n \wedge \varphi_{n+1}^\delta$, with $\delta \in \{0, 1\}$, such that $\mathcal{T}_{\psi_{n+1}}$ is infinite. Finally, the set $\{\psi_n \mid n \in \omega\}$ forces a complete theory T being an accumulation point both for $\mathcal{T}$ and for each $\mathcal{T}_{\psi_n}$. Thus, $\mathcal{T} \setminus \{T\}$ is a required approximating family.

If Σ is uncountable we find an accumulation point T_0 for infinite $\mathcal{T} \upharpoonright \Sigma_0$, where Σ_0 is a countable sublanguage of Σ. Now we extend T_0 till a complete Σ-theory T adding Σ-sentences χ such that $\mathcal{T}_\chi$ are infinite. Again $\mathcal{T} \setminus \{T\}$ is a required approximating family. □

Using the construction for the proof of Theorem 6.1 we observe that having infinite $\mathcal{T}_\varphi$ we obtain an accumulation point T for $\mathcal{T}_\varphi$ such that $\varphi \in T$. So having infinite $\mathcal{T}_\varphi$ and $\mathcal{T}_{\neg\varphi}$ we have at least two accumulation points for $\mathcal{T}$. Therefore we obtain the following:

Proposition 6.2. *If a family $\mathcal{T}$ has infinite subfamilies $\mathcal{T}_\varphi$ and $\mathcal{T}_{\neg\varphi}$ then e-*Sp$(\mathcal{T}) \geq 2$.

Similarly we have the following:

Proposition 6.3. *If a family $\mathcal{T}$ has infinite subfamilies $\mathcal{T}_{\varphi_i}$ for pairwise inconsistent formulas φ_i, $i \in I$, then e-*Sp$(\mathcal{T}) \geq |I|$.

7. Single-valued approximations

Definition. An approximating family $\mathcal{T}$ is *single-valued*, or *e-categorical*, if e-Sp$(\mathcal{T}) = 1$.

Clearly, if $\mathcal{T}$ is single-valued then $\mathcal{T}$ has a single accumulation point, i.e., approximating some unique theory T.

If T is the (unique) accumulation point for $\mathcal{T}$ then the family $\mathcal{T} \cup \{T\}$ is also called *single-valued*, or *e-categorical*.

Proposition 7.1. *Any E-closed family $\mathcal{T}$ with finite* e-Sp$(\mathcal{T}) > 0$ *is represented as a disjoint union of e-categorical families $\mathcal{T}_1, \ldots, \mathcal{T}_n$.*

Proof. Let e-Sp$(\mathcal{T}) = n$ and $T_1, \ldots, T_n$ be accumulation points for $\mathcal{T}$ witnessing that equality. Now we consider pairwise inconsistent formulas $\varphi_i \in T_i$ separating T_i from T_j, $j \neq i$, i.e., with $\neg\varphi_i \in T_j$. By Proposition 1.1 each family $\mathcal{T}_i = \mathcal{T}_{\varphi_i}$ is infinite, with unique accumulation point T_i, and thus $\mathcal{T}_i$ is e-categorical. Besides, the families $\mathcal{T}_i$ are disjoint by the choice of φ_i, and $\mathcal{T}' = \mathcal{T} \setminus \left(\bigcup\limits_{i=1}^{n} \mathcal{T}_i\right)$ does not have accumulation points. Therefore $\mathcal{T}' \cup \mathcal{T}_1$ is e-categorical, too. Thus, $\mathcal{T}' \cup \mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n$ is the required partition of $\mathcal{T}$ on e-categorical families. □

Remark 7.2. An arbitrary partition of a family $\mathcal{T}$ into disjoint e-categorical families $\mathcal{T}_i$, $i \in I$, does not imply e-Sp$(\mathcal{T}) = |I|$. Indeed, taking a language $\{Q_n^k \mid k = 1, 2, n \in \omega\}$ of unary predicates we can form a family $\mathcal{T} = \{T_n^k \mid k \in \{1, 2\}, n \in \omega\}$ of theories T_n^k such that the predicates Q_m^k, $m \geq n$, are complete and the others are empty. All families $\mathcal{T}$, $\{T_n^1 \mid n \in \omega\}$, $\{T_n^2 \mid n \in \omega\}$ are e-categorical, whose common accumulation point consists of all empty predicates, whereas $\{T_n^1 \mid n \in \omega\}$, $\{T_n^2 \mid n \in \omega\}$ form a partition of $\mathcal{T}$.

Similarly, having an arbitrary infinite family $\mathcal{T}$ of theories in the empty language (which is e-categorical) we can arbitrarily divide $\mathcal{T}$ into two infinite parts each of which is again e-categorical, with the common accumulation point T having infinite models.

More generally, by Theorem 6.1, if $\mathcal{T}$ is e-categorical then each infinite $\mathcal{T}' \subseteq \mathcal{T}$ is again e-categorical with the same accumulation point.

Definition. An approximating family $\mathcal{T}$ is called *e-minimal* if for any sentence $\varphi \in \Sigma(\mathcal{T})$, $\mathcal{T}_\varphi$ is finite or $\mathcal{T}_{\neg\varphi}$ is finite.

Theorem 7.3. *A family $\mathcal{T}$ is e-minimal if and only if it is e-categorical.*

Proof. Let $\mathcal{T}$ be e-minimal. We consider the set $T = \{\varphi \in \Sigma(\mathcal{T}) \mid \mathcal{T}_\varphi \text{ is infinite}\}$. By compactness T is consistent and by e-minimality of $\mathcal{T}$, T is a complete theory. Thus, by the definition T is an accumulation point for $\mathcal{T}$. This accumulation point is unique since if $T' \neq T$ is a $\Sigma(\mathcal{T})$-theory then there is $\varphi \in T$ such that $\neg\varphi \in T'$, so $\mathcal{T}_{\neg\varphi}$ is finite and $T' \notin \mathrm{Cl}_E(\mathcal{T} \setminus \{T'\})$ by Proposition 1.1. Thus, $\mathcal{T}$ is e-categorical.

Conversely, if $\mathcal{T}$ is not e-minimal then e-Sp$(\mathcal{T}) \geq 2$ by Corollary 6.2. Thus, $\mathcal{T}$ is not e-categorical. □

Remark 7.4. As shown in Remark 7.2 e-categorical families can be always divided into e-categorical parts. So the condition for e-minimality, on divisibilities only with respect to neighbourhoods $\mathcal{T}_\varphi$, is essential counting e-spectra.

References

[1] P. Bankston, *Ulptraproducts in topology*, General Topology and its Applications, **7**:3 (1977), 283–308.

[2] P. Bankston, *A survey of ultraproduct constructions in general topology*, Topology Atlas Invited Contributions, **8**:2 (2003), 1–32.

[3] G. Cherlin, L. Harrington, A.H. Lachlan, *$\aleph_0$-categorical, $\aleph_0$-stable structures*, Ann. Pure Appl. Logic, **28**:2 (1985), 103–135. Zbl 0566.03022

[4] G. Cherlin, A.H. Lachlan, *Stable finitely homogeneous structures*, Trans. Amer. Math. Soc., **296**:2 (1986), 815–850. Zbl 0641.03022

[5] G. Ahlbrandt, M. Ziegler, *Quasi finitely axiomatizable totally categorical theories*, Annals of Pure and Applied Logic, **30**:1 (1986), 63–82. Zbl 0592.03018

[6] E. Rosen, *Some Aspects of Model Theory and Finite Structures*, The Bulletin of Symbolic Logic, **8**:3 (2002), 380–403. MR1931349

[7] J. Väänänen, *Pseudo-finite model theory*, Mat. Contemp., **24** (2003), 169–183. Zbl 1078.03029

[8] G. Cherlin, E. Hrushovski, *Finite Structures with Few Types*, Annals of Mathematics Studies, **152**, Princeton University Press, Princeton, 2003. Zbl 1024.03001

[9] D. Macpherson, Ch. Steinhorn, *Definability in the classes of finite structures*, London Matheematical Society Lecture Notes series **379** (2011), 140–176. Zbl 1248.03051

[10] U. Andrews, H. J. Keisler, *Separable models of randomizations*, J. Symb. Log., **80**:4 (2015), 1149–1181. Zbl 1373.03051

[11] S.V. Sudoplatov, *Combinations of structures*, Izv. Irkutsk. Gos. Univ., Ser. Mat., **24** (2018), 65–84. Zbl 1409.03002

[12] S.V. Sudoplatov, *Closures and generating sets related to combinations of structures*, Izv. Irkutsk. Gos. Univ., Ser. Mat., **16** (2016), 131–144. Zbl 1402.03039

[13] S. V. Sudoplatov, S.V. Sudoplatov, *Families of language uniform theories and their generating sets*, Izv. Irkutsk. Gos. Univ., Ser. Mat., **16** (2016), 62–76. Zbl 1362.03031

[14] S.V. Sudoplatov, *Combinations related to classes of finite and countably categorical structures and their theories*, Sib. Electron. Mat. Izv., **14** (2017), 135–150. MR3622899

[15] S.V. Sudoplatov, *Relative e-spectra and relative closures for families of theories*, Sib. Elektron. Mat. Izv., **14** (2017), 296–307. Zbl 1423.03127

[16] S.V. Sudoplatov, *On semilattices and lattices for families of theories*, Sib. Elektron. Mat. Izv., **14**, (2017), 980–985. MR3711758

[17] A. Ehrenfeucht, G. Fuhrken, *A finitely axiomatizable complete theory with atomless* $F_1(T)$, Arch. Math. Logik, **14**:3-4 (1971), 162–166. Zbl 0254.02043

[18] E.A. Palyutin, *Models with countable-categorical universal theories*, Algebra Logika, **10** (1971), 23–32. Zbl 0221.02040

[19] J.A. Makowsky, *On some conjecture connected with complete sentences*, Fund. Math., **81**:3 (1974), 193–202. Zbl 0285.02042

[20] E.A. Palyutin, *Categorical Horn classes.* I, Algebra Logic, **19**:5 (1980), 377–400. Zbl 0491.03011

[21] B.I. Zilber, *On a solution of the problem of finite axiomatizability for theories categorical in all infinite powers*, Transl., II. Ser., Am. Math. Soc., **135** (1987), 13–17. Zbl 0614.03025

[22] E. Hrushovski, *Finitely Axiomatizable* $\aleph_1$ *Categorical Theories*, J. Symbolic Logic, **59**:3 (1994), 838–844. Zbl 0808.03015

[23] M.G. Peretyatkin, *Finitely Axiomatizable Theories*, Plenum Publ. Corp, New York, 1997. Zbl 0884.03031

[24] T. Blossier, E. Bouscaren, *Finitely Axiomatizable Strongly Minimal Groups*, J. Symb. Log., **75**:1 (2010), 25–50. Zbl 1196.03046

[25] S.V. Sudoplatov, *Classification of Countable Models of Complete Theories*, NSTU, Novosibirsk, 2018.

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