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EXPONENTIAL CONVEXITY AND TOTAL POSITIVITY

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Abstract. Class of exponentially convex functions is a sub-class of convex functions on a given interval (a, b). For exponentially convex function $f(x)$ S. N. Bernstein's integral representation holds. A condition for $f(x)$, providing the kernel $K(x, y) = f(x+y)$ to be totally positive is given. New examples of totally positive kernels are obtained. For example the kernel $(x + y)^{-\alpha}$ is totally positive for any $\alpha > 0$.

Keywords: exponential convexity, total positivity, kernel.

1. Introduction

Function $f : \mathbb{R} \to \mathbb{R}$ is called exponentially convex (e. c.) on $\mathbb{R}$ if it is continuous and

$$\sum_{i,\,j=1}^{n} f(x_i + x_j)\xi_i\xi_j \geq 0 \tag{1}$$

for any x_i, $\xi_i \in \mathbb{R}$ and any n.

The classic examples include the functions $f(x) = e^{ax}$ for any $a \in \mathbb{R}$, for them condition (1) can be checked directly.

E. c. functions were introduced by S. N. Bernstein [1]. An overview of properties of such functions is given in [2] and [4]. Inequality (1) means that the matrix

$$A = \big[f(x_i + x_j)\big]_{i,j=1}^{n}$$

is non-negative definite for any values $x_1, \dots, x_n \in \mathbb{R}$. Therefore all principal minors of matrix A are non-negative. Hence simplest properties follow. For $n = 1$ we get $f(2x_1)\xi_1^2 \geq 0$. So $f(x) \geq 0$ for all x.

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Proposition. *If f is exponentially convex and does not vanish identically then $f(x) > 0$ for all $x \in \mathbb{R}$.*

This statement can be found in [2] (Remark 2.7.).

The most important property of e. c. functions is their integral representation ([2], p. 184).

Theorem 1. *Function f is exponentially convex iff*

$$f(x) = \int_{-\infty}^{\infty} e^{xt}\, d\mu(t), \qquad x \in \mathbb{R}, \tag{2}$$

where μ is a non-negative Borel measure on $\mathbb{R}$.

A class of e. c. functions is rather wide. It contains functions e^x, $\frac{e^x-1}{x}$, $\frac{\sinh x}{x}$, e^{x^2}. However function e^{-x^2} is not e. c. because it's not convex.

2. E. c. functions and totally positive kernels

2.1. **Totally non-negative matrices.** Consider the matrix $\left[f(x_i+x_j)\right]_{i,j=1}^n$ where f is e. c. function from perspective of the theory of totally non-negative matrices [5].

Definition. *A matrix $A = [a_{ij}]_{i,j=1}^n$ is called totally non-negative if all its minors (of arbitrary order) are non-negative. If all minors are positive then A is called totally positive.*

Let A, B be totally non-negative matrices (TNM). Then, their product is also a TNM ([5]) and their sum may not be a TNM. Let's give an example (we haven't found such examples in literature):

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 1 \\ 4 & 1 \end{bmatrix}, \quad C = A + B = \begin{bmatrix} 10 & 5 \\ 5 & 2 \end{bmatrix}.$$

Here A, B are total positive matrices and C is not even a TNM since $\det(C) < 0$. Now let f be an e. c. function. Then the matrix

$$A = \left[f(x_i + x_j)\right]_{i,j=1}^n$$

is non-negative definite. That's why principal minors are non-negative. But will all its minors be non-negative? Let's check it with the help of an example of "typical" e. c. function $f(x) = e^{ax}$, $a \in \mathbb{R}$. Calculate in the matrix $A = \left[e^{a(x_i+x_j)}\right]_{i,j=1}^n$ one minor of order 2

$$\begin{vmatrix} e^{ax_1}e^{ax_3} & e^{ax_1}e^{ax_4} \\ e^{ax_2}e^{ax_3} & e^{ax_2}e^{ax_4} \end{vmatrix} = e^{ax_1}e^{ax_2}e^{ax_3}e^{ax_4} \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0.$$

Such method of taking out a factor from the row or from the column can be applied to any minor. Finally we obtain that all minors of the matrix A, starting from 2nd order, are equal to zero. So, A is a TNM.

2.2. **Totally positive kernels and e. c. functions.** A kernel $K(x, y)$ is called totally positive on $(a, b) \times (a, b)$, if the determinant

$$K\begin{bmatrix} x_1, & \dots, & x_n \\ y_1, & \dots, & y_n \end{bmatrix} = \det\bigl[K(x_i, y_j)\bigr]_{i,j=1}^{n}$$

is positive for any n and for any points $x_1 < \dots < x_n$, $y_1 < \dots y_n$ from (a, b).

In this section we examine the kernels $K(x, y) = f(x+y)$ where f is e. c. function on (a, b). The example of the function $f(x) = \exp ax$ shows that the kernel $f(x+y)$ will not always be totally positive.

Let's consider measures defined by positive continuous density $p(t)$ on (c, d). Let integral

$$f(x) = \int_c^d e^{xt} p(t)\, dt \tag{3}$$

be finite for any $x \in (a, b)$. Then function $f(x)$ will be exponentially convex on (a, b).

Now let the interval (a, b) be either $(-\infty, \infty)$, or $(0, \infty)$. Then $x + y \in (a, b)$ for all $x, y \in (a, b)$ and the kernel can be defined as $K(x, y) = f(x + y)$.

Theorem 2. *Let a function $f(x)$ be represented by formula* (3). *Then the kernel $K(x, y) = f(x + y)$ is totally positive on $(a, b) \times (a, b)$.*

Proof. We have

$$f(x + y) = \int_c^d e^{xt} p(t) e^{yt}\, dt.$$

We see that the kernel $K(x, y)$ is a composition of two kernels

$$L(x, t) = e^{xt} p(t) \text{ and } M(t, y) = e^{ty}.$$

It means that

$$K(x, y) = \int_c^d L(x, t) M(t, y)\, dt.$$

It is well-known that the kernel $M(t, y) = e^{ty}$ is totally positive:

$$M\begin{bmatrix} t_1, & \dots, & t_n \\ y_1, & \dots, & y_n \end{bmatrix} > 0$$

for any $t_1 < \dots < t_n$ on (c, d) and $y_1 < \dots < y_n$ on (a, b) ([6], p. 21). The kernel $L(x, t)$ is also totally positive. Take $x_1 < \dots < x_n$ and calculate the determinator

$$L\begin{bmatrix} x_1, & \dots, & x_n \\ t_1, & \dots, & t_n \end{bmatrix} = \det\bigl[e^{x_i t_j} p(t_j)\bigr]_{i,j=1}^{n} = \prod_{j=1}^{n} p(t_j) \det\bigl[e^{x_i t_j}\bigr]_{i,j=1}^{n} > 0.$$

Here we took into consideration that $p(t_j) > 0$. By composition formula ([6], p. 14, Eq.(3.12)),

$$K\begin{bmatrix} x_1, & \dots, & x_n \\ y_1, & \dots, & y_n \end{bmatrix} = \\ = \int\limits_{c \le t_1 < t_2 < \dots < t_n \le d} \cdots \int L\begin{bmatrix} x_1, & \dots, & x_n \\ t_1, & \dots, & t_n \end{bmatrix} M\begin{bmatrix} t_1, & \dots, & t_n \\ y_1, & \dots, & y_n \end{bmatrix} dt_1 \dots dt_n.$$

This formula is an integral analogue of the Binet-Cauchy formula. As determinators L and M are positive then

$$K\begin{bmatrix} x_1, & \dots, & x_n \\ y_1, & \dots, & y_n \end{bmatrix} > 0.$$

Theorem is proved. □

From the theorem we can get some interesting examples of totally positive kernels. First example is well-known.

Example 1. *For the function $f(x) = e^{x^2}$ the representation holds*

$$e^{x^2} = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{xt} e^{-t^2/4}\, dt, \qquad x \in \mathbb{R}.$$

To calculate the integral we can use the identity $xt - t^2/4 = -\frac{1}{4}(t-2x)^2 + x^2$.

Since the density $e^{-t^2/4}$ is positive then the kernel

$$K(x, y) = \exp\big((x+y)^2\big)$$

is totally positive on $\mathbb{R} \times \mathbb{R}$.

Example 2. *For any $\alpha > 0$ we have the equality*

$$x^{-\alpha} = \int_0^{\infty} e^{-xt} \frac{t^{\alpha-1}}{\Gamma(\alpha)}\, dt, \qquad x > 0.$$

This can also be written as

$$x^{-\alpha} = \int_{-\infty}^{0} e^{x\tau} \frac{|\tau|^{\alpha-1}}{\Gamma(\alpha)}\, d\tau, \qquad x > 0.$$

By Theorem 2 the kernel $K(x, y) = (x+y)^{-\alpha}$ is totally positive for $x > 0$, $y > 0$, any $\alpha > 0$. For $\alpha = 1$ we obtain the well-known Cauchy kernel.

Example 3. *For $n \geq 1$ consider the function*

$$F_n(x) = \frac{1}{x^n}\left[e^x - \sum_{k=0}^{n-1} \frac{x^k}{k!}\right], \qquad x \neq 0; \qquad F_n(0) = \frac{1}{n!}.$$

This function is exponentially convex on $\mathbb{R}$. Indeed, let's write the Taylor formula

$$e^x - \sum_{k=0}^{n-1} \frac{x^k}{k!} = \frac{1}{(n-1)!} \int_0^x e^{\tau} (x-\tau)^{n-1}\, d\tau.$$

Here in the integral we make the substitution $\tau = xt$, $t \in [0, 1]$, and then divide by x^n. Finally we get

$$F_n(x) = \frac{1}{(n-1)!} \int_0^1 e^{xt} (1-t)^{n-1}\, dt, \qquad x \in \mathbb{R}.$$

Since $(1-t)^{n-1} > 0$ on $(0, 1)$ then F_n is e. c. function. For the function $F_n(t)$ Theorem 2 holds i. e. the kernel of $F_n(x+y)$ is totally positive on $\mathbb{R} \times \mathbb{R}$.

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