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СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 17, стр. 814–823 (2020) УДК 519.175.3, 519.172
DOI 10.33048/semi.2020.17.059 MSC 05C30, 39A10

COUNTING ROOTED SPANNING FORESTS IN COBORDISM OF TWO CIRCULANT GRAPHS

N.V. ABROSIMOV, G.A. BAIGONAKOVA, L.A. GRUNWALD, I.A. MEDNYKH

Abstract. We consider a family of graphs $H_n(s_1, \ldots, s_k; t_1, \ldots, t_\ell)$, which is a generalization of the family of I-graphs, which in turn, includes the generalized Petersen graphs and the prism graphs. We present an explicit formula for the number $f_H(n)$ of rooted spanning forests in these graphs in terms of Chebyshev polynomials and find its asymptotics. Also, we show that the number of rooted spanning forests can be represented in the form $f_H(n) = p\,a(n)^2$, where $a(n)$ is an integer sequence and p is a prescribed integer depending on the number of odd elements in the sequence $s_1, \ldots, s_k, t_1, \ldots, t_\ell$ and the parity of n.

Keywords: circulant graph, I-graph, Petersen graph, prism graph, spanning forest, Chebyshev polynomial, Mahler measure.

1. Introduction

A *tree* is a connected undirected graph without cycles. A *spanning tree* in a graph G is a subgraph that is a tree containing all the vertices of G. A *rooted tree* is a tree with one marked vertex called *root*. A *rooted forest* is a graph whose connected components are rooted trees. A *rooted spanning forest* F in a graph G is a subgraph that is a rooted forest containing all the vertices of G.

The number $\tau(G)$ of spanning trees in a connected graph G is a well studied invariant. It is also called the *complexity of a graph*. In some simplest cases it can be calculated directly, e. g. for a tree, cycle graph, fan graph, wheel graph,

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Abrosimov, N.V., Baigonakova, G.A., Grunwald, L.A., Mednykh, I.A., Counting rooted spanning foresrs in cobordism of two circulant graphs.

Parts 1–4 of the work were supported by Mathematical Center in Akademgorodok under agreement No. 075-15-2019-1613 with the Ministry of Science and Higher Education of the Russian Federation, part 5 was supported by Russian Foundation for Basic Research (project 18-01-00420).

Received January, 4, 2020, published June, 19, 2020.

complete graph (see, for example, [2], [8] or [9]). More complicated formulas for the number of spanning trees are known for some special graphs, such as prism, ladder, and Möbius ladder [4], grids [17], lattices [19], anti-prism [20]. We mention that the number of spanning trees for circulant graphs is expressed is terms of the Chebyshev polynomials; it was found in [24], [23] and [22]. Similar results are also true for the I-graph [15]. The number of spanning trees in cobordism of two circulant graphs was investigated in [1].

At the same time, the problem for enumeration of spanning forests f_G in a graph G is not so widely investigated. The number of unrooted spanning forests in the complete graphs were calculated in [13] and [21]. Enumeration of rooted spanning forests in the complete, the star and other graphs was done in [11]. In paper [7], the two last named authors suggested a new method for counting rooted spanning forests in circulant graphs. In the present paper, we develop this method to find the number of rooted spanning forests in the cobordism graphs.

We start with some basic definitions.

Let $s_1, s_2, \ldots, s_k$ be integers such that $1 \leq s_1 < s_2 < \ldots < s_k \leq \frac{n}{2}$. The graph $C_n(s_1, s_2, \ldots, s_k)$ with n vertices $0, 1, 2, \ldots, n-1$ is called *circulant graph* if the vertex i, $0 \leq i \leq n-1$ is adjacent to the vertices $i \pm s_1, i \pm s_2, \ldots, i \pm s_k \pmod{n}$. All vertices of the graph are of even degree $2k$. If n is even and $s_k = \frac{n}{2}$, then the vertices i and $i + s_k$ are connected by two edges.

Let $G = C_n(s_1, s_2, \ldots, s_k)$ and $G' = C_n(t_1, t_2, \ldots, t_\ell)$ be circulant graphs. A *cobordism* $H(G, G')$ of graphs G and G' is the graph with the following vertex set and edge set

$$V(H(G, G')) = \{u_i, v_i \mid i = 1, 2, \ldots, n\},$$

$$E(H(G, G')) = \{u_i u_{i+s_j}, u_i v_i, v_i v_{i+t_h} \mid i = 1, 2, \ldots, n, j = 1, 2, \ldots, k, h = 1, 2, \ldots, \ell\}$$

where all subscripts are given modulo n. An example of cobordism $H(G, G')$ of graphs $G = C_6(1)$ and $G' = C_6(1, 2)$ is shown in Fig. 1.

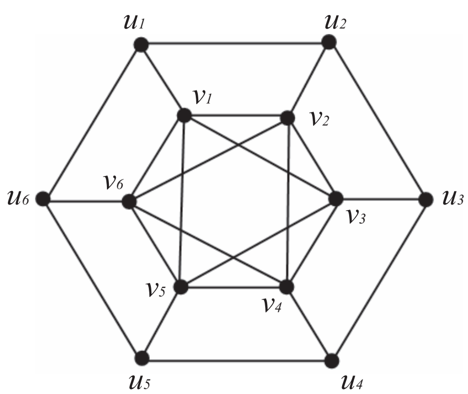

FIG. 1. Cobordism of graphs $C_6(1)$ and $C_6(1, 2)$

To emphasize the dependence of $H(G, G')$ from the parameters, we also will write it in the form $H_n(s_1, \ldots, s_k; t_1, \ldots, t_\ell)$. In the above definition, all vertices u_i are of valency $2k + 1$, while all vertices v_i are of valency $2l + 1$. In the case of even n when at least one of s_j or t_h is equal to $n/2$, the graph under consideration has multiple edges. Repeating the arguments from the papers [3], [10], [18] we conclude that the graph $H_n(s_1,\dots,s_k;t_1,\dots,t_\ell)$ is connected if the greatest common divisor $\gcd(n,s_1,\dots,s_k,t_1,\dots,t_\ell)=1$. If $\gcd(n,s_1,\dots,s_k,t_1,\dots,t_\ell)=m>1$ then $H_n(s_1,\dots,s_k;t_1,\dots,t_\ell)$ is a union of m copies of the graph $H_{n/m}(s_1/m,\dots,s_k/m;t_1/m,\dots,t_\ell/m)$. If $m=1$ and $\gcd(s_1,\dots,s_k,t_1,\dots,t_\ell)=d$ then the graphs $H_n(s_1,\dots,s_k;t_1,\dots,t_\ell)$ and $H_n(s_1/d,\dots,s_k/d;t_1/d,\dots,t_\ell/d)$ are isomorphic. Thus, for connected graphs without loss of generality we can assume that $\gcd(s_1,\dots,s_k,t_1,\dots,t_\ell)=1$. One can see that graph $H_n(1;1)$ isomorphic to the prism graph $Pr(n)$, $H_n(k;1)$ is isomorphic to the generalized Petersen graph $GP(n,k)$ and $H_n(k;\ell)$ is isomorphic to the I-graph $I(n,k,\ell)$. The number of spanning trees in the generalized Petersen graph and I-graph were investigated in [12] and [16] respectively.

In this paper, we obtain a closed formula for the number $f_H(n)$ of rooted spanning forests in the graph $H_n(s_1,\dots,s_k;t_1,\dots,t_\ell)$, investigate some arithmetical properties of this number and provide its asymptotic behavior.

2. Basic definitions and preliminary facts

We need the following basic properties of Chebyshev polynomials.

Let $T_n(z)=\cos(n\arccos z)$ and $U_{n-1}(z)=\frac{\sin(n\arccos z)}{\sin(\arccos z)}$ be the *Chebyshev polynomials* of the first and second kind respectively.

For any $z\neq 0$ we have the following identity $T_n(\frac{1}{2}(z+z^{-1}))=\frac{1}{2}(z^n+z^{-n})$. Also, the polynomials $T_n(z)$ and $U_{n-1}(z)$ admit the following well known presentation $T_n(z)=(q^n+q^{-n})/2$ and $U_{n-1}(z)=(q^n-q^{-n})/(q-q^{-1})$, where $q=z+\sqrt{z^2-1}$. See monograph [14] for other properties.

We denote the vertex and edge set of G by $V(G)$ and $E(G)$ respectively. Given $u,v\in V(G)$, we denote by a_{uv} the number of edges between vertices u and v. The matrix $A=A(G)=\{a_{uv}\}_{u,v\in V(G)}$, called *the adjacency matrix* of the graph G. The degree $d(v)$ of a vertex $v\in V(G)$ is defined by $d(v)=\sum_u a_{uv}$. Let $D=D(G)$ be the diagonal matrix indexed by the elements of $V(G)$ with $d_{vv}=d(v)$. Matrix $L=L(G)=D(G)-A(G)$ is called *the Laplacian matrix*, or simply *Laplacian*, of the graph G.

In what follows, by I_n we denote the identity matrix of order n.

We refer to an $n\times n$ matrix to be *circulant,* and denote it by $circ(a_0,a_1,\dots,a_{n-1})$ if it is of the form

$$circ(a_0,a_1,\dots,a_{n-1})=\begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1}\\ a_{n-1} & a_0 & a_1 & \dots & a_{n-2}\\ \vdots & \vdots & & \ddots & \vdots\\ a_1 & a_2 & a_3 & \dots & a_0\end{pmatrix}.$$

Recall [5] that the eigenvalues of matrix $C=circ(a_0,a_1,\dots,a_{n-1})$ are given by the following simple formulas $\lambda_j=p(\varepsilon_n^j)$, $j=0,1,\dots,n-1$, where $p(x)=a_0+a_1x+\dots+a_{n-1}x^{n-1}$ and ε_n is the order n primitive root of the unity. Moreover, the circulant matrix $C=p(T_n)$, where $T_n=circ(\underbrace{0,1,0,\dots,0}_{n})$ is the matrix shift operator $T_n:(x_0,x_1,\dots,x_{n-2},x_{n-1})\to(x_1,x_2,\dots,x_{n-1},x_0)$.

Denote by $L=L(H(G,G'))$ the Laplacian of $H(G,G')$, where $G=C_n(s_1,s_2,\dots,s_k)$ and $G'=C_n(t_1,t_2,\dots,t_\ell)$.

Then we have

$$L = \begin{pmatrix} (2k+1)I_n - \sum_{j=1}^{k}(T_n^{s_j} + T_n^{-s_j}) & -I_n \\ -I_n & (2l+1)I_n - \sum_{h=1}^{\ell}(T_n^{t_h} + T_n^{-t_h}) \end{pmatrix}.$$

3. Counting the number of rooted spanning forests

The main result of this section is the following theorem.

Theorem 1. *The number of rooted spanning forests in the graph $H = H_n(s_1, \dots, s_k; t_1, \dots, t_\ell)$ is given by the formula*

$$f_H(n) = \prod_{s=0}^{s_k+t_\ell-1} |2\,T_n(w_s) - 2|,$$

where $w_s,\ s = 0, 1, \dots, s_k + t_\ell - 1$ are all the roots of the algebraic equation

$$(2k + 2 - \sum_{j=1}^{k} 2\,T_{s_j}(w))(2\ell + 2 - \sum_{h=1}^{\ell} 2\,T_{t_h}(w)) = 1$$

and $T_n(w)$ is the Chebyshev polynomial of the first kind.

Proof. By ([7], formula (1)), the number of rooted spanning forests $f_H(n)$ is equal to the product of all eigenvalues of the matrix $I_{2n} + L(H)$ of graph $H(G, G')$. To investigate the spectrum of the matrix

$$I_{2n} + L(H) = \begin{pmatrix} (2k+2)I_n - \sum\limits_{j=1}^{k}(T_n^{s_j} + T_n^{-s_j}) & -I_n \\ -I_n & (2\ell+2)I_n - \sum\limits_{j=1}^{\ell}(T_n^{t_j} + T_n^{-t_j}) \end{pmatrix},$$

we note that the eigenvalues of circulant matrix T_n are ε_n^j, where $\varepsilon_n = e^{\frac{2\pi i}{n}}$. Recall that the matrix T_n is conjugate to the diagonal matrix $\mathbb{T}_n = diag(1, \varepsilon_n, \dots, \varepsilon_n^{n-1})$. To find spectrum of $I_{2n}+L(H)$, without loss of generality, one can replace T_n by $\mathbb{T}_n$. In this case, all $n \times n$ blocks of $I_{2n} + L(H)$ are diagonal matrices. If λ is eigenvalue of $I_{2n} + L(H)$ and $(x, y) = (x_1, \dots, x_n, y_1, \dots, y_n)$ is the respective eigenvector, we have the following system of equations

$$\begin{cases} C(T_n)x - y &= \lambda x \\ -x + D(T_n)y &= \lambda y \end{cases},$$

where $C(z) = 2k + 2 - \sum\limits_{j=1}^{k}(z^{s_j} + z^{-s_j})$ and $D(z) = 2\ell + 2 - \sum\limits_{h=1}^{\ell}(z^{t_h} + z^{-t_h})$.

From the first equation the have $y = C(T_n)x - \lambda x = (C(T_n) - \lambda)x$. Substituting y in the second equation we obtain $((C(T_n) - \lambda)(D(T_n) - \lambda) - 1)x = 0$.

Recall the matrices under consideration are diagonal and the $(j + 1, j + 1)$-th entry of $\mathbb{T}_n$ is equal to ε_n^j. Hence, $((C(\varepsilon_n^j) - \lambda)(D(\varepsilon_n^j) - \lambda) - 1)x_{j+1} = 0$ and $y_{j+1} = (C(\varepsilon_n^j) - \lambda)x_{j+1}$.

As a result, for any $j = 0, \dots, n - 1$ the matrix $I_{2n} + L(H)$ has exactly two eigenvalues $\lambda_{1,j}$ and $\lambda_{2,j}$ which are the roots of quadratic equation $(C(\varepsilon_n^j) - \lambda)(D(\varepsilon_n^j) - \lambda) - 1 = 0$. The corresponding eigenvectors are (x, y), where $x = \mathbf{e}_{j+1} =$

$(0,\dots,\underbrace{1}_{(j+1)-th},\dots,0)$ and $y=(C(\varepsilon_n^j)-\lambda)\mathbf{e}_{j+1}$. Since $\lambda_{1,j}$ and $\lambda_{2,j}$ are roots of the same quadratic equation, we obtain $\lambda_{1,j}\lambda_{2,j}=P(\varepsilon_n^j)$, where

$$P(z)=C(z)D(z)-1=(2k+2-\sum_{j=1}^{k}(z^{s_j}+z^{-s_j}))(2\ell+2-\sum_{h=1}^{\ell}(z^{t_h}+z^{-t_h}))-1. \tag{1}$$

In particular, for $j=0$ eigenvalues $\lambda_{1,0}$ and $\lambda_{2,0}$ satisfy the equation $(2-\lambda)(2-\lambda)-1=0$. So $\lambda_{1,0}=1$ and $\lambda_{2,0}=3$.

Now we have

$$f_H(n)=\lambda_{1,0}\lambda_{2,0}\prod_{j=1}^{n-1}\lambda_{1,j}\lambda_{2,j}=3\prod_{j=1}^{n-1}P(\varepsilon_n^j). \tag{2}$$

To continue we need the following elementary lemma. Its proof easily follows from the equality $T_n(\frac{1}{2}(z+z^{-1}))=\frac{1}{2}(z^n+z^{-n})$.

Lemma 1. *The following identity holds* $P(z)=Q(w)$, *where*

$$Q(w)=(2k+2-\sum_{j=1}^{k}2\,T_{s_j}(w))(2\ell+2-\sum_{h=1}^{\ell}2\,T_{t_h}(w))-1,$$

here $T_m(w)$ *is the Chebyshev polynomial of the first kind and* $w=\frac{1}{2}(z+z^{-1})$.

The following lemma is needed to finish the prove of the theorem.

Lemma 2.

$$\prod_{j=0}^{n-1}P(\varepsilon_n^j)=(-1)^{n\,m}\prod_{p=1}^{m}(2T_n(w_p)-2), \tag{3}$$

where $m=s_k+t_\ell$ *and* w_p, $j=1,\dots,m$ *are all the roots of the algebraic equation* $Q(w)=0$.

To prove the above formula we use some statements from theory of resultants. We introduce integer polynomial $\widetilde{P}(z)=z^mP(z)$. This is a monic polynomial of degree $2m$ with the same roots as $P(z)$. As $P(z)=P(\frac{1}{z})$, its roots look like $z_1,\frac{1}{z_1},\dots,z_m,\frac{1}{z_m}$.

$$\begin{aligned}\prod_{j=0}^{n-1}P(\varepsilon_n^j) &= \prod_{j=0}^{n-1}\varepsilon_n^{-mj}\widetilde{P}(\varepsilon_n^j)=\mathrm{Res}\,(\widetilde{P}(z),\,z^n-1)\prod_{j=0}^{n-1}\varepsilon_n^{-mj}\\ &= (-1)^{(n-1)m}\mathrm{Res}\,(z^n-1,\,\widetilde{P}(z))=(-1)^{(n-1)m}\prod_{z:P(z)=0}(z^n-1)\\ &= (-1)^{(n-1)m}\prod_{p=1}^{m}(z_p^n-1)(z_p^{-n}-1)=(-1)^{n\,m}\prod_{p=1}^{m}(2T_n(w_p)-2).\end{aligned}$$

We use the identity $T_n(\frac{1}{2}(z+z^{-1}))=\frac{1}{2}(z^n+z^{-n})$. Here $w_p=\frac{1}{2}(z_p+\frac{1}{z_p})$, $p=1,\dots,s_k$. These numbers are the roots of algebraic equation $Q(w)=0$. The lemma is proved.

Taking into account formula (2) we conclude that $\prod_{j=0}^{n-1} P(\varepsilon_n^j)$ is positive. To finish the proof of the theorem we take the absolute value of the righthand side of formula (3) and put it into equation (2). □

Theorem 2. *Let $f_H(n)$ be the number of rooted spanning forests of the graph $H = H_n(s_1, \ldots, s_k; t_1, \ldots, t_\ell)$. Denote by s and t the number of odd numbers in the sequences $s_1, \ldots, s_k$ and $t_1, \ldots, t_\ell$ respectively. Let p be the square-free part of the number $u = 3(8(2st + s + t) + 3)$. Then there exist an integer sequence $a(n)$, $n \in \mathbb{N}$ such that*

1° *$f_H(n) = 3\,a(n)^2$, if n is odd.*
2° *$f_H(n) = p\,a(n)^2$, if n is even.*

Proof. By theorem 1 we have $f_H(n) = \prod_{j=0}^{n-1} \lambda_{1,j}\lambda_{2,j}$. Note that $\lambda_{1,j}\lambda_{2,j} = P(\varepsilon_n^j) = P(\varepsilon_n^{n-j}) = \lambda_{1,n-j}\lambda_{2,n-j}$, $j = 1, \ldots, n-1$, where $P(z)$ is given by the formula (1). Therefore, we have $f_H(n) = 3(\prod_{j=1}^{(n-1)/2} \lambda_{1,j}, \lambda_{2,j})^2$ if n is odd, and $f_H = 3\lambda_{1,\frac{n}{2}}\lambda_{2,\frac{n}{2}}(\prod_{j=1}^{n/2-1} \lambda_{1,j}, \lambda_{2,j})^2$ if n is even. Moreover, if n is even we get

$$\begin{aligned}
&\lambda_{1,\frac{n}{2}}\lambda_{2,\frac{n}{2}} = P(-1)\\
&= \left(2k + 2 - \sum_{j=1}^{k}\left((-1)^{s_j} + (-1)^{-s_j}\right)\right)\left(2\ell + 2 - \sum_{h=1}^{\ell}\left((-1)^{t_h} + (-1)^{-t_h}\right)\right) - 1\\
&= \left(2 + 4\sum_{j=1}^{k}\frac{1-(-1)^{s_j}}{2}\right)\left(2 + 4\sum_{h=1}^{\ell}\frac{1-(-1)^{t_h}}{2}\right) - 1\\
&= (2+4s)(2+4t) - 1 = 8(2st + s + t) + 3.
\end{aligned}$$

Let $u = 3(8(s + t + 2st) + 3)$. We present number u in the form $u = p\,r^2$, where p is the square free part of u.

Each algebraic number $\lambda_{i,j}$ comes into the products $\prod_{j=1}^{(n-1)/2} \lambda_{1,j}, \lambda_{2,j}$ and $\prod_{j=1}^{n/2-1} \lambda_{1,j}, \lambda_{2,j}$ together with all of its Galois conjugate elements. Hence we conclude that both products are integer numbers. We set $a(n) = \prod_{j=1}^{(n-1)/2} \lambda_{1,j}, \lambda_{2,j}$ if n is odd and $a(n) = r\prod_{j=1}^{n/2-1} \lambda_{1,j}, \lambda_{2,j}$ if n is even to finish the proof. □

4. Asymptotic for the number of rooted spanning forests

The asymptotic formula for the number of rooted spanning forests in the graph $H_n(s_1, \ldots, s_k; t_1, \ldots, t_\ell)$ is given in the following theorem.

Theorem 3. *The number $f_H(n)$ of rooted spanning forests in the graph $H = H_n(s_1, \ldots, s_k; t_1, \ldots, t_\ell)$ has the following asymptotic*

$$f_H(n) \sim A^n,\ n \to \infty,$$

where $A = \exp\left(\int_0^1 \log(P(e^{2\pi \mathrm{i} t}))\,dt\right)$ is the Mahler measure of Laurent polynomial

$$P(z) = (2k + 2 - \sum_{j=1}^{k}(z^{s_j} + z^{-s_j}))(2\ell + 2 - \sum_{h=1}^{\ell}(z^{t_h} + z^{-t_h})) - 1.$$

Proof. By theorem 1 we have

$$f_H(n) = \prod_{s=0}^{s_k+t_\ell-1} |2T_n(w_s) - 2|,$$

where $w_s, s = 0, 1, \ldots, s_k + t_\ell - 1$ are all the roots of the polynomial

$$Q(w) = (2k + 2 - 2\sum_{j=1}^{k} T_{s_j}(w))(2\ell + 2 - 2\sum_{h=1}^{\ell} T_{t_h}(w)) - 1.$$

By lemma 1, $T_n(w_s) = \frac{1}{2}(z_s^n + z_s^{-n})$ where the z_s and $1/z_s$, $s = 1, \ldots, s_k + t_\ell$ are all the roots of the polynomial $P(z)$. Since $P(e^{i\varphi}) = (2k + 2 - 2\sum_{i=1}^{k} \cos(s_i\varphi))(2\ell + 2 - 2\sum_{j=1}^{\ell} \cos(t_j\varphi)) - 1 = (2 + 2\sum_{i=1}^{k}(1 - \cos(s_i\varphi)))(2 + 2\sum_{j=1}^{\ell}(1 - \cos(t_j\varphi))) - 1 \geq 2 \cdot 2 - 1 = 3 > 0$. Hence, all the roots z_s of polynomial $P(z)$ satisfy the property $|z_s| \neq 1$. Replacing z_s by $1/z_s$, if it is necessary, we can assume that all $|z_s| > 1$ for all s. Then $T_n(w_s) \sim \frac{1}{2}z_s^n$ and $|2T_n(w_s) - 2| \sim |z_s|^n$ as $n \to \infty$. So

$$\prod_{s=0}^{s_k+t_\ell-1} |2T_n(w_s) - 2| \sim \prod_{s=0}^{s_k+t_\ell-1} |z_s|^n = \prod_{P(z)=0,\, |z|>1} |z|^n = A^n,$$

where $A = \prod_{P(z)=0,\, |z|>1} |z|$. By ([6], p. 67) this value coincides with the Mahler measure of the polynomial $P(z)$ and can be calculated by the formula

$$A = \exp\left(\int_0^1 \log|P(e^{2\pi it})|\mathrm{d}t\right).$$

The theorem is proved. □

5. Examples and Tables

Examples.

1° The prism graph $H = Pr(n) = H_n(1;1)$. By Theorem 1 we have the following formula for the number of rooted spanning forests

$$f_H(n) = (2T_n(\frac{3}{2}) - 2)(2T_n(\frac{5}{2}) - 2).$$

By Theorem 2, there is an integer sequence $a(n)$ such that $f_H(n) = 3\,a(n)^2$ if n odd and $f_H(n) = 105\,a(n)^2$ if n is even. Also, by Theorem 3 we get $f_H(n) \sim A^n$, $n \to \infty$, where $A = \frac{1}{4}(3 + \sqrt{5})(5 + \sqrt{21}) \cong 12.54375$.

2° The generalized Petersen graph $H = GP(n, 2) = H_n(1;2)$. By Theorem 1 we have the following formula for the number of rooted spanning forests

$$f_H(n) = |2T_n(w_1) - 2| \cdot |2T_n(w_2) - 2| \cdot |2T_n(w_3) - 2|,$$

where w_1, w_2, w_3 are thge roots of the equation $23 - 12w - 16w^2 + 8w^3 = 0$. By Theorem 2, for some integer sequence $a(n)$ we have $f_H(n) = 3\,a(n)^2$ if n odd and $f_H(n) = 33\,a(n)^2$ if n is even. Also, by Theorem 3 we get $f_H(n) \sim A^n$, $n \to \infty$, where $A \cong 12.69197$.

3° The graph $H = H_n(1,2;1)$. By Theorem 1 we have the following formula for the number of rooted spanning forests

$$f_H(n) = |2T_n(w_1) - 2| \cdot |2T_n(w_2) - 2| \cdot |2T_n(w_3) - 2|,$$

where w_1, w_2, w_3 are thge roots of the equation $Q(w) = 0$, where $Q(w) = 31 - 24w - 12w^2 + 8w^3$. By Theorem 2, there is an integer sequence $a(n)$ such that $f_H(n) = 3\,a(n)^2$ if n odd and $\tau(n) = 105\,a(n)^2$ if n is even. Also, by Theorem 3 we get $f_H(n) \sim A^n$, $n \to \infty$, where $A \cong 19.24864$.

4° The graph $H = H_n(1,2;1,2)$. By Theorem 1 we have the formula

$$f_H(n) = (2\,T_n(w_1) - 2)(2\,T_n(w_2) - 2))(2\,T_n(w_3) - 2))(2\,T_n(w_4) - 2)),$$

where w_1, w_2, w_3, w_4 are all the roots of the equation $(-9+2w+4w^2)(-7+2w+4w^2) = 0$. By Theorem 3 we obtain $f_H(n) \sim A^n$, $n \to \infty$, where $A = \frac{1}{16}(7+\sqrt{5}+\sqrt{38+14\sqrt{5}})(9+\sqrt{21}+\sqrt{86+18\sqrt{21}}) \cong 29.15438$.

Table of $f_H(n)$ for prism graph $H = H_n(1;1)$

Table 1

n	$f_H(n)$	$a(n)$
3	1728	24
4	23625	15
5	305283	319
6	3870720	192
7	48746883	4031
8	612383625	2415
9	7685938368	506161
10	96431267625	30305
11	1209709290243	635009
12	15174770688000	380160
13	190350868481283	7965569
14	2387725428997545	4768673
15	29951093548271808	99918456
16	375699412158863625	59817135
17	4712682363364886403	1253353151
18	59114736024534650880	750331584
19	741520759611850588803	15721755199
20	9301454448258671015625	9411975375
21	116675161113660560411328	197209838424
22	1463544572644729649289705	118061508289
23	18358343737788441927052803	2473751788801
24	230282555741777804525568000	1480934568960
25	2888607830124833179158458883	31030134977281

We note that $a(2k)$, $k = 1, 2, \ldots$ is sequence $A006238$ in the On-Line Encyclopedia of Integer Sequences.

Acknowledgements

The authors are grateful to Prof. A.D. Mednykh who attracted our attention to the problem.

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Nikolay Vladimirovich Abrosimov
Sobolev Institute of Mathematics,
4, Acad. Koptyug ave.,
Novosibirsk, 630090, Russia

Novosibirsk State University,
1, Pirogova str.,
Novosibirsk, 630090, Russia
E-mail address: abrosimov@math.nsc.ru

Galya Amanboldynovna Baigonakova
Gorno-Altaysk State University,
34, Socialisticheskaya str.,
Gorno-Altaysk, 639000, Russia
E-mail address: galyaab@mail.ru

Liliya Aleksandrovna Grunwald
Sobolev Institute of Mathematics,
4, Acad. Koptyug ave.,
Novosibirsk, 630090, Russia

Novosibirsk State University,
1, Pirogova str.,
Novosibirsk, 630090, Russia
E-mail address: lfb_o@yahoo.co.uk

Ilya Aleksandrovich Mednykh
Sobolev Institute of Mathematics,
4, Acad. Koptyug ave.,
Novosibirsk, 630090, Russia

Novosibirsk State University,
1, Pirogova str.,
Novosibirsk, 630090, Russia
E-mail address: ilyamednykh@mail.ru