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# ON THE COMPLEXITY OF THE LATTICES OF SUBVARIETIES AND CONGRUENCES. II. DIFFERENTIAL GROUPOIDS AND UNARY ALGEBRAS

A.V. KRAVCHENKO, M.V. SCHWIDEFSKY

Abstract. We prove that certain lattices can be represented as the lattices of relative subvarieties and relative congruences of differential groupoids and unary algebras. This representation result implies that there are continuum many quasivarieties of differential groupoids such that the sets of isomorphism types of finite sublattices of their lattices of relative subvarieties and congruences are not computable. A similar result is obtained for unary algebras and their lattices of relative congruences.

**Keywords:** quasivariety, variety, congruence lattice, differential groupoid, unary algebra, undecidable problem, computable set.

## 1. Introduction

G. Birkhoff and A. I. Maltsev have initiated the study of the lattices of (quasi)varieties by raising the problem on finding a description of lattices that are isomorphic to such lattices, see [2] and [13]. Many results concerning the lattices of (quasi)varieties demonstrate their complicated inner structure; some of those results can be found in the monograph of V. A. Gorbunov [3, Sec. 5.4.5]. The present paper concerns the structural and algorithmic complexity of the lattices of relative varieties and relative congruences; it continues [8] and [12].

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In [14], A. M. Nurakunov constructed a quasivariety $\mathbf{R}$ of unars whose quasi-equational theory is undecidable and the finite membership problem is undecidable for both $\mathbf{R}$ and the class of sublattices of the lattice $\mathrm{Lq}(\mathbf{R})$ of subquasivarieties of $\mathbf{R}$. Later, he established a similar result for the variety of pointed Abelian groups, see [15]. Using ideas from [14], sufficient conditions on a quasivariety $\mathbf{M}$ were found in [19] for existence of continuum many subclasses $\mathbf{K} \subseteq \mathbf{M}$ such that the finite membership problem is undecidable for the class of sublattices of the lattice $\mathrm{Lq}(\mathbf{K})$ of relative subquasivarieties (or $\mathbf{K}$-quasivarieties). As is proven in [7], there are continuum many axiomatizable classes $\mathbf{K}$ of unary algebras such that the finite membership problem is undecidable for the class of sublattices of the lattice $\mathrm{Lq}(\mathbf{K})$ of $\mathbf{K}$-quasivarieties. For similar results, we also refer to [16, 20, 21].

Using ideas from [14], sufficient conditions on a quasivariety $\mathbf{M}$ were found in [12] for existence of continuum many subquasivarieties $\mathbf{K} \subseteq \mathbf{M}$ satisfying the following conditions:

(a) the quasi-equational theory of $\mathbf{K}$ is undecidable,

(b) the finite membership problem is undecidable for $\mathbf{K}$ and the classes of sublattices of the lattice $\mathrm{Lv}(\mathbf{K})$ of relative subvarieties and the lattice $\mathrm{Con}_{\mathbf{K}}\,\mathcal{A}$ of relative congruences, where $\mathcal{A} \in \mathbf{K}$.

Although the variety $\mathbf{Dm}$ of differential groupoids and a certain quasivariety $\mathbf{V}$ of unary algebras with two unary operations do not satisfy these sufficient conditions, a similar result holds for them, see Theorem 19. Along with that result, we prove representation results for the lattices of relative varieties and relative congruences of differential groupoids and unary algebras, see Theorems 13, 15, and 17. We also note that the quasivarieties $\mathbf{Dm}$ and $\mathbf{V}$ are $Q$-universal according to [6] and [5].

## 2. Basic definitions and auxiliary results

For all definitions and notation concerning algebraic structures and quasivarieties, we refer to the monograph [3, Ch. 1], as well as to the papers [8, 9, 10, 11, 12].

2.1. **Differential groupoids.** A *differential groupoid* is an algebra endowed with one binary operation $\cdot$ that satisfies the following identities:

$$\forall x\ [x \cdot x = x], \quad \forall x\,\forall y\ [x \cdot (x \cdot y) = x],$$
$$\forall x\,\forall y\,\forall z\,\forall t\ [(x \cdot y) \cdot (z \cdot t) = (x \cdot z) \cdot (y \cdot t)].$$

Let $\mathbf{Dm}$ denote the variety of all differential groupoids. For brevity, we write $x_1x_2\ldots x_n$ for $(\ldots(x_1 \cdot x_2)\cdot\ldots)\cdot x_n$ and $xy^n$ for $x\underbrace{y\ldots y}_{n}$.

We use the following representation of differential groupoids which is due to [17]. A groupoid $\mathcal{G}$ is an **Lz**-**Lz**-*sum* (of orbits $\mathcal{G}_i$ over a groupoid $\mathcal{I}$) *satisfying the left normal law* if there is a partition $G = \bigcup_{i\in I} G_i$ such that, for every pair $(i,j) \in I^2$, there is a mapping $h_i^j \colon G_i \to G_i$ satisfying the following conditions:

(i) for every $i \in I$, $h_i^i$ is the identity mapping;

(ii) we have $h_i^j(h_i^k(x)) = h_i^k(h_i^j(x))$ for all $i,j,k \in I$ and $x \in G_i$;

(iii) we have $a_i \cdot a_j = h_i^j(a_i)$ for all $i,j \in I$, $a_i \in G_i$ and $a_j \in G_j$.

According to [17, Theorem 2.2], a groupoid is differential if and only if it can be represented as an **Lz**-**Lz**-sum satisfying the left normal law. For more detailed information on differential groupoids, we refer to the monograph [18, Secs. 5.6 and 8.4].

Let $n > 0$. The structure defined in **Dm** by the generators $\{x, y\}$ and the defining relations $\{yx = y, xy^n = x\}$ is called the *cycle* of length $n$ and is denoted by $\mathcal{D}_n$. It is convenient to regard $D_n$ as $G_0 \cup G_1$, where $G_1$ is the singleton orbit $\{b\}$ and $G_0 = \{a, ab, ab^2, \dots, ab^{n-1}\}$. We denote the trivial groupoid by $\mathcal{D}_0$.

Let $\mathbb{P}$ denote the set of all primes; we assume that $\mathbb{P} = \{p_i \mid i < \omega\}$, where $p_i \leqslant p_j$ if and only if $i \leqslant j$ for all $i, j < \omega$. Let $\mathcal{P}_{\mathrm{fin}}(\omega)$ denote the set of all finite subsets of the set of all natural numbers $\omega$. For an arbitrary set $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$, we put $[F] = \prod_{i \in F} p_i$ if $F \neq \varnothing$ and $[F] = 1$ if $F = \varnothing$.

**Lemma 1.** *Let $n > 0$.*

(i) *The class $\{\mathcal{D}_m \mid m$ divides $n\}$ coincides with the class of nontrivial homomorphic images of $\mathcal{D}_n$.*

(ii) *If $m \in \mathbb{N}$ and $\varphi\colon \mathcal{D}_n \to \mathcal{D}_m$ is a homomorphism then either $\varphi(\mathcal{D}_n) \cong \mathcal{D}_0$ or $m$ divides $n$.*

(iii) *If $n > 0$ and $X, X_1, \dots, X_n \in \mathcal{P}_{\mathrm{fin}}(\omega)$ are such that the set $\{1, \dots, n\}$ is minimal with respect to the property that $\mathcal{D}_{[X]} \in \mathbf{SP}(\mathcal{D}_{[X_1]}, \dots, \mathcal{D}_{[X_n]})$, then $X = X_1 \cup \dots \cup X_n$. Conversely, if $X = X_1 \cup \dots \cup X_n$ then $\mathcal{D}_{[X]} \in \mathbf{SP}(\mathcal{D}_{[X_1]}, \dots, \mathcal{D}_{[X_n]})$.*

*Proof.* Statements (i)–(ii) can be deduced from [6, Lemma 3]. Statement (iii) follows from (i)–(ii) and [1, Lemma 3], see also [21, Lemma 4.3]. □

The structure of the lattice of varieties of differential groupoids is explicitly described in [17] (see also [18, Theorem 8.4.14]). In particular, each subvariety of **Dm** is defined by a single identity and is locally finite. In contrast to that description, the structure of the lattice $\mathrm{Lq}(\mathbf{Dm})$ of quasivarieties is much more complicated. Namely, the variety **Dm** is $Q$-universal [6], there exist $2^\omega$ classes **K** of differential groupoids such that the set of (isomorphism types of) finite sublattices of $\mathrm{Lq}(\mathbf{K})$ is not computable [19], and there exist continuum many quasivarieties of differential groupoids with no independent quasi-equational basis [1].

The following statement is proven in [8, Theorem 4], see also [1, Theorem 5].

**Theorem 2.** *There exist $2^\omega$ quasivarieties of differential groupoids with an $\omega$-independent quasi-equational basis and no independent quasi-equational basis relative to* **Dm**.

2.2. **Unary algebras.** As is proven in [5], the variety $\mathbf{K}_3$ of unary algebras of signature $\sigma = \{f, g\}$ defined by the identities

$$\forall x\, \forall y\ [f(f(x)) = f(f(y)) = f(g(y))],$$
$$\forall x\, \forall y\ [g(g(x)) = g(g(y)) = g(f(y))],$$
$$\forall x\ [f(f(x)) = g(g(x))]$$

is a minimal $Q$-universal variety. It follows from the proof that the proper subquasivariety $\mathbf{V} \subset \mathbf{K}_3$ defined by the quasi-identities

$$\forall x\ [f(x) = f(f(x)) \longrightarrow f(x) = g(x)],$$
$$\forall x\ [g(x) = g(g(x)) \longrightarrow f(x) = g(x)],$$
$$\forall x\ [f(x) = g(x) \longrightarrow f(x) = f(f(x))],$$
$$\forall x\, \forall y\ [f(x) = f(y) \longrightarrow g(x) = g(y)],$$
$$\forall x\, \forall y\ [g(x) = g(y) \longrightarrow f(x) = f(y)]$$

is $Q$-universal; moreover, so is the lattice of $\mathbf{W}$-quasivarieties, where $\mathbf{W}$ denotes the subclass of $\mathbf{V}$ defined by the sentences

$$\forall x\,\forall y\ [g(x) = g(y)\,\&\,x \neq y \longrightarrow g(x) = g(g(x))], \tag{1}$$

$$(\forall x\ [g(x) = g(g(x))] \longrightarrow \forall x\,\forall y\ [x = y]). \tag{2}$$

We recall the notation for certain unary algebras, see [5, 7].

For $n \geqslant 2$, let $\mathcal{C}_n$ denote the algebra whose universe is

$$C_n = \{0\} \cup A_n \cup B_n, \quad A_n = \{a^n_0, \ldots, a^n_{n-1}\},\ B_n = \{b^n_0, \ldots, b^n_{n-1}\},$$

and the operations are defined as follows: $f(0) = g(0) = f(a^n_i) = g(a^n_i) = 0$ and $g(b^n_i) = a^n_i$ for $0 \leqslant i \leqslant n-1$, $f(b^n_i) = a^n_{i+1}$ for $0 \leqslant i \leqslant n-2$, and $f(b^n_{n-1}) = a^n_0$. Let $\mathcal{C}_1$ denote the trivial algebra.

It is clear that, for $n \geqslant 1$, we have $\mathcal{C}_n \in \mathbf{W}$.

We recall necessary properties of the algebras $\mathcal{C}_n$ from [5, Lemma 3] and [7, Lemmas 1–3].

**Lemma 3.** *If $n > 1$ then the following statements hold.*

- (i) *If $m$ divides $n$ then there exists a homomorphism $\varphi$ from $\mathcal{C}_n$ onto $\mathcal{C}_m$; moreover, the kernels of all such homomorphisms coincide and we have*

$$\ker\varphi = \{(x,x) : x \in C_n\} \cup \{(a_i,a_j),(b_i,b_j) : i \equiv j \pmod{m}\}.$$

- (ii) *If $\mathcal{A} \in \mathbf{W}$ and there exists a homomorphism from $\mathcal{C}_n$ onto $\mathcal{A}$ then $\mathcal{A}$ is isomorphic to $\mathcal{C}_m$ for a suitable divisor $m$ of $n$.*

**Lemma 4.** *If $X, Y, Z \in \mathcal{P}_{\mathrm{fin}}(\omega)$ and $n > 1$ then the following statements hold.*

- (i) *There exists a homomorphism from $\mathcal{C}_{[X]}$ onto $\mathcal{C}_{[Y]}$ if and only if $Y \subseteq X$.*
- (ii) *We have $\mathcal{C}_{[X]} \leq \mathcal{C}_{[Y]} \times \mathcal{C}_{[Z]}$ if and only if $X = Y \cup Z$.*
- (iii) *If $\mathcal{C}_n \leq \mathcal{C}_m \times \mathcal{C}_k$ then each prime divisor of $n$ divides either $m$ or $k$.*

**Lemma 5.** *If $n > 1$ and $m > 0$ then there exists a quasi-identity $q(n,m)$ such that, for every $k > 1$, the structure $\mathcal{C}_k$ satisfies $q(n,m)$ if and only if either $k$ is not a divisor of $n$ or $k$ divides $m$.*

The following statement is proven in [8, Theorem 8], see also [7, Theorem 2].

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**Theorem 6.** *There exist $2^\omega$ $\mathbf{W}$-subquasivarieties with an $\omega$-independent quasi-equational basis and no independent quasi-equational basis relative to $\mathbf{W}$.*

2.3. **Lattices.** We use the following construction which is similar to the one described in [12]. Let $\mathbf{2} = \{0,1\}$ be a two-element lattice. Consider an arbitrary sequence $\mathbb{L} = \langle L_i \mid i < \omega\rangle$ such that $L_i$ is a finite lattice for all $i < \omega$. For each $i < \omega$, let $1_i$ denote the greatest and $0_i$ the least element of $L_i$; let also $M_i = L_i \backslash \{1_i\}$. We assume that $0, 1 \notin \bigcup_{i<\omega} M_i$ and that $M_j \cap M_j = \varnothing$ whenever $i < j < \omega$. On the set $L(\mathbb{L}) = \{0,1\} \cup \bigcup_{i<\omega} M_i$, we define a partial order $\leq$ as follows. For $a, b \in L(\mathbb{L})$, we put $a \leq b$ if one of the following conditions hold:

- $a = 0$;
- $b = 1$;
- $a \in M_i$ and $b \in M_j$ for some $i < j < \omega$;
- $a$, $b \in M_i$ for some $i < \omega$ and $a \leq b$ in $L_i$.

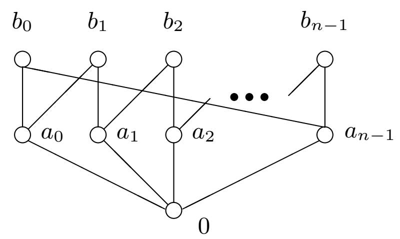


FIG. 1. Meet semilattice $\mathcal{K}_n$, a crown

Then 0 is a least element of $L(\mathbb{L})$, and $L(\mathbb{L})$ is a complete algebraic lattice, where each element $a$ with $a \neq 1$ is compact.

We define a mapping $\xi\colon L(\mathbb{L}) \to \prod_{i<\omega} L_i \times \mathbf{2} \times \mathbf{2}$. We put

$$\xi(a)(i) = \begin{cases} 0_i & \text{if } a \in M_j \text{ for some } j < i, \\ a & \text{if } a \in M_i, \\ 1_i & \text{if } a \in M_j \text{ for some } j > i \text{ or } a = 1; \end{cases}$$

$$\xi(a)(\omega) = \begin{cases} 0 & \text{if } a = 0, \\ 1 & \text{if } a \in M_i \text{ for some } i < \omega \text{ or } a = 1; \end{cases}$$

$$\xi(a)(\omega+1) = \begin{cases} 0 & \text{if } a \in M_i \text{ for some } i < \omega \text{ or } a = 0, \\ 1 & \text{if } a = 1, \end{cases}$$

where $i < \omega$. It is straightforward to verify that $\xi$ is a subdirect embedding. Let $L^*(\mathbb{L}) = L(\mathbb{L}) \setminus \{0\}$.

For every natural number $n > 2$, we consider the semilattice $\mathcal{K}_n$ displayed on Figure 1 and the lattice $\operatorname{Sub}(\mathcal{K}_n)$ of its subsemilattices.

The next lemma is essentially [12, Lemma 2.2].

**Lemma 7.** *Let $f\colon \omega \to \omega\setminus\{0,1,2\}$ be a strictly increasing function such that the set $f(\omega)$ is not computable* (*not computably enumerable*) *and let $\mathbb{L} = \langle \operatorname{Sub}(\mathcal{K}_{f(i)}) \mid i < \omega \rangle$. Then the sets of isomorphism types of finite sublattices of $L(\mathbb{L})$, $L^*(\mathbb{L})$, and $\prod_{i<\omega} \operatorname{Sub}(\mathcal{K}_{f(i)}) \times \mathbf{2} \times \mathbf{2}$ are not computable* (*not computably enumerable*).

2.4. **Operators on classes.** *Quasi-identities* are universal sentences of the form

$$\forall \overline{x}\ [\varphi_1(\overline{x})\ \&\ldots\&\ \varphi_k(\overline{x}) \longrightarrow \varphi_0(\overline{x})],$$

where $\varphi_i(\overline{x})$ is an atomic formula for each $i \leqslant k$. A class $\mathbf{K}$ is a *quasivariety* if it coincides with the class of all models of some set $\Phi$ of quasi-identities. Then the set $\Phi$ is called a *quasi-equational basis* for $\mathbf{K}$.

We denote structures by calligraphic letters. The universe of a structure is denoted by the corresponding italic letter. For classes of algebraic structures, we use boldface letters. We assume that all classes are *abstract*, i.e., closed under isomorphism.

Let $\mathbf{K}(\sigma)$ denote the class of all structures of signature $\sigma$. For a class $\mathbf{K} \subseteq \mathbf{K}(\sigma)$, let $\mathbf{V}(\mathbf{K})$ denote the least variety and let $\mathbf{Q}(\mathbf{K})$ denote the least quasivariety containing $\mathbf{K}$. Let $\mathbf{H}(\mathbf{K})$ denote the class of structures from $\mathbf{K}(\sigma)$ which are homomorphic images of structures from $\mathbf{K}$; let $\mathbf{P}(\mathbf{K})$ denote the class of structures from $\mathbf{K}(\sigma)$ which are isomorphic to Cartesian products of structures from $\mathbf{K}$; let $\mathbf{S}(\mathbf{K})$

denote the class of structures from $\mathbf{K}(\sigma)$ which are isomorphic to substructures of structures from $\mathbf{K}$; let $\mathbf{L}_s(\mathbf{K})$ denote the class of structures from $\mathbf{K}(\sigma)$ which are isomorphic to limits of superdirect spectra of structures from $\mathbf{K}$; cf. for example, [3, Sec. 1.2.7]. Finally, let $\mathbf{T}$ denote the trivial (quasi)variety.

According to G. Birkhoff [2] and V. A. Gorbunov and V. I. Tumanov [4], we have

$$\mathbf{V}(\mathbf{K}) = \mathbf{HSP}(\mathbf{K}); \quad \mathbf{Q}(\mathbf{K}) = \mathbf{L_s SP}(\mathbf{K}),$$

see also [3, Theorem 2.1.12 and Corollary 2.3.4]. Let $\mathbf{K}' \subseteq \mathbf{K} \subseteq \mathbf{K}(\sigma)$. Then $\mathbf{K}'$ is a $\mathbf{K}$-*variety*, if $\mathbf{K}' = \mathbf{K} \cap \operatorname{Mod}(\Sigma)$ for some set $\Sigma$ of identities of signature $\sigma$. The following statement is straightforward to prove.

**Lemma 8.** *A class $\mathbf{K}' \subseteq \mathbf{K}$ is a $\mathbf{K}$-variety, if and only if $\mathbf{K}' = \mathbf{K} \cap \mathbf{V}(\mathbf{K}')$. A class $\mathbf{K}' \subseteq \mathbf{K}$ is a $\mathbf{K}$-quasivariety, if and only if $\mathbf{K}' = \mathbf{K} \cap \mathbf{Q}(\mathbf{K}')$.*

Let $\operatorname{Lv}(\mathbf{K})$ denote the set of all $\mathbf{K}$-varieties of $\mathbf{K}$. Ordered with respect to inclusion, $\operatorname{Lv}(\mathbf{K})$ forms a complete lattice which is called the *lattice of* $\mathbf{K}$-*varieties* (or *relative varieties*). The following two results concerning the lattices of relative varieties and congruences were obtained in [11, Theorem 5.1] and [11, Theorem 3.1] respectively.

**Theorem 9.** *Let $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ be a quasivariety of a finite type $\sigma$ and let $\mathbf{A} = \{\mathcal{A}_F \mid F \in \mathcal{P}_{\mathrm{fin}}(\omega)\} \subseteq \mathbf{M}$ be a* B-*class with respect to $\mathbf{M}$ such that the following conditions hold for every nonempty set $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$ and every $G \in \mathcal{P}_{\mathrm{fin}}(\omega)$:*

(1) *there are no proper subdirectly irreducible substructures of $\mathcal{A}_F$ in $\mathbf{Q}(\mathcal{A}_F)$;*
(2) *if $\mathbf{V}(\mathcal{A}_F) = \mathbf{V}(\mathcal{A}_G)$ then $F = G$.*

*Then, for every finite lattice $L$, there is a locally finite quasivariety $\mathbf{K} \subseteq \mathbf{Q}(\mathbf{A})$ such that $\operatorname{Lv}\big(\mathbf{K}\big) \cong L$.*

**Theorem 10.** *Let $\mathbf{M}$ be a quasivariety of finite similarity type and let $\mathbf{A} = \{\mathcal{A}_F \mid F \in \mathcal{P}_{\mathrm{fin}}(\omega)\} \subseteq \mathbf{M}$ be a* B-*class with respect to $\mathbf{M}$. Then, for every finite lattice $L$, there is a quasivariety $\mathbf{K} \subseteq \mathbf{Q}(\mathbf{A})$ and a set $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$ such that $\mathcal{A}_F \in \mathbf{K}$ and $\operatorname{Con}_{\mathbf{K}} \mathcal{A}_F \cong L$. Moreover, if $\mathbf{A}$ is a finite* B-*class then $\mathcal{A}_F$ is finite and the quasivariety $\mathbf{K}$ is locally finite.*

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The following two stronger results were obtained in [12, Theorem 3.1] and [12, Theorem 3.8] respectively. It is easy to see that Theorem 11 implies Theorem 9 and Theorem 12 implies Theorem 10.

**Theorem 11.** *Let $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ be a quasivariety of a finite type $\sigma$ and let $\mathbf{A} = \{\mathcal{A}_F \mid F \in \mathcal{P}_{\mathrm{fin}}(\omega)\} \subseteq \mathbf{M}$ be a finite* B-*class with respect to $\mathbf{M}$. Assume that the following conditions hold for all nonempty finite sets $F, G \in \mathcal{P}_{\mathrm{fin}}(\omega)$:*

(1) *if $\mathbf{V}(\mathcal{A}_F) = \mathbf{V}(\mathcal{A}_G)$ then $F = G$;*
(2) *for every $F \in \mathcal{P}_{\mathrm{fin}}(\omega)$, we have $\mathbf{S}(\mathcal{A}_F) \subseteq \mathbf{A}$;*
(3) *for every proper relative subvariety $\mathbf{W} \subset \mathbf{Q}(\mathbf{A})$, each finitely generated structure $\mathcal{A} \in \mathbf{W}$ is $l$-projective in $\mathbf{Q}(\mathbf{A})$.*

*Then, for every sequence $\mathbb{L} = \langle L_i \mid i < \omega \rangle$ of finite lattices, there is a subquasivariety $\mathbf{K}$ of $\mathbf{Q}(\mathbf{A})$ such that $\operatorname{Lv}(\mathbf{K}) \cong L^*(\mathbb{L})$.*

For the definition and main properties of $l$-projective structures, the reader is referred to [3, Sec. 2.4].

**Theorem 12.** *Let* $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ *be a quasivariety of a finite type* $\sigma$ *and let* $\mathbf{A} = \{\mathcal{A}_F \mid F \in \mathcal{P}_{\text{fin}}(\omega)\} \subseteq \mathbf{M}$ *be a* B*-class with respect to* $\mathbf{M}$*. If there is a structure* $\mathcal{A} \in \mathbf{K}(\sigma)$ *with the properties*

(4) $\mathcal{A} \in \mathbf{Q}\big(\mathcal{A}_F \mid F \in \mathcal{I}\big)$ *for every infinite subset* $\mathcal{I} \subseteq \mathcal{P}_{\text{fin}}(\omega)$;
(5) $\mathbf{H}(\mathcal{A}) \cap \mathbf{Q}(\mathbf{A}) = \mathbf{A} \cup \{\mathcal{A}\}$;
(6) *for every nonempty set* $F \in \mathcal{P}_{\text{fin}}(\omega)$*, if* $f,\ g\colon \mathcal{A} \to \mathcal{A}_F$ *are onto homomorphisms then* $\ker f = \ker g$*; moreover, if* $f(\mathcal{A}) \cong \mathcal{A}$ *for some homomorphism* $f$ *then* $f$ *is an embedding*

*then, for every sequence* $\mathbb{L} = \langle L_i \mid i < \omega\rangle$ *of finite lattices, there is a subquasivariety* $\mathbf{K} \subseteq \mathbf{Q}(\mathbf{A})$ *such that* $\mathcal{A} \in \mathbf{K}$ *and* $\mathrm{Con}_{\mathbf{K}}\,\mathcal{A}$ *is dually isomorphic to* $L^*(\mathbb{L})$*.*

Theorems 9 and 10 apply to many classical quasivarieties, see Corollaries 3.2 and 5.2 in [11]. The same applies to Theorems 11 and 12, see Corollaries 5.1 and 5.2 in [12]. It is essential in those cases that a quasivariety contains a B-class for the definition of which we refer to [9]. According to [10, Proposition 10.6], the variety **Dm** and the quasivariety **V** contain no B-classes. In the present paper, we prove that, nevertheless, analogues of Theorems 11 and 12 hold also for **Dm** and **W**, see Theorems 13–17.

## 3. Main results

We use the notation introduced in Subsection 2.3.

**Theorem 13.** *For every sequence* $\mathbb{L} = \langle L_i \mid i < \omega\rangle$ *of finite lattices, there is a quasivariety* $\mathbf{K} \subseteq \mathbf{Dm}$ *such that* $\mathrm{Lv}(\mathbf{K}) \cong L(\mathbb{L})$*.*

*Proof.* Let $\omega = \bigcup_{i<\omega} N_i$ be a partition of the set $\omega$ such that $|N_i| = |M_i|$ for all $i < \omega$. For each $i < \omega$, let also $a \mapsto n_a$ be a bijection of $M_i$ onto $N_i$. For each $i < \omega$ and each $a \in M_i$, let

$$X(a) = \bigcup_{0<j<i} N_j \cup \{n_b \in N_i \mid b \ngeq a\}.$$

For brevity, we write $L$ instead of $L(\mathbb{L})$. For the proof of the following claim, see the proof of [12, Claim 3.2].

**Claim 1.** *The following conditions hold for all* $a,\ b \in L \setminus \{0,1\}$:

(i) *we have* $X(0_0) = \varnothing$;
(ii) *we have* $X(a \vee b) = X(a) \cup X(b)$;
(iii) *we have* $X(a) \subseteq X(b)$ *if and only if* $a \leq b$*.*

We put

$$\mathcal{A}_a = \mathcal{D}_{[X(a)]} \text{ for all } a \in L\backslash\{0,1\},$$
$$\mathbf{M} = \big\{\mathcal{A}_a \mid a \in L,\ a \notin \{0,1\}\big\}, \quad \mathbf{K} = \mathbf{Q}(\mathbf{M}).$$

In particular, we have $\mathcal{A}_{0_0} \cong \mathcal{D}_1$.

**Claim 2.** *For each* $a \in L \setminus \{0,1\}$*, we have* $\mathbf{V}(\mathcal{A}_a) \cap \mathbf{M} = \big\{\mathcal{A}_b \mid 0 < b \leq a\big\}$*.*

*Proof of Claim.* If $a = 0_0$, then the classes on the both sides of the equality are equal to $\{\mathcal{D}_1\}$ by Claim 1(i). Hence, we may assume that $a \neq 0_0$.

Let $0 < b \leq a$. By Claim 1(iii), we have $X(b) \subseteq X(a)$. By Lemma 1(i), we obtain $\{\mathcal{A}_b \mid 0 < b \leq a\} \subseteq \mathbf{H}(\mathcal{A}_a) \cap \mathbf{M} \subseteq \mathbf{V}(\mathcal{A}_a) \cap \mathbf{M}$.

Let $\mathcal{B}$ be a nontrivial structure in $\mathbf{V}(\mathcal{A}_a) \cap \mathbf{M}$. Since $\mathcal{B} \in \mathbf{M}$, we have $\mathcal{B} \cong \mathcal{A}_b$ for a suitable element $b \in L \backslash \{0, 1\}$. It remains to show that $b \leq a$.

By Claim 1(ii), we have $X(a \vee b) = X(a) \cup X(b)$. According to Lemma 1(iii), we have $\mathcal{A}_{a \vee b} \in \mathbf{SP}(\mathcal{A}_a, \mathcal{A}_b) \subseteq \mathbf{V}(\mathcal{A}_a, \mathcal{A}_b)$. Since $\mathcal{A}_b \in \mathbf{V}(\mathcal{A}_a)$, we conclude that $\mathcal{A}_{a \vee b} \in \mathbf{V}(\mathcal{A}_a)$. Since

$$\mathcal{A}_a \models \forall x \, \forall y \, \left[ xy^{[X(a)]} = x \right],$$

we conclude that

$$\mathcal{A}_{a \vee b} \models \forall x \, \forall y \, \left[ xy^{[X(a)]} = x \right],$$

which implies that $[X(a \vee b)]$ divides $[X(a)]$. This means that $X(b) \subseteq X(a \vee b) \subseteq X(a)$. Therefore, $b \leq a$ in view of Claim 1(iii). □

**Claim 3.** *For every $a \in L \setminus \{0, 1\}$, we have*

$$\mathbf{V}(\mathcal{A}_a) \cap \mathbf{K} = \mathbf{V}(\mathcal{A}_a) \cap \mathbf{Q}(\mathbf{M}) = \mathbf{Q}\big(\mathcal{A}_b \mid 0 < b \leq a\big) = \mathbf{SP}\big(\mathcal{A}_b \mid 0 < b \leq a\big).$$

*Proof of Claim.* If $a = 0_0$, then

$$\mathbf{V}(\mathcal{A}_a) \cap \mathbf{K} = \mathbf{V}(\mathcal{A}_a) \cap \mathbf{Q}(\mathbf{M}) = \mathbf{V}(\mathcal{D}_1) = \mathbf{Q}(\mathcal{D}_1) = \mathbf{SP}(\mathcal{D}_1),$$

so that the desired statement holds. We assume now that $a \neq 0_0$. By Claim 2, we have $\{\mathcal{A}_b \mid 0 < b \leq a\} \subseteq \mathbf{V}(\mathcal{A}_a)$ and $\{\mathcal{A}_b \mid 0 < b \leq a\} \subseteq \mathbf{M}$. Hence, $\mathbf{Q}\big(\mathcal{A}_b \mid 0 < b \leq a\big) \subseteq \mathbf{V}(\mathcal{A}_a)$ and

$$\mathbf{Q}\big(\mathcal{A}_b \mid 0 < b \leq a\big) = \mathbf{SP}\big(\mathcal{A}_b \mid 0 < b \leq a\big) \subseteq \mathbf{SP}(\mathbf{M}) \subseteq \mathbf{K}.$$

Therefore,

$$\mathbf{Q}\big(\mathcal{A}_b \mid 0 < b \leq a\big) \subseteq \mathbf{V}(\mathcal{A}_a) \cap \mathbf{K} = \mathbf{V}(\mathcal{A}_a) \cap \mathbf{Q}(\mathbf{M}).$$

Let $\mathcal{B}$ be a nontrivial structure in $\mathbf{V}(\mathcal{A}_a) \cap \mathbf{Q}(\mathbf{M})$. Since $\mathcal{A}_a$ is a finite structure, the variety $\mathbf{V}(\mathcal{A}_a)$ is locally finite. Since every (quasi)variety is generated by its finitely generated structures, we may assume that $\mathcal{B}$ is finitely generated and thus finite; in particular, $\mathcal{B}$ is $l$-projective in $\mathbf{Q}(\mathbf{M}) = \mathbf{L}_s\mathbf{SP}(\mathbf{M})$. Therefore, $\mathcal{B} \in \mathbf{SP}(\mathbf{M})$. By Lemma 1(iii), we conclude that $\mathcal{B}$ can be represented as a subdirect product of the structures from the set $\{\mathcal{A}_{b_0}, \ldots, \mathcal{A}_{b_n}\}$ for suitable $n < \omega$ and some elements $b_0, \ldots, b_n \in L \backslash \{0, 1\}$. It remains to show that $b_0, \ldots, b_n \leq a$.

By the definition of a subdirect product, we have $\mathcal{A}_{b_i} \in \mathbf{H}(\mathcal{B}) \subseteq \mathbf{V}(\mathcal{B})$ for each $i \leqslant n$. Since $\mathcal{B} \in \mathbf{V}(\mathcal{A}_a)$, we conclude that $\{\mathcal{A}_{b_0}, \ldots, \mathcal{A}_{b_n}\} \subseteq \mathbf{V}(\mathcal{A}_a)$. Let $b = b_0 \vee \cdots \vee b_n$. By Claim 1(ii), we have $X(a \vee b) = X(a) \cup X(b_0) \cup \cdots \cup X(b_n)$. According to Lemma 1(iii), we conclude that

$$\mathcal{A}_{a \vee b} \in \mathbf{SP}(\mathcal{A}_a, \mathcal{A}_{b_0}, \ldots, \mathcal{A}_{b_n}) \subseteq \mathbf{V}(\mathcal{A}_a, \mathcal{A}_{b_0}, \ldots \mathcal{A}_{b_n}) = \mathbf{V}(\mathcal{A}_a).$$

Since $\mathcal{A}_a \models \forall x \, \forall y \, \left[ xy^{[X(a)]} = x \right]$, we conclude that

$$\mathcal{A}_{a \vee b} \models \forall x \, \forall y \, \left[ xy^{[X(a)]} = x \right],$$

which implies that $[X(a \vee b)]$ divides $[X(a)]$. This means that $X(b_i) \subseteq X(a \vee b) \subseteq X(a)$ for all $i \leqslant n$. We use Claim 1(iii) and obtain $b_i \leq a$ for all $i \leqslant n$. □

We can therefore define a mapping $\varphi \colon L \to \mathrm{Lv}(\mathbf{K})$ by

$$\varphi(a) = \begin{cases} \mathbf{V}(\mathcal{A}_a) \cap \mathbf{K} & \text{if } a \notin \{0, 1\}, \\ \mathbf{T} & \text{if } a = 0, \\ \mathbf{K} & \text{if } a = 1. \end{cases}$$

**Claim 4.** *For each $a \in L$, the following conditions hold*:

(i) $\varphi(a) \in \mathrm{Lv}(\mathbf{K})$;

(ii) $\varphi(a) \subseteq \varphi(b)$ *if and only if* $a \leq b$.

*In particular,* $\varphi$ *is one-to-one.*

*Proof of Claim.* Condition (i) is immediate from the definition of $\varphi$. Condition (ii) is immediate from Claims 2–3. Since the equality $\varphi(a) = \mathbf{K}$ is valid for $a = 1$ only and the equality $\varphi(a) = \mathbf{T}$ is valid for $a = 0$ only, the particular statement is immediate from (ii). □

**Claim 5.** *The mapping* $\varphi$ *is onto.*

*Proof of Claim.* We have $\mathbf{T} = \varphi(0)$. Let $\mathbf{V} \subseteq \mathbf{Dm}$ be a nontrivial variety and let $B = \{b \in L\backslash\{0,1\} \mid \mathcal{A}_b \in \mathbf{V}\}$. We notice that $B \neq \varnothing$, as $0_0 \in B$.

We consider the case in which the set $B$ is infinite. In this case, for every $n < \omega$, there is $b \in B$ such that $b \in M_j$ for some $j > n$. For every $a \in M_n$, we obtain $X(a) \subseteq \bigcup_{k<j} N_k \subseteq X(b)$. We conclude by Lemma 1(i) that $\mathcal{A}_a \in \mathbf{H}(\mathcal{A}_b) \subseteq \mathbf{V}$, i.e., $\mathbf{M} \subseteq \mathbf{V}$, whence $\mathbf{K} = \mathbf{Q}(\mathbf{M}) \subseteq \mathbf{V}$. Thus, $\mathbf{V} \cap \mathbf{K} = \mathbf{K} = \varphi(1)$.

We consider the case in which the set $B$ is finite. We put $a = \bigvee B$. Then $a \in L\backslash\{0,1\}$ and $X(a) = \bigcup\{X(b) \mid b \in B\}$. According to Lemma 1(iii), we have $\mathcal{A}_a \in \mathbf{SP}(\mathcal{A}_{X(b)} \mid b \in B) \subseteq \mathbf{V}$, whence $a \in B$ and $\varphi(a) = \mathbf{V}(\mathcal{A}_a) \cap \mathbf{K} \subseteq \mathbf{V} \cap \mathbf{K}$.

To prove the reverse inclusion, let $\mathcal{C} \in \mathbf{V} \cap \mathbf{M} \subseteq \mathbf{M}$. Then there is $b \in B$ such that $\mathcal{C} \cong \mathcal{A}_b$. Since $X(b) \subseteq X(a)$, it follows from Lemma 1(i) that $\mathcal{C} \in \mathbf{H}(\mathcal{A}_a) \cap \mathbf{M} \subseteq \mathbf{V}(\mathcal{A}_a) \cap \mathbf{M}$. Therefore, $\mathbf{V} \cap \mathbf{M} = \mathbf{V}(\mathcal{A}_a) \cap \mathbf{M}$.

Suppose now that $\mathcal{C} \in \mathbf{V} \cap \mathbf{K} \subseteq \mathbf{Q}(\mathbf{M})$. Since each (quasi)variety is generated by its finitely generated structures, we may assume that $\mathcal{C}$ is finitely generated. Since each proper subvariety of $\mathbf{Dm}$ is locally finite (see [17] or [18, Theorem 8.4.14]), we obtain $\mathcal{C} \in \mathbf{SP}(\mathbf{M})$. This means that there is a set $J$ such that $\mathcal{C} \leq \prod_{j \in J} \mathcal{C}_j$ and $\mathcal{C}_j \in \mathbf{M}$ for every $j \in J$. Taking into account that $\mathcal{C}_j$ is a finite structure and each substructure of a structure from $\mathbf{M}$ belongs to $\mathbf{P}_s(\mathbf{M})$, we may assume that $\pi_j(\mathcal{C}) \in \mathbf{M}$ for all $j \in J$; in particular, $\pi_j(\mathcal{C}) \in \mathbf{H}(\mathcal{C}) \subseteq \mathbf{V}$ for all $j \in J$. Therefore, $\pi_j(\mathcal{C}) \in \mathbf{V} \cap \mathbf{M} = \mathbf{V}(\mathcal{A}_{X(a)}) \cap \mathbf{M}$ for all $j \in J$ according to what we have just proved above. This implies that $\mathcal{C} \in \mathbf{V}(\mathcal{A}_{X(a)}) \cap \mathbf{K}$ and thus $\mathbf{V} \cap \mathbf{K} = \mathbf{V}(\mathcal{A}_{X(a)}) \cap \mathbf{K} = \varphi(a)$. □

It follows from Claims 4 and 5 that $\varphi$ is an isomorphism. □

From Theorem 13, we obtain the following

**Corollary 14.** *For every finite lattice $L$ such that the least element is meet-irreducible, there is a subquasivariety* $\mathbf{K}$ *of* $\mathbf{Dm}$ *such that* $\mathrm{Lv}(\mathbf{K}) \cong L$.

*Proof.* Let $L_0 = L\backslash\{0_L\}$ and let $L_i$ be a one-element lattice for all positive $i < \omega$. It is straightforward that $L \cong L(\mathbb{L})$, where $\mathbb{L} = \langle L_i \mid i < \omega\rangle$. Moreover, the class $\mathbf{M}$ constructed in the proof of Theorem 13 contains only finitely many non-isomorphic structures. Hence $\mathbf{K} = \mathbf{SP}(\mathbf{M}) = \mathbf{Q}(\mathbf{M})$, and it remains to refer to Theorem 13. □

**Theorem 15.** *For every sequence $\mathbb{L} = \langle L_i \mid i < \omega\rangle$ of finite lattices, there is a subquasivariety* $\mathbf{K}$ *of* $\mathbf{Dm}$ *and a differential groupoid* $\mathcal{D} \in \mathbf{K}$ *such that* $\mathrm{Con}_{\mathbf{K}} \mathcal{D}$ *is dually isomorphic to* $L(\mathbb{L})$.

*Proof.* We use the notation introduced in the proof of Theorem 13. We put

$$\mathbf{K} = \mathbf{Q}\big(\mathcal{A}_{X(a)} \mid a \in L,\ a \notin \{0,1\}\big).$$

**Claim 1.** *For a positive integer $n < \omega$, we have $\mathcal{D}_n \in \mathbf{K}$ if and only if $n = [X(a)]$ for some $a \in L \backslash \{0,1\}$.*

*Proof of Claim.* Suppose that

$$\mathcal{D}_n \in \mathbf{K} = \mathbf{L}_s\mathbf{SP}\big(\mathcal{A}_{X(a)} \mid a \in L,\ a \notin \{0,1\}\big)$$

for some positive integer $n < \omega$. Since $\mathcal{D}_n$ is a finite structure, it is $l$-projective in $\mathbf{K}$, whence $\mathcal{D}_n \in \mathbf{SP}\big(\mathcal{A}_{X(a)} \mid a \in L,\ a \notin \{0,1\}\big)$. By Lemma 1(iii), we conclude that $\mathcal{D}_n$ can be represented as a subdirect product of structures from the set $\{\mathcal{A}_{b_0}, \ldots, \mathcal{A}_{b_m}\}$ for suitable $m < \omega$ and elements $b_0, \ldots, b_m \in L\backslash\{0,1\}$. According to Lemma 1(i), $[X(b_i)]$ divides $n$ for all $j \leqslant m$. Consider the set $F = \big\{b \in L\backslash\{0,1\} \mid [X(b)] \text{ divides } n\big\}$. The set $F$ is finite and nonempty, as $n$ is a positive integer and $\{b_0, \ldots, b_m\} \subseteq F$; we put $a = \bigvee F$. Then $a \in L\backslash\{0,1\}$ and $X(a) = \bigcup_{b \in F} X(b)$ by Claim 1(ii) from the proof of Theorem 13, whence $[X(a)]$ divides $n$ and $a$ is the greatest element of $F$. Since

$$\mathbf{K} \models \forall x\, \forall y\ \Big[xy^n = x \longrightarrow xy^{[X(a)]} = x\Big]$$

and $\mathcal{D}_n \in \mathbf{K}$, we conclude that $n = [X(a)]$. □

We continue the proof of the theorem. There are two cases to consider.

*Case* 1: the set $\{i < \omega \mid |L_i| > 1\}$ is finite. In this case, $L = L(\mathbb{L})$ is a finite lattice and we may assume without loss of generality that $L_0 \cong L(\mathbb{L})$, that $L_i$ is a trivial lattice for all positive $i < \omega$, and that $a = 1 = 1_0$. Then $X(a) = L\backslash\{0,1\} = L_0\backslash\{1_0\}$; we put $\mathcal{D} = \langle\{\infty, 0, \ldots, [X(a)] - 1\}; \cdot\rangle \cong \mathcal{A}_a$. Consider an arbitrary non-trivial congruence $\theta \in \mathrm{Con}_{\mathbf{K}}\,\mathcal{D}$. Two cases are possible.

*Case* 1.1: $(\infty, n) \in \theta$ for some $n < [X(a)]$. In this case, we have for all $m < [X(a)]$

$$m = (m \cdot n)\theta(m \cdot \infty) = (m+1) \mod [X(a)],$$

whence $\theta = 1_{\mathcal{D}}$.

*Case* 1.2: $(\infty, n) \notin \theta$ for all $n < [X(a)]$. Since $\theta$ is a nontrivial congruence, we conclude that $(n, m) \in \theta$ for some $n < m < [X(a)]$. We choose $n$ and $m$ so that $k = m - n > 0$ is minimal. Since $m = n \cdot \infty^k \theta n$, $\theta \in \mathrm{Con}_{\mathbf{K}}\,\mathcal{D}$, and

$$\mathbf{K} \models \forall x\, \forall y\, \forall z\ \left[xz^k = x \longrightarrow yz^k = y\right],$$

we conclude that $(s, s \cdot \infty^k) \in \theta$ for all $s < \omega$. Moreover, the differential groupoid $\mathcal{D}$ is generated by the set $\{0, \infty\}$, whence $\mathcal{D}_k \cong \mathcal{D}/\theta \in \mathbf{K}$ because of minimality of $k$. According to Claim 1, $k = [X(b)]$ and $\mathcal{D}/\theta \cong \mathcal{A}_b$ for some $b \in L\backslash\{0,1\}$. We have by Lemma 1(ii) that $[X(b)]$ divides $[X(a)]$, whence $X(b) \subseteq X(a)$ and $b \leq a$ by Claim 1(iii) from the proof of Theorem 13. We put in this case $\theta = \theta_b$.

It is clear that the mapping

$$\varphi\colon L \to \mathrm{Con}_{\mathbf{K}}\,\mathcal{D}, \quad \varphi(b) = \begin{cases} \theta_b & \text{if } b \notin \{0,1\}, \\ 0_{\mathcal{D}} & \text{if } b = 1, \\ 1_{\mathcal{D}} & \text{if } b = 0 \end{cases}$$

is one-to-one and reverses the ordering, whence it is a dual isomorphism.

*Case* 2: the set $\{i < \omega \mid |L_i| > 1\}$ is infinite. We consider the differential groupoid $\mathcal{D} = \langle \omega \cup \{\infty\}; \cdot \rangle$, where

$$\begin{aligned} &\infty \cdot x = \infty \text{ for all } x \in \omega \cup \{\infty\}, \\ &x \cdot y = x \text{ for all } x, y \in \omega, \\ &x \cdot \infty = x + 1 \text{ for all } x \in \omega. \end{aligned}$$

Notice that $\mathcal{D} \in \mathbf{K}$ in view of the standard arguments about ultraproducts of structures of arbitrarily large finite cardinalities over nonprincipal ultrafilters.

Consider an arbitrary non-trivial congruence $\theta \in \operatorname{Con}_{\mathbf{K}} \mathcal{D}$. Again, two cases are possible.

*Case* 2.1: $(\infty, n) \in \theta$ for some $n < \omega$. In this case, we have for all $m < \omega$

$$m = (m \cdot n)\theta(m \cdot \infty) = m + 1,$$

whence $\theta = 1_{\mathcal{D}}$.

*Case* 2.2: $(\infty, n) \notin \theta$ for all $n < \omega$. Since $\theta$ is a nontrivial congruence, we conclude that $(n, m) \in \theta$ for some $n < m < \omega$. We choose $n$ and $m$ so that $k = m - n > 0$ is minimal. Since $m = n \cdot \infty^k \theta n$, $\theta \in \operatorname{Con}_{\mathbf{K}} \mathcal{D}$, and

$$\mathbf{K} \models \forall x \, \forall y \, \forall z \ \left[ xz^k = x \longrightarrow yz^k = y \right],$$

we conclude that $(s, s \cdot \infty^k) \in \theta$ for all $s < \omega$. Moreover, the differential groupoid $\mathcal{D}$ is generated by the set $\{0, \infty\}$, whence $\mathcal{D}_k \cong \mathcal{D}/\theta \in \mathbf{K}$ because of minimality of $k$. According to Claim 1, $k = [X(a)]$ and $\mathcal{D}/\theta \cong \mathcal{A}_a$ for some $a \in L \backslash \{0, 1\}$. We put in this case $\theta = \theta_a$.

We prove that the mapping

$$\varphi\colon L \to \operatorname{Con}_{\mathbf{K}} \mathcal{D}, \quad \varphi(a) = \begin{cases} \theta_a & \text{if } a \notin \{0,1\}, \\ 0_{\mathcal{D}} & \text{if } a = 1, \\ 1_{\mathcal{D}} & \text{if } a = 0 \end{cases}$$

is a dual isomorphism.

**Claim 2.** *For all $a, b \in L$, we have $a \leq b$ in $L$ if and only if $\varphi(b) \subseteq \varphi(a)$.*

*Proof of Claim.* If $a = 0$, then $\varphi(b) \subseteq 1_{\mathcal{D}} = \varphi(a)$ for all $b \in L$. If $b = 1$, then obviously $\varphi(b) = 0_{\mathcal{D}} \subseteq \varphi(a)$ for all $a \in L$. Let $a, b \in L \backslash \{0, 1\}$. Then the inequality $a \leq b$ implies that $X(a) \subseteq X(b)$. There is an onto homomorphism $f\colon \mathcal{D} \to \mathcal{A}_b$. According to Lemma 1(i), there is an onto homomorphism $g\colon \mathcal{A}_b \to \mathcal{A}_a$. Hence $\varphi(b) = \theta_b = \ker f \subseteq \ker gf = \theta_a = \varphi(a)$.

Conversely, suppose that $a, b \in L \backslash \{0, 1\}$ and $\theta_b = \varphi(b) \subseteq \varphi(a) = \theta_a$. By the isomorphism theorem, see [3, Proposition 1.4.3], we conclude that $\mathcal{A}_a \in \mathbf{H}(\mathcal{A}_b)$. Lemma 1(ii) yields the inclusion $X(a) \subseteq X(b)$, whence $a \leq b$. □

Since the mapping $\varphi$ is onto according to cases 2.1–2.2, our desired statement follows. □

Theorem 15 immediately implies the following statement, cf. [21, Proposition 4.6(i)].

**Corollary 16.** *For every finite lattice $L$ such that the greatest element is join-irreducible, there is a subquasivariety $\mathbf{K}$ of $\mathbf{Dm}$ and a differential groupoid $\mathcal{D} \in \mathbf{K}$ such that $\operatorname{Con}_{\mathbf{K}} \mathcal{D} \cong L$.*

Analogues of Theorem 13 and Corollary 14 do not hold for the quasivariety $\mathbf{V}$ of unary algebras. Indeed, according to [5], the lattice $\mathrm{Lv}(\mathbf{V})$ is finite. However, the following analogues of Theorem 15 and Corollary 16 are valid.

**Theorem 17.** *For every sequence $\mathbb{L} = \langle L_i \mid i < \omega \rangle$ of finite lattices, there is a $\mathbf{W}$-quasivariety $\mathbf{K}$ and a unary algebra $\mathcal{A} \in \mathbf{K}$ such that $\mathrm{Con}_{\mathbf{K}} \mathcal{A}$ is a lattice that is dually isomorphic to $L^*(\mathbb{L})$.*

*Proof.* We adopt the notation introduced in the proof of Theorem 13 with slight modifications. Namely, we consider a partition $\omega = \bigcup_{i<\omega} N_i$, use the same definition of $X(a)$, write $L$ instead of $L^*(\mathbb{L})$ for brevity, and put $\mathcal{A}_a = \mathcal{C}_{[X(a)]}$ for every $a \in L \setminus \{1\}$, $\mathbf{M} = \{\mathcal{A}_a \mid a \in L,\ a \neq 1\}$, and $\mathbf{K} = \mathbf{Q}(\mathbf{M}) \cap \mathbf{W}$.

We introduce an algebra $\mathcal{C}_\infty$. We put $C_\infty = \{0\} \cup \{a_i \mid i < \omega\} \cup \{b_i \mid i < \omega\}$. On $C_\infty$, we introduce unary functions $f$ and $g$ as follows. We put $f(0) = g(0) = f(a_i) = g(a_i) = 0$, $f(b_i) = a_i$, and $g(b_i) = a_{i+1}$ for all $i < \omega$.

We describe homomorphic images of $\mathcal{C}_\infty$ within the class $\mathbf{W}$. Our arguments follow the lines of the proof of [5, Lemma 3]. We consider a homomorphism $\varphi : \mathcal{C}_\infty \to \mathcal{B}$, where $\mathcal{B} \in \mathbf{W}$ and $\varphi$ is not an isomorphism. There exist $x, y \in C_\infty$ such that $x \neq y$ and $\varphi(x) = \varphi(y)$. We consider the five possible cases (up to symmetry); namely,

$$(1)\ x = 0,\ y = a_i;\ \ (2)\ x = 0,\ y = b_i;\ \ (3)\ x = a_i,\ y = b_j;$$
$$(4)\ x = a_i,\ y = a_j;\ \ (5)\ x = b_i,\ y = b_j.$$

In cases (1)–(3), we repeat the arguments from [5] verbatim and find that $\mathcal{B}$ is a trivial algebra. In a similar way, we find that either $\mathcal{B}$ is a trivial algebra or condition (4) holds if an only if condition (5) holds for the same $i$ and $j$. This allows us to repeat the final part of the proof from [5] almost verbatim and find that $\mathcal{B}$ is a homomorphic image of $\mathcal{C}_{|i-j|}$ (possibly, a trivial algebra). We conclude that the following statement is valid.

**Claim 1.** *We have $\mathbf{H}(\mathcal{C}_\infty) \cap \mathbf{W} = \{\mathcal{C}_n \mid n \in \{1, 2, \dots\} \cup \{\infty\}\}$.*

We describe structures of the form $\mathcal{C}_n$ with $n \in \{1, 2, \dots\} \cup \{\infty\}$ in $\mathbf{K}$.

**Claim 2.** *Let $n < \omega$ and let $n > 0$. We have $\mathcal{C}_n \in \mathbf{K}$ if and only if $n = [X(a)]$ for some $a \in L \backslash \{1\}$, i.e., we have $\mathcal{C}_n \in \mathbf{M}$.*

*Proof of Claim.* If $n = [X(a)]$ for some $a \in L \backslash \{1\}$ then $\mathcal{C}_n \in \mathbf{K}$ by the definition.

We prove the reverse implication. Let

$$\mathcal{C}_n \in \mathbf{K} = \mathbf{L}_s\mathbf{SP}(\mathcal{A}_a \mid a \in L,\ a \neq 1)$$

for some positive integer $n < \omega$. Since $\mathcal{C}_n$ is a finite structure, it is $l$-projective in $\mathbf{K}$, whence $\mathcal{C}_n \in \mathbf{SP}(\mathcal{A}_a \mid a \in L,\ a \neq 1)$. By Lemma 4(ii), we represent $\mathcal{C}_n$ as a subdirect product of $\mathcal{C}_{b_0}, \dots, \mathcal{C}_{b_m}$ for suitable $m < \omega$ and elements $b_0, \dots, b_m \in L \backslash \{1\}$. By Lemma 3(ii), we find that $[X(b_j)]$ divides $n$ for each $j \leqslant m$. Consider the set $F = \{b \in L \backslash \{1\} \mid [X(b)] \text{ divides } n\}$. Since $n$ is a positive integer and $\{b_0, \dots, b_m\} \subseteq F$, we conclude that $F$ is a nonempty finite set. We put $a = \bigvee F$. Then $a \in L \backslash \{1\}$ and $X(a) = \bigcup_{b \in F} X(b)$ by Claim 1(ii) from the proof of Theorem 13. We find that $[X(a)]$ divides $n$. Moreover, by Lemma 4(iii), each prime divisor of $n$ divides $[X(b)]$ for some $b \in F$. Since $X(b) \subseteq \mathbb{P}$, we conclude that each prime divisor of $n$ belongs to $X(a)$. Let $n = p_0^{s_0} \cdots p_k^{s_k}$, where $p_0, \dots, p_k \in \mathbb{P}$ and $s_0, \dots, s_k > 0$. Put $n^* = p_0 \cdots p_k$. By Lemma 5, there exists a quasi-identity

$q(n, n^*)$ that holds in $\mathcal{C}_k$ if and only if either $k$ does not divide $n$ or $k$ divides $n^*$. Since each $[X(a)]$ is the product of a family of pairwise distinct prime numbers, we find that $\mathbf{K} \models \varphi(n, n^*)$. Since $\mathcal{C}_n \in \mathbf{K}$, we conclude that $n = [X(a)]$. □

**Claim 3.** *We have $\mathcal{C}_\infty \in \mathbf{K}$ if $L$ is an infinite lattice.*

*Proof of Claim.* If $L$ is infinite then, for every natural number $N$, there exists $\mathcal{A}_a \in \mathbf{M}$ with $[X(a)] > N$. For a structure of the form $\mathcal{C}_n$ with $n \in \{1, 2, \dots\} \cup \{\infty\}$, we consider the following partition of the universe: $C_n = \{0\} \cup A \cup B$, where $B = \{x \in C_n \mid g(x) \neq 0\}$ and $A = C_n \setminus (\{0\} \cup B)$. It is clear that the condition $|B| > N$ is first-order definable for every natural $N$. Hence, there exists an ultraproduct $\mathcal{A}$ of structures $\mathcal{A}_a \in \mathbf{M}$ over a nonprincipal ultrafilter such that the set $B$ is infinite. We conclude that $\mathcal{C}_\infty$ is a subalgebra of $\mathcal{A}$. □

It is easy to see that $L$ is a finite lattice if and only if the set $\{i \in I \mid |L_i| > 1\}$ is finite. Without loss of generality, we may assume that $I = \{0\}$, $L = L_0$, $a = 1 = 1_0$, and $X(a) = L \setminus \{1\} = L_0 \setminus \{1_0\}$. We put $\mathcal{A} = \mathcal{A}_a$. If the set $\{i < \omega \mid |L_i| > 1\}$ is infinite then we put $\mathcal{A} = \mathcal{C}_\infty$.

Then $\mathcal{A} \in \mathbf{K}$. According to Lemma 3(ii) and Claim 1, for every $\mathcal{B} \in \mathbf{K}$, there is a unique congruence $\theta_\mathcal{B} \in \operatorname{Con}_\mathbf{K} \mathcal{A}$ such that $\mathcal{A}/\theta_\mathcal{B} \cong \mathcal{B}$; moreover, $\mathcal{B} \cong \mathcal{C}_a$ for some $a \in L$. We write $\theta_a$ instead of $\theta_\mathcal{B}$ for every $a \in L \backslash \{1\}$. It is clear that $\theta_0 = \theta_{X(0)} = 1_\mathcal{A}$. We prove that the mapping

$$\varphi\colon L \to \operatorname{Con}_\mathbf{K} \mathcal{A}, \quad \varphi(a) = \begin{cases} \theta_a & \text{if } a \neq 1, \\ 0_\mathcal{A} & \text{if } a = 1 \end{cases}$$

is a dual isomorphism.

**Claim 4.** *For all $a, b \in L$, we have $a \leq b$ in $L$ if and only if $\varphi(b) \subseteq \varphi(a)$.*

*Proof of Claim.* If $b = 1$ then we have $\varphi(b) = 0_\mathcal{A} \subseteq \varphi(a)$ for each $a \in L$. Let $a, b \in L \backslash \{1\}$. Then we have $a \leq b$ if and only if $X(a) \subseteq X(b)$ in view of Claim 1 from the proof of Theorem 13. Consider a homomorphism $f : \mathcal{A} \to \mathcal{A}_b$. If $X(a) \subseteq X(b)$ then, according to Lemma 4(i), there is an onto homomorphism $g\colon \mathcal{A}_b \to \mathcal{A}_a$. Hence $\varphi(b) = \theta_b = \ker f \subseteq \ker gf = \theta_a = \varphi(a)$.

Conversely, suppose that $a, b \in L \backslash \{1\}$ and $\theta_b = \varphi(b) \subseteq \varphi(a) = \theta_a$. By the isomorphism theorem, see [3, Proposition 1.4.3], we conclude that $\mathcal{A}_a \in \mathbf{H}(\mathcal{A}_b)$. Lemma 4(i) yields the inclusion $X(a) \subseteq X(b)$, whence $a \leq b$. □

It remains to notice that the mapping $\varphi$ is onto by the definition. □

From Theorem 17, we immediately obtain the following

**Corollary 18.** *For every finite lattice $L$, there is a $\mathbf{W}$-quasivariety $\mathbf{K}$ and a unary algebra $\mathcal{A} \in \mathbf{K}$ such that $\operatorname{Con}_\mathbf{K} \mathcal{A} \cong L$.*

**Theorem 19.** *The following statements hold.*

(i) *There are continuum many prevarieties $\mathbf{K} \subseteq \mathbf{Dm}$ such that the finite membership problem is undecidable for both $\mathbf{K}$ and $\mathbf{S}\big(\operatorname{Lv}(\mathbf{K})\big)$, and the quasi-equational theory $\operatorname{Th}_q(\mathbf{K})$ is also undecidable.*

(ii) *There are continuum many quasivarieties $\mathbf{K} \subseteq \mathbf{Dm}$ such that the finite membership problem is undecidable for both $\mathbf{K}$ and $\mathbf{S}\big(\operatorname{Con}_\mathbf{K} \mathcal{A}\big)$, where $\mathcal{A} \in \mathbf{K}$, and the quasi-equational theory $\operatorname{Th}_q(\mathbf{K})$ is also undecidable.*

(iii) *There are continuum many axiomatizable classes* $\mathbf{K} \subseteq \mathbf{W}$ *such that the set of isomorphism types of the class of finite sublattices of* $\operatorname{Con}_{\mathbf{K}} \mathcal{A}$ *is not computable* (*not computably enumerable*) *for some structure* $\mathcal{A} \in \mathbf{K}$; *thus the problem if a finite lattice embeds into* $\operatorname{Con}_{\mathbf{K}} \mathcal{A}$ *is undecidable. Moreover, both the finite membership problem and the quasi-equational theory* $\mathrm{Th}_q(\mathbf{K})$ *are undecidable.*

*Proof.* We prove (i); statement (ii) has a similar proof. The proof of statement (iii) repeats the arguments from the proof of [12, Theorem 4.2]. The proof of the "moreover" statement is similar to the proof of Claims 2 and 3 below (it suffices to use quasi-identities of the form $q(n,m)$ from Lemma 5 instead of quasi-identities of the form $\psi_n$).

For every $i < \omega$, let $\mathcal{K}_i$ be the meet semilattice displayed on Figure 1. Let $f\colon \omega \to \omega\backslash\{0,1,2\}$ be a strictly increasing function such that the set $f(\omega)$ is not computable (not computably enumerable, respectively). Then the set $\mathcal{S} = \{\mathrm{Sub}(\mathcal{K}_{f(i)}) \mid i < \omega\}$ is also not computable; let $\mathbb{L} = \langle \mathrm{Sub}(\mathcal{K}_{f(i)}) \mid i < \omega \rangle$. According to Theorem 13, there is a quasivariety $\mathbf{K} \subseteq \mathbf{Dm}$ such that

$$\mathrm{Lv}(\mathbf{K}) \cong L(\mathbb{L}).$$

**Claim 1.** *The finite membership problem for* $\mathbf{S}\big(\mathrm{Lv}(\mathbf{K})\big)$ *is undecidable.*

*Proof of Claim.* According to Lemma 7, the set of isomorphism types of the class of finite sublattices of $\mathrm{Lv}(\mathbf{K})$ is not computable (not computably enumerable, respectively), whence the desired statement follows. □

For the proof of the following two claims, we refer to the constructions from the proof of Theorem 13. Moreover, we assume herein that

$$N_0 = \{p_j \mid j < |\,\mathrm{Sub}(\mathcal{K}_0)|\}, \quad m_0 = |N_0|, \quad s_0 = [N_0];$$
$$N_{i+1} = \{p_{s_i+j} \mid j < |\,\mathrm{Sub}(\mathcal{K}_{i+1})|\}, \quad m_{i+1} = |N_{i+1}|, \quad s_{i+1} = s_i[N_{i+1}], \quad i < \omega.$$

**Claim 2.** *The finite membership problem for* $\mathbf{K}$ *is undecidable.*

*Proof of Claim.* If the finite membership problem for $\mathbf{K}$ is decidable, then the set $\{n < \omega \mid n > 0,\ \mathcal{D}_n \in K\}$ is computable. According to Claim 1 from the proof of Theorem 13, the set $\{s_n \mid n < \omega\}$ is computable. Since we can effectively calculate the number of primes which divide $s_{i+1}$ but do not divide $s_i$, we can effectively decide if the lattice $\mathrm{Sub}(\mathcal{K}_i)$ belongs to $\mathcal{S}$ or not, which is a contradiction. □

**Claim 3.** *The quasi-equational theory* $\mathrm{Th}_q(\mathbf{K})$ *is undecidable.*

*Proof of Claim.* For each $n < \omega$ consider the following quasi-identity which we denote by $\psi_n$:

$$\forall x\,\forall y\ \ [xy^{s_{i+1}} = x \longrightarrow xy^{s_i} = x]\,.$$

If the quasi-equational theory $\mathrm{Th}_q(\mathbf{K})$ is decidable, then the set $\{n < \omega \mid \mathbf{K} \models \psi_n\}$ is computable. For $n < \omega$, $\mathbf{K} \models \psi_n$ if and only if $\mathcal{D}_{ks_i} \notin \mathbf{K}$ for each positive integer $k > 1$ which divides $[N_{i+1}]$. According to Claim 1 from the proof of Theorem 13, the last condition is satisfied if and only if $\mathcal{D}_{s_{i+1}} \notin \mathbf{K}$. This implies that the set $\{s_n \mid n < \omega\}$ is computable. We arrive to the same contradiction as in the proof of Claim 2. □

The fact that there are uncountably many subsets of $\omega$ which are not computable (not computably enumerable) finishes the proof. □

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Aleksandr Vladimirovich Kravchenko
Sobolev Institute of Mathematics,
4, Koptyuga ave.,
630090, Novosibirsk, Russia
Novosibirsk State University,
1, Pirogova str.,
630090, Novosibirsk, Russia
Siberian Institute of Management,
6, Nizhegorodskaya str.,
Novosibirsk, 630102,Russia
Novosibirsk State Technical University,
20, Karl Marx ave.,
Novosibirsk, 630073, Russia
*E-mail address*: a.v.kravchenko@mail.ru

Marina Vladimirovna Schwidefsky
Sobolev Institute of Mathematics,
4, Koptyuga ave.,
Novosibirsk, 630090, Russia
Novosibirsk State University,
1, Pirogova str.,
Novosibirsk, 630090, Russia
Novosibirsk State Technical University,
20, Karl Marx ave..,
Novosibirsk, 630073, Russia
*E-mail address*: semenova@math.nsc.ru