S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 17, стр. 726–731 (2020) UDC 510.67, 512.56

DOI 10.33048/semi.2020.17.050 MSC 18D35

STABILITY OF THE CLASS OF DIVISIBLE S-ACTS

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Abstract. We describe monoids S such that the theory of the class of all divisible S-acts is stable, superstable or, for commutative monoid, ω-stable. More precisely, we prove that the theory of the class of all divisible S-acts is stable (superstable) iff S is a linearly ordered (well ordered) monoid. We also prove that for a commutative monoid S the theory of the class of all divisible S-acts is ω-stable iff S is either an abelian group with at most countable number of subgroups or is finite and has only one proper ideal. Classes of regular, projective and strongly flat S-acts were considered in [1, 2]. Using results from [3] we obtain necessary and sufficient conditions for stability, superstability and ω-stability of theories of classes of all divisible S-acts.

Keywords: monoid, divisible S-act, stability, superstability, ω-stability.

1. Preliminaries

Let us recall some definitions and facts from the theory of S-acts [4]. Let S be a monoid, i.e. a semigroup with the unit element 1. A monoid S is called *linearly (well) ordered*, if the set $\{Sa \mid a \in S\}$ is linearly ordered (well ordered) by $\supseteq$. An element $c \in S$ is said to be *right cancellable* if for all $a, b \in S$ the equality $ac = bc$ implies that $a = b$. An element $s \in S$ is called *right invertible* if there exists an element $t \in S$ such that $st = 1$.

A *(left) S-act* ${}_SA$ over a monoid S is a set A, on which an action of S is defined and the unit element of S acts on A as identity. By *S-Act* we denote the class of all S-acts.

Remark 1. *If t is a right invertible element of S and ${}_SA$ is an S-act, then $tA = A$.*

Krasitskaya, A.I., Stability of the class of divisible S-acts.

This research was partially supported by Ministry of Education and Science of the Russian Federation, contract №075-02-2020-1482-1, 21.04.2020.

Received April 6, 2019, published May 27, 2019.

A *congruence* on an S-act ${}_SA$ is an equivalence relation ρ on A such that $(a,a')\in\rho$ implies $(sa,sa')\in\rho$ for all $a,a'\in A$, $s\in S$. The smallest congruence on the S-act ${}_SA$ with respect to $\subseteq$ which contains a set X is called *a congruence of the S-act ${}_SA$, generated by X*, and is denoted by $\rho(X)$. Denote by a/ρ the class of a congruence ρ with a representative $a\in A$.

We say that an element $a\in A$ is *divisible* by $s\in S$ in ${}_SA$, if there exists $b\in A$ such that $sb=a$. A *divisible S-act* is an S-act ${}_SA$, such that $cA=A$ for every right cancellable element $c\in S$. We use *S-Div* to denote the class of all divisible S-acts. It is clear that *S-Div* is an elementary class.

By the *coproduct* of S-acts ${}_SA_i$, $i\in I$, we mean their disjoint union. The coproduct of S-acts ${}_SA_i, i\in I$, is denoted by $\coprod\limits_{i\in I}{}_SA_i$. Note that a coproduct of divisible S-acts is also a divisible S-act. A map $\theta: A\to B$ such that $\theta(sa)=s\theta(a)$ for all $a\in A, s\in S$, is called an *S-morphism* from the S-act A to the S-act B. An S-act F is said to be *free* in *S-Act* (with a set of free generators X) if for every S-act A and every map $\theta: X\to A$ there exists a unique S-morphism $\bar\theta: F\to A$ such that $\iota\bar\theta=\theta$, where $\iota: X\to F$ is an embedding.

Let T be the set of all right cancellable but not right invertible elements of S. Let ${}_SA$ be an S-act, $X=\{(t,a)\in T\times A \mid a \text{ is not divisible by } t\}$, ${}_SF(X)$ be a free S-act with a set of free generators X, $H=\{(t(t,a),a)|(t,a)\in X, a\in A\}\subseteq (F(X)\coprod A)\times(F(X)\coprod A)$, $\rho(H)$ be a congruence of the S-act ${}_S(F(X)\coprod A)$, generated by H, ${}_SU(T,A)={}_S(F(X)\coprod A)/\rho(H)$. Note that there exists a natural embedding $\pi: A\to U(T,A)$. Therefore we can identify elements $a\in A$ with $\pi(a)$.

We introduce the following notations: ${}_SA_0={}_SA$, ${}_SA_i={}_SU(T,A_{i-1})$ for $i\in\mathbb{N}$, $D(A)=\bigcup_{i\in\omega}A_i$.

Fact 1. [4] *The S-act ${}_SD(A)$ is divisible.*

The S-act ${}_SD(A)$ is called the *divisble extension* of the S-act ${}_SA$.

Fact 2. [5] *Let ${}_SA$ be an S-act, $a,b\in A$, $d\in D(A)\backslash A$, $a,b\in Sd$. Then $a,b\in Sc$ for some $c\in A$.*

The following facts from model theory can be found in [6, 7]. Let T be a consistent theory in a language L, $X=\{x_i \mid 1\le i\le n\}$, $L_n=L_X$. Every set p of sentences of a language L_n is called an *n–type* of L. If a theory $p\cup T$ is consistent, then p is said to be an *n–type* over T. If p is a complete theory, **then** p is called a *complete n–type* of L. If also $T\subseteq p$, we say that p is a *full n–type* over T. The set of all complete n–types over T is denoted by $S_n(T)$.

Let $\mathcal{A}$ be an algebraic system of a language L, $X\subseteq A$, and $a\in A$. The *type* of an element a over the set X is the set $tp(a,X)=\{\Phi(x)\mid \mathcal{A}_X\models\Phi(a)\}$. It is easy to see that $tp(a,X)$ is a complete 1–type over $Th(\mathcal{A}_X)$. Denote $S_1(Th(\mathcal{A}_X))$ by $S(X)$.

A theory T is called *stable for the cardinality* κ or κ–stable, if $|S(X)|\le\kappa$ for every model $\mathcal{A}$ of a theory T and every $X\subseteq A$ of cardinality κ. If a theory T is κ–stable for some infinite κ then T is called *stable.* If a theory T is κ–stable for every $\kappa\ge 2^{|T|}$, then T is said to be *superstable.* If a theory T is not stable, then T is called *unstable.*

Fact 3. [1] *A complete theory is unstable iff there exists a formula $\Phi(\bar x,\bar y)$ with $2n$ variables, a model $\mathcal{A}$ of a theory T and $\bar a_i\in A^n$, $i\in\omega$, such that for every i,j, $i\ne j$,*

$$i<j \iff \mathcal{A}\models\Phi(\bar a_i,\bar a_j).$$

Let K be a class of S-acts. A monoid S is called a K*-stabiliser* (K-superstabiliser, K-ω-stabiliser), if $Th({}_SA)$ is stable (superstable, ω-stable) for every S-act ${}_SA \in K$. If $K = S\text{-}Act$, then K-stabiliser (K-superstabiliser, K-ω-stabiliser) is called the stabiliser (superstabiliser, ω-superstabiliser).

Let us introduce the following notation:

$$\exists^n y\phi(x,y) \leftrightharpoons \exists y_1 \ldots \exists y_n (\bigwedge_{1\le i<j\le n} \neg(y_i = y_j) \wedge \bigwedge_{1\le i\le n} \phi(x, y_i) \wedge$$
$$\wedge \forall y(\phi(x,y) \to \bigvee_{1\le i\le n} y = y_i)).$$

Fact 4. [3] *A monoid S is a stabiliser iff S is a linearly ordered monoid.*

Fact 5. [3] *A monoid S is a superstabiliser iff S is a well ordered monoid.*

Fact 6. [7] *If a theory T is stable for a countable cardinality (ω-stable), then it is stable for all infinite cardinalities.*

Fact 7. [8] *Let S be an arbitrary countable commutative monoid. Then the following statements are equivalent:*
1) S is an ω-stabliser;
2) Either S is an abelian group with at most countable number of subgroups, or S is finite and has a unique proper ideal.

2. Stability of the S-*Div* class

Theorem 1. *Given a monoid S, the following statements are equivalent:*
1) S is a S-Div-stabiliser;
2) S is a stabiliser;
3) S is a linearly ordered monoid.

Proof. The implication $2 \Rightarrow 1$ is trivial.

The implication $2 \Leftrightarrow 3$ follows from Fact 4.

Let us prove $1 \Rightarrow 3$.

Assume that S is a S-*Div*-stabiliser but not a linearly ordered monoid, i.e. there exist $t, s \in S$ such that $St \not\subseteq Ss$ and $Ss \not\subseteq St$. Let $K = \{\langle i,j\rangle \mid j \leqslant i < \omega\}$; ${}_SS_{ij}$ – a copy of an S-act ${}_SS$ ($\langle i,j\rangle \in K$) and $S_{ij} \cap S_{kl} = \emptyset$, if $\langle i,j\rangle \neq \langle k,l\rangle$; d_{ij} is a copy of $d \in S$ in S_{ij}. Let ${}_SA$ be an S-act $\bigcup_{\langle i,j\rangle\in K} {}_SS_{ij}/\theta$, where θ is a congruence of the S-act $\bigcup_{\langle i,j\rangle\in K} {}_SS_{ij}$, generated by the set $\{\langle t_{ij}, t_{il}\rangle \mid \langle i,j\rangle \in K, \langle i,l\rangle \in K\} \cup \{\langle s_{ij}, s_{lj}\rangle \mid \langle i,j\rangle, \langle l,j\rangle \in K\}$; let t_i be an equivalence class of θ with a representative t_{ij}; let s_j be an equivalence class of θ with a representative s_{ij}; let $\varphi(x,y)$ be a formula $\exists z(x = tz \wedge y = sz)$. It is clear that the restriction of θ on the S-act ${}_SA_{ij}$ ($\langle i,j\rangle \in K$) is a zero-congruence.

Let us prove that

$$(1) \qquad {}_SD(A) \models \varphi(t_i, s_j) \Longleftrightarrow i \geqslant j.$$

If $i \geqslant j$, then $t_i = t1_{ij}/\theta$ and $s_i = s1_{ij}/\theta$, i.e. ${}_SD(A) \models \varphi(t_i, s_j)$. Let $i < j$. Assume that $t_i = tu$ and $s_j = su$ for some $u \in D(A)$. Then Fact 2 implies that $t_i, s_j \in Sc/\theta$ for some $c/\theta \in A$, which is not true.

From Fact 3 it follows that (1) contradicts the stability of $Th({}_SD(A))$. Therefore S is a linearly ordered monoid. □

3. Superstability of the S-*Div* class

Theorem 2. *Given a monoid S, the following statements are equivalent:*
(1) S is a S-Div-superstabiliser;
(2) S is a superstabiliser;
(3) S is a well ordered monoid.

Proof. The implication $2 \Rightarrow 1$ is trivial.
The implication $2 \Leftrightarrow 3$ follows from Fact 5.
Let us prove $1 \Rightarrow 3$.
Let S be a S-*Div*-superstabiliser. Then S is a S-*Div*-stabiliser and from Theorem 1 it is linearly ordered. We want to show that S is a well ordered monoid. Assume the contrary, i.e. there exist $a_i \in S$ such that $Sa_i \subset Sa_{i+1}$ $(i \in \omega)$. Let T be a theory S-*Div*, κ a cardinal number, such that $\kappa \geqslant 2^{|T|}$. For $\eta \in \kappa^\omega$, we denote copies of an S-act by ${}_SS_\eta$ and copies of elements $c \in S$ by $c_\eta \in S_\eta$. Let $\eta|0 = \emptyset$, $\eta|i = (\eta(0), \eta(1), \ldots, \eta(i-1))$, where $i \in \omega \setminus \{0\}$.

Let ${}_SA = \bigsqcup_{\eta\in\kappa^\omega} {}_SS_\eta/\Theta$, where Θ is a congruence of the S-act $\bigsqcup_{\eta\in\kappa^\omega} {}_SS_\eta$ generated by the set $\{((a_i)_\eta, (a_i)_\varepsilon) \mid \eta, \varepsilon \in \kappa^\omega, \eta|i = \varepsilon|i\}$, $b_{\eta|i} = (a_i)_\eta/\Theta$, $b_\eta = 1_\eta/\Theta$, where $\eta \in \kappa^\omega$, $B = \{b_{\eta|i} \mid \eta \in \kappa^\omega, i \in \omega\}$. It is clear that $|b_\eta| = 1$ for every $\eta \in \kappa^\omega$ and $|B| = \kappa^\omega$. Let $\eta, \varepsilon \in \kappa^\omega$, $\eta \neq \varepsilon$. We show that $tp(b_\eta, B)$ and $tp(b_\varepsilon, B)$ are distinct 1-types over a theory $Th(D({}_SA))$. Since $\eta \neq \varepsilon$, it follows that $\eta|i \neq \varepsilon|i$ for some $i \in \omega$ and $b_{\eta|i} \neq b_{\varepsilon|i}$. Furthermore, $b_{\eta|i} = a_ib_\eta$ and $b_{\varepsilon|i} = a_ib_\varepsilon$. Then $b_{\eta|i} = a_ix \in tp(b_\eta, B) \backslash tp(b_\varepsilon, B)$. Consequently, $|S(B)| \geq |\{b_\eta \mid \eta \in \kappa^\omega\}| = \kappa^\omega > \kappa$ and it follows that the theory $Th(D({}_SA))$ is not superstable. This implies that monoid S is not a S-*Div*-superstabiliserб and with that we obtain a contradiction. Therefore, S is a well ordered monoid. □

4. ω-stability of the S-*Div* class

Theorem 3. *Given a commutative countable monoid S, the following statements are equivalent:*
(1) S is a S-Div-ω-superstabiliser;
(2) S is an ω-stabiliser;
(3) Either S is an abelian group with at most countable number of subgroups or S is finite and has a unique proper ideal.

Proof. Let S be a commutative countable monoid.
The implication $2 \Rightarrow 1$ is trivial.
The implication $2 \Leftrightarrow 3$ follows from Fact 7.
We shall prove that $1 \Rightarrow 3$.
For a proof assume that S is a $S - Div$-ω-stabiliser. Then Fact 6. implies that S is a $S - Div$-superstabiliser and from theorem 2 we obtain that S is a well ordered monoid.

If S contains no cancellable and non-invertible elements then by definition of divisible S-acts it follows that $S - Div = S - Act$. Then S is an ω-stabiliser.

Assume that such element exists in S. Since S is a well ordered monoid, there exists a cancellable and non-invertible element $g \in S$ such that

(2) $$\forall a \in S(Sg \subset Sa \Rightarrow a \text{ is non-cancellable or invertible}).$$

We denote $1 \in S$ by g^0.

Consider an element $i \in \omega$. We will show that

(3) $$\forall a \in S(Sg^{i+1} \subset Sa \subseteq Sg^i \Rightarrow Sa = Sg^i).$$

Assume that $Sg^{i+1} \subset Sa \subseteq Sg^i$. Then $a = sg^i$ and $s \notin Sg$. Since S is linearly ordered, we have that $Sg \subset Ss$ and $g = ts$ for some $t \in S$. From (2) it follows that s is either non-cancellable or invertible. Let us check that s is a cancellable element of S. Indeed, if $xs = ys$ for some $x, y \in S$, then $xts = yts$, i.e. $xg = yg$ and $x = y$. Therefore, s is invertible and $Ss = S$. Hence $Sa = Ssg^i = Sg^i$ and (3) is proved.

We will now show that $Sg^{i+1} \subset Sg^i$. Assume that $Sg^i = Sg^{i+1}$. Then $g^i = sg^{i+1}$ for some $s \in S$. Since g is cancellable, it follows that $1 = sg$. This contradicts the fact that g is not invertible.

Consider elements $s, t \in S$. We shall prove that

(4) $$s \in Sg^i \setminus Sg^{i+1} \wedge t \in Sg^j \setminus Sg^{j+1} \Longrightarrow st \in Sg^{i+j} \setminus Sg^{i+j+1}.$$

Let $s \in Sg^i \setminus Sg^{i+1}$, $t \in Sg^j \setminus Sg^{j+1}$. As $s \in Sg^i$, then $s = s_1 g^i$ for some $s_1 \in S$. Since $t \in Sg^j$, it follows that $t = t_1 g^j$ for some $t_1 \in S$. Then $st = s_1 t_1 g^{i+j} \in Sg^{i+j}$. Because $s \notin Sg^{i+j}$, we obtain that $s_1 \notin Sg$; also $t \notin Sg^{i+j}$, implies that $t_1 \notin Sg$. Furthermore since S is linearly ordered we have that $Sg \subset Ss_1$, $Sg \subset St_1$ and from (3), we derive that $Ss_1 = S$ and $St_1 = S$. Assume $st \in Sg^{i+j+1}$, i.e. $st = rg^{i+j+1}$ for some $r \in S$. Then $rg^{i+j+1} = s_1 t_1 g^{i+j}$. From the fact that g is cancellable we see that $rg = s_1 t_1$. Thus, $s_1 t_1 \in Sg$, i.e. $S = Ss_1 t_1 \subseteq Sg$, which is a contradiction. Hence $st \in Sg^{i+j} \setminus Sg^{i+j+1}$.

Let $\sim$ be the following relation on the set S:

$$a \sim b \Longleftrightarrow \exists i \in \omega : a, b \in Sg^i \setminus Sg^{i+1}.$$

It is easy to show that $\sim$ is an equivalence relation. Let us prove that $\sim$ is also a congruence of the S-act ${}_S S$. Consider elements $a, b, s \in S$, such that $s \in Sg^i \setminus Sg^{i+1}$, where $i \in \omega$, and $a \sim b$. We want to show that $sa \sim sb$. Since $a \sim b$, it follows that $a, b \in Sg^j \setminus Sg^{j+1}$ for some $j \in \omega$. From (4), we obtain that $sa, sb \in Sg^{i+j} \setminus Sg^{i+j+1}$, i.e. $sa \sim sb$.

Consider ${}_S\bar{S} =_S S/_\sim$ (${}_S\bar{S}$ is an S-act). For $a \in S$ denote by $\bar{a}$ a congruence class of $\sim$ with a representative a. Then (3) yields that $\bar{S} = \{\bar{g}^i \mid i \in \omega\}$.

Define the action of the S-act S on the set $A = \{\bar{g}^n \mid n \in \mathbb{Z}\}$. Let $s \in S$. Then $s \sim g^k$ for some $k \in \omega$. Assume that $s\bar{g}^n = \bar{g}^{n+k}$ for every $n \in \mathbb{Z}$. We will prove that $s(t\bar{g}^n) = (st)\bar{g}^n$ for every $s, t \in S$, $n \in \mathbb{Z}$. Consider elements $s \sim g^k$, $t \sim g^m$. From (4) we have $st \sim g^{m+k}$. By definition of the action of the monoid S on the set A, we have that $(st)\bar{g}^n = \bar{g}^{n+m+k}$, $t\bar{g}^n = \bar{g}^{m+n}$ and $s(t\bar{g}^n) = s\bar{g}^{m+n} = \bar{g}^{k+m+n}$. Hence ${}_SA$ is an S-act. Note that ${}_S\bar{S}$ is a sub-S-act of ${}_SA$.

We will see that ${}_SA \in S\text{-}Div$. Let t be a cancellable element of S. We want to check that $tA = A$. By Remark 1. it can be assumed that t is not invertible. Let $\bar{g}^n \in A$ и $t \sim g^i$. Then $\bar{g}^n = t\bar{g}^{n-i} \in tA$. Therefore $tA = A$.

We will now show that a theory $Th({}_SA)$ is not ω-stable. Let ${}_SA_i$ $(i \in \mathbb{N})$ be copies of the S-act ${}_SA$, and $a_i \in A_i$ be copies of an element $a \in A$. We assign to every $n \in \mathbb{N}$ an S-act ${}_SA^n = \bigsqcup\limits_{i \leq n} {}_SA_i/\Theta^n$, where Θ^n is a congruence of an S-act $\bigsqcup\limits_{i \leq n} {}_SA_i$ generated by the set $\{\bar{g}_i^{-2} \mid i \leq n\}$. Note that for $m \geq -2$ elements $\bar{g}_1^m/\Theta^n$, ... ,$\bar{g}_n^m/\Theta^n$ are the same. Denote by $(g^m)_n$ an element $\bar{g}_i^m/\Theta^n$ $(m \geq -2)$. To each $K \subseteq \mathbb{N}$ we assign an S-act ${}_SA^K = \bigsqcup\limits_{n \in K} {}_SA_n/\eta^K$, where η^K is a congruence of an S-act $\bigsqcup_{n\in K} {}_SA_n$ generated by the set $\{(\bar{g}^{-1})_n \mid n \in K\}$. Note that for $m \geq -1$ elements $(g^m)_n/\eta^K$, $n \in K$, coincide. Denote an element $(g^m)_n/\eta^K$ by $(g^m)_K$.

Let ${}_SB = \bigsqcup_{K\subseteq\mathbb{N}} {}_SA^K/\xi$, where ξ is a congruence of an S-act $\bigsqcup_{K\subseteq\mathbb{N}} A^K$ generated by the set $\{ (g^0)_K \mid K \subseteq \mathbb{N}\}$. Note that for $m \geq 0$ elements $(g^m)_K/\xi$, $K \subseteq \omega$, coincide. For $m \geq 0$, we will denote element $(g^m)_K/\xi$ by g^m. For every $n \in \mathbb{N}$, we define:

$$\varphi_n(y) \leftrightharpoons \exists^n z(gz = y).$$

Let $K_1 \neq K_2$. We want to check that $tp((g^{-1})_{K_1}, \emptyset) \neq tp((g^{-1})_{K_2}, \emptyset)$. Assume there exists n such that $n \in K_1 \backslash K_2$. Then

$${}_SD(B) \models \varphi_n((g^{-1})_{K_1}) \wedge \neg\varphi_n((g^{-1})_{K_2}).$$

Hence, $|S(\emptyset)| \geq 2^{\mathbb{N}}$ and a theory $Th({}_SD(B))$ is not ω-stable. Therefore, a monoid S is not a *Div*-ω-stabiliser. □

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