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ON AN ANALOG OF THE BINET INTEGRAL REPRESENTATION

V.I. KUZOVATOV, A.M. KYTMANOV

Abstract. We obtain an analog of the Binet integral representation, which is essential for obtaining the functional equation for the classical Riemann zeta-function.

Keywords: Binet formula, integral representation, interpolation problem.

1. Introduction

The aim of the paper is to obtain an analog of the Binet integral representation (see, e.g., [1], Chap. 12, section 12.32), which is used for finding a functional equation (see, e.g., [2], Chap. 2, section 9) for the classical Riemann zeta-function.

The classical Binet formula expresses the logarithmic derivative of the Euler gamma function $\Gamma(z)$ (if the real part of z is positive) in terms of the following integral:

$$\frac{d}{dz}\ln\Gamma(z) = -\frac{1}{2z} + \ln z - 2\int\limits_0^\infty \frac{t\,dt}{(z^2+t^2)\,(e^{2\pi t}-1)}.$$

Concerning generalizations of the zeta-function, we note that in 1950s Gelfand, Levitan, and Dikii (see, e.g., [3, 4, 5]) studied the zeta-function associated with the eigenvalues of the Sturm – Lioville operator. As it turned out, its value is connected with the trace of the operator. Later on, their approach was developed by Lidskii and

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Sadovnichii [6], who considered a class of entire functions in one variable, defined the zeta-function of their zeros and investigated its domain of analytic continuation. Smagin and Shubin [7] constructed the zeta-functions for elliptic operators, as well as for operators of more general type, proved the possibility of a meromorphic continuation of the zeta-function and gave some information on its poles.

Multidimensional results were obtained by A.M. Kytmanov and Myslivets [8]. They introduced the concept of zeta-function associated with a system of meromorphic functions $f = (f_1, \dots, f_n)$ in $\mathbb{C}^n$. With the help of the residue theory, an integral representation for the zeta-function was given provided some strong conditions on the system $f_1, \dots, f_n$ are fulfilled.

In [9], with the help of the residue theory, Kuzovatov and A.A. Kytmanov obtained two integral representations for the zeta-function constructed from the zeros of an entire function of finite order on the complex plane. With the use of these representations, they described a domain the zeta-function can be extended to.

2. Auxiliary results

Let us formulate some results from [9]. Let $f(z)$ be an entire function of order ρ in $\mathbb{C}$. Consider the equation

$$f(z) = 0. \tag{1}$$

Denote by $N_f = f^{-1}(0)$ the set of all solutions to (1) (we take every zero as many times as its multiplicity). The numbers of roots is at most countable (by the uniqueness theorem for holomorphic functions).

The zeta-function $\zeta_f(s)$ of roots of Eq. (1) is defined as follows:

$$\zeta_f(s) = \sum_{a \in N_f} (-a)^{-s}$$

where $s \in \mathbb{C}$. The minus sign in the definition is taken for the convenience of representing the integrals below.

Now we give an integral representation for the zeta-function $\zeta_f(s)$ of the zeros z_n of f, which are $z_n = -q_n + is_n$, $q_n > 0$. Put

$$F(f, x) = \sum_{n=1}^{\infty} e^{z_n x}. \tag{2}$$

We assume that $\operatorname{Re} s = \sigma > 1$ and the following conditions hold:

$$\varliminf_{n \to \infty} \frac{q_n}{n} > 0, \tag{3}$$

$$\text{the series} \quad \sum_{n=1}^{\infty} \left(\frac{1}{q_n}\right)^{\sigma - 1} \quad \text{converges.} \tag{4}$$

Condition (3) implies that, for the convergence of series (2), it is necessary and sufficient (for real x) that $x > 0$ [9].

Theorem 1 ([9]). *Let* (3) *and* (4) *hold, and* $\operatorname{Re} s > 1$. *Then*

$$\zeta_f(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} F(f, x)\, dx$$

where $F(f, x)$ *is defined by* (2).

3. An analog of the Plana formula

Kuzovatov and A.M. Kytmanov [10] obtained an analog of the Plana formula under certain constraints. In [11], Kuzovatov removed these constraints. Let us formulate the result (an analog of the Plana formula) from [11], which we need to obtain the main result.

In the paper, we assume that $z_n = -q_n$, $q_n > 0$, where q_n is a sequence of naturals.

Consider the function $(z = x + iy)$

$$F(f, 2\pi iz) = \sum_{n=1}^{\infty} e^{z_n 2\pi iz} = \sum_{n=1}^{\infty} e^{-q_n 2\pi iz}. \tag{5}$$

The domain of convergence for $F(f, 2\pi iz)$, defined by (5), is the set $y < 0$ (see [11]).

With the help of the change of variable $e^{-2\pi iz} = w$, the series (5) is reduced to the form $\sum\limits_{n=1}^{\infty} w^{q_n}$ or

$$G(w) = \sum_{n=1}^{\infty} f_n w^n \tag{6}$$

where

$$f_n = \begin{cases} 1, & n = q_k; \\ 0, & n \neq q_k, \end{cases} \quad \text{therefore,} \quad \overline{\lim_{n\to\infty}} \sqrt[n]{|f_n|} = 1.$$

We should note that infinitely many of the coefficients f_n in (6) are nonzero.

The function $G(w)$ is unbounded as $|w| \to 1 - 0$ but it is holomorphic in the unit disk. Therefore, $G(w)$ cannot be extended to the point 1. From the Fabry gap theorem (see, e.g., [12, §2.3]) it follows that we can take the coefficients of the series in such a way that the whole circle is the natural boundary for $G(w)$.

Further on, we restrict ourselves by considering classes of rational functions $G(w)$ satisfying (6). We recall

Theorem 2 (Szegő, [12], § 6.1). *Let*

$$G(w) = \sum_{n=0}^{\infty} f_n w^n \tag{7}$$

be a power series whose coefficients f_n take only a finite number of dictinct values. Then either $G(w)$ is a rational function or it cannot be extended outside the unit disk. If the sum of (7) is a rational function then

$$G(w) = \frac{P(w)}{1 - w^N}$$

where $P(w)$ is a polynomial and N is a natural number.

By the theorem, $G(w)$ can have singularities (simple poles) only at the points

$$w_k = e^{i\frac{2\pi}{N}k}, \quad k = 0, 1, \dots, N-1, \quad N \in \mathbb{N}.$$

In particular, $w_0 = 1$. In terms of the z-variable, the singularities of $F(f, 2\pi iz)$ are

$$e^{-2\pi iz} = w_k, \quad -2\pi iz = i\left(\frac{2\pi}{N}k + 2\pi l\right), \quad z = -\left(\frac{k}{N} + l\right), \quad l = 0, \pm 1, \pm 2, \dots,$$

or, equivalently,

$$z_{k,l} = l - \frac{k}{N}, \quad l = 0, \pm 1, \pm 2, \ldots, \quad k = 0, 1, \ldots, N-1.$$

Let q_n satisfy the inequality (3). Assume additionally that $\deg P(w) = N$, i.e.,

$$P(w) = d_1 w + d_2 w^2 + \ldots + d_{N-1} w^{N-1} + w^N \tag{8}$$

where, because of (6), the coefficients $d_j \in \{0, 1\}$, $j = 1, \ldots, N-1$.

It should be noted that if $\deg P(w) > N$ then, in the expansion of $G(w)$, we can separate a part which contains only a finite number of terms; these terms do not influence the other terms with sufficiently large indices. If $\deg P(w) \leq N$ then, in the expansion of $G(w)$, the coefficients of $P(w)$ occur; they are repeated periodically because the expansion of $\left(1 - w^N\right)^{-1}$ is a geometric series. We write separately the monomial w^N for convenience of the calculations.

We have an explicit expression for $F(f, 2\pi i z)$:

$$F(f, 2\pi i z) = \frac{P\left(e^{-2\pi i z}\right)}{1 - e^{-2\pi i z N}}. \tag{9}$$

We note that (9) gives an analytic continuation of the function $F(f, 2\pi i z)$, introduced above and defined by (5). The domain of definition for (9) is the complex plane $\mathbb{C}$ except for the points $z_{k,l}$.

Theorem 3 (an analog of the Plana formula, [11])**.** *Let x_1 and x_2 be integers and $\varphi(z)$ be a function holomorphic and bounded on the set $\{x_1 \leq \operatorname{Re} z \leq x_2\}$. Then*

$$\frac{1}{N} \sum_{k=0}^{N-1} P(w_k) \Big(\varphi\big(x_1 + 1 - \frac{k}{N}\big) + \varphi\big(x_1 + 2 - \frac{k}{N}\big) + \ldots + \varphi\big(x_2 - 1 - \frac{k}{N}\big)$$
$$+ \varphi\big(x_2 - \frac{k}{N}\big) \Big) = \frac{P(w_0)}{N} \left(-\frac{1}{2} \varphi(x_1) + \frac{1}{2} \varphi(x_2) \right) + \int_{x_1}^{x_2} \varphi(z)\, dz$$
$$+ \frac{1}{i} \int_0^\infty \Big(\varphi(x_1 - iy) F(f, 2\pi y) + \varphi(x_1 + iy) \left[1 + F(f, -2\pi y)\right] \Big) dy$$
$$- \frac{1}{i} \int_0^\infty \Big(\varphi(x_2 - iy) F(f, 2\pi y) + \varphi(x_2 + iy) \left[1 + F(f, -2\pi y)\right] \Big) dy.$$

Note that the functions $F(f, 2\pi y)$ and $1 + F(f, -2\pi y)$ are defined [11] as follows ($t = e^{2\pi y}$):

$$F(f, 2\pi y) = -\frac{d_1 t^{N-1} + d_2 t^{N-2} + \ldots + d_{N-1} t + 1}{1 - t^N}, \tag{10}$$

$$1 + F(f, -2\pi y) = \frac{1 + d_1 t + d_2 t^2 + \ldots + d_{N-1} t^{N-1}}{1 - t^N}. \tag{11}$$

4. On one interpolation problem

To solve the following interpolation problem, we give some definitions and statements.

A function $\rho(r)$ that satisfies the conditions

$$\lim_{r\to\infty}\rho(r)=\rho \quad \text{and} \quad \lim_{r\to\infty} r\rho'(r)\ln r=0$$

is called a *proximate order*. Clearly, the proximate order of the given function is not uniquely determined (see, e.g., [13, Chap. I, §12]).

We say (see, e.g., [13, Chap. II, §1]) that a point set of the complex plane has *angular density of index* $\rho(r)$ if, for all but a countable set of values ϑ and θ ($0<\vartheta<\theta\le 2\pi$), there exists a limit

$$\Delta(\vartheta,\theta)=\lim_{r\to\infty}\frac{n(r,\vartheta,\theta)}{r^{\rho(r)}},$$

where $n(r,\vartheta,\theta)$ is the number of points of the set within the sector $|z|\le r$, $\vartheta<\arg z\le\theta$. The quantity $\Delta(\vartheta,\theta)$ is called the angular density of the set in the angle $\vartheta<\arg z<\theta$, or, simply, in the angle (ϑ,θ). For a fixed ϑ, up to an additive constant, the equation

$$\Delta(\theta)-\Delta(\vartheta)=\Delta(\vartheta,\theta)$$

determines a nondecreasing function $\Delta(\theta)$ (see, e.g., [13, Chap. II, §1]).

A set is said to be *regularly distributed relative to* $\rho(r)$ if it has angular density $\rho(r)$ with ρ nonintegral. (see, e.g., [13, Chap. II, §1]) .

A regular point set $\{a_n\}$ satisfying one of the conditions (C) or (C$'$) is called (see, e.g., [13, Chap. II, §1]) a regular set, or, briefly, an *R-set*:

(C) *There exists a number* $d>0$ *such that the circles of radii*

$$r_n=d\,|a_n|^{1-\frac{\rho(|a_n|)}{2}}$$

with centers at the points a_n *do not intersect.*

(C$'$) *The points* a_n *lie inside angles with a common vertex at the origin but with no other points in common that are such that if one arranges the points of the set* $\{a_n\}$ *within any one of these angles in the order of increasing moduli then, for all points which lie inside the same angle, it is true that*

$$|a_{k+1}|-|a_k|>d\,|a_k|^{1-\rho(|a_k|)}$$

for some $d>0$.

Below we introduce (see, e.g., [13, Chap. II, §1]) the quantity $H(\theta)$, called the *indicator* of the set and defined by the formula (if ρ is not an integer)

$$H(\theta)=\frac{\pi}{\sin\pi\rho}\int\limits_{\theta-2\pi}^{\theta}\cos\rho(\theta-\psi-\pi)\,d\Delta(\psi),$$

where $\Delta(\psi)$ is the density of the set in the angle $(0,\psi)$.

Let us consider some particular case (see, e.g., [13, Chap. II, §2]) of regular distribution of the points $\{a_n\}$. We have the simplest case when all points $\{a_n\}$ lie on one ray $\arg z=\psi$ with some density

$$\Delta=\lim_{r\to\infty}\frac{n(r)}{r^{\rho(r)}},$$

where $n(r)$ is the number of points $\{a_n\}$ in the interval $(0,r)$ on the ray $\arg z = \psi$. In this case

$$H(\theta) = \frac{\pi\Delta}{\sin \pi\rho} \cos \rho(\theta - \psi - \pi). \tag{12}$$

We also note that if $\{a_n\}$ is a sequence of natural numbers then $\{a_n\}$ is an R-set.

To characterize the dependence of the growth of a function of finite order ρ in an angle $\theta_1 \le \arg z \le \theta_2$ in a direction in which z tends to infinity, Phragmén and Lindelöf (see, e.g., [13, Chap. I, §15]) introduced the function

$$h_f(\theta) = \overline{\lim_{r\to\infty}} \frac{\ln\left|f\left(re^{i\theta}\right)\right|}{r^\rho} \qquad (\theta_1 \le \theta \le \theta_2),$$

called the *indicator function of* $f(z)$.

Let $\psi(z)$ be an entire function having simple zeros at all the nodes $\{a_n\}$ ($|a_n| \to \infty$), and let $\{b_n\}$ be values assigned at the nodes. It is necessary to define an entire function $F(z)$ such that

$$F(a_n) = b_n \quad \text{for all } n = 1, 2, \dots$$

If the series (*the Lagrange interpolation series*)

$$\sum_{n=1}^{\infty} \frac{b_n \psi(z)}{\psi'(a_n)(z - a_n)} \tag{13}$$

converges uniformly in every bounded domain then it represents an entire function solving the interpolation problem.

We will investigate the question of the representation of an entire function by the Lagrange interpolation series in the case when the nodes form an R-set with index $\rho(r)$. Then the following theorem about the convergence of the Lagrange interpolation series holds (see, e.g., [13, Chap. IV, §4]):

Theorem 4 ([13])**.** *Let $\{a_n\}$ be an R – set of points with index $\rho(r)$ and $\{b_n\}$ a sequence of numbers. A necessary and sufficient condition for the existence of an entire function $F(z)$ satisfying the conditions*

$$F(a_n) = b_n, \qquad h_F(\theta) \le H(\theta),$$

is given by

$$\overline{\lim_{n\to\infty}} \frac{\ln|b_n|}{H(\psi_n)|a_n|^{\rho(|a_n|)}} \le 1 \qquad (\psi_n = \arg a_n).$$

We note that if

$$\overline{\lim_{n\to\infty}} \frac{\ln|b_n|}{H(\psi_n)|a_n|^{\rho(|a_n|)}} < 1, \tag{14}$$

then the interpolating function $F(z)$ is the Lagrange series (13), which converges uniformly in every bounded domain, and the limit is an entire function with the indicator not exceeding $H(\theta)$.

Corollary 1. *If $\psi_n = \arg a_n = 0$, then the inequality* (14) *takes the form*

$$\frac{1}{H(0)} \overline{\lim_{n\to\infty}} \frac{\ln|b_n|}{|a_n|^{\rho(|a_n|)}} < 1. \tag{15}$$

5. The main result

Let $f(z)$ be an entire function of order ρ with zeros $z_1, z_2, \dots$ such that $f(0) \neq 0$ and let ρ_1 be the index of convergence of the zeros, which is defined by the formula

$$\rho_1 = \overline{\lim_{n\to\infty}} \frac{\ln n}{\ln |z_n|}.$$

Then, according to the Hadamard factorization theorem (see, e.g., [14], Chap. VIII, section 8.2.4), the function $f(z)$ is represented in the form

$$f(z) = e^{T(z)} R(z), \qquad (16)$$

where $R(z)$ is the canonical product constructed by the zero set of the function $f(z)$, and $T(z)$ is a polynomial of degree at most $[\rho]$. In this case, the canonical product $R(z)$ has the form

$$R(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n} + \left(\frac{z}{z_n}\right)^2/2 + \ldots + \left(\frac{z}{z_n}\right)^{\kappa}/\kappa}$$

and $\kappa \leq [\rho_1] \leq [\rho]$.

In the particular case when $\kappa = 0$ (it is true for $\rho < 1$, or for $\rho \geq 1$ and $\rho_1 < 1$, or, finally, for $\rho_1 = 1$, and the series $\sum_{n=1}^{\infty} \frac{1}{|z_n|}$ converges), the polynomials

$$p_n(z) = \frac{z}{z_n} + \left(\frac{z}{z_n}\right)^2/2 + \ldots + \left(\frac{z}{z_n}\right)^{\kappa}/\kappa$$

are equal to zero. Then the formula (16) takes a particularly simple form

$$f(z) = e^{T(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right).$$

We consider entire functions $f(z)$ of order ρ, which have the form

$$f(z) = C \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right). \qquad (17)$$

Representation (17) holds, for example, for entire functions of order less than 1 or for entire functions of the first order with an additional condition (the series $\sum_{n=1}^{\infty} \frac{1}{|z_n|}$ is convergent). In particular, (17) holds for functions of zero genus.

It is easy to show that, in this case, we obtain

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty} \frac{1}{z - z_n} \qquad (18)$$

if $z \neq z_n$.

Let $z_n = -q_n$, $q_n > 0$, q_n satisfy condition (3) and form some sequence of natural numbers, which is an R-set.

To justify the change of the order of summation and differentiation in (18), it is necessary to prove the uniform convergence of the series (18) for given z_n. We have

$$\frac{1}{|z+q_n|} = \frac{1}{|x+iy+q_n|} = \frac{1}{\sqrt{(x+q_n)^2 + y^2}} \leq \frac{1}{\sqrt{(x+q_n)^2}} \leq \frac{1}{\sqrt{q_n^2}} = \frac{1}{|z_n|},\ x \geq 0.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{|z_n|}$ converges, for given z_n, the series (18) converges uniformly on the set Re $z \geq 0$ in accordance with the Weierstrass criterion for a uniform convergence of functional series.

In view of (18), our goal is to obtain an integral representation for

$$-\frac{d}{dz}\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty} \frac{1}{(z+q_n)^2}.$$

Lemma 1. *The function*

$$\varphi(\zeta) = \frac{Q(\zeta)\, e^{\zeta^2}}{(z+\zeta)^2} \tag{19}$$

is holomorphic and bounded on the set $\{0 \leq \text{Re}\ \zeta \leq x_2\}$*, where* $Q(\zeta)$ *is an entire function of the first order and the real part of* z *is positive.*

Proof. Since $Q(\zeta)$ is an entire function of the first order, for every positive value of ε ([14], Chap. VIII, section 8.2), we have

$$Q(\zeta) = O\left(e^{r^{1+\varepsilon}}\right), \quad |\zeta| = r \to \infty,$$

but not for any negative value. Thus, for $\zeta = u + iv$

$$|Q(\zeta)| \leq Ce^{|\zeta|^{1+\varepsilon}} = Ce^{\left(u^2+v^2\right)^{\frac{1+\varepsilon}{2}}} \leq Ce^{\left(x_2^2+v^2\right)^{\frac{1+\varepsilon}{2}}},$$
$$\left|e^{\zeta^2}\right| = e^{u^2-v^2} \leq e^{x_2^2-v^2}.$$

To prove the boundedness of the function $\varphi(\zeta)$, we consider the limit

$$\lim_{v\to\pm\infty} Ce^{\left(x_2^2+v^2\right)^{\frac{1+\varepsilon}{2}}} \cdot e^{x_2^2-v^2} = Ce^{x_2^2} \lim_{v\to\pm\infty} \frac{e^{\left(x_2^2+v^2\right)^{\frac{1+\varepsilon}{2}}}}{e^{v^2}}$$
$$= Ce^{x_2^2} \lim_{v\to\pm\infty} \frac{e^{\left(x_2^2+v^2\right)^{\frac{1+\varepsilon}{2}}}}{e^{v^2}} \cdot \frac{e^{\left(v^2\right)^{\frac{1+\varepsilon}{2}}}}{e^{\left(v^2\right)^{\frac{1+\varepsilon}{2}}}} = Ce^{x_2^2} \lim_{v\to\pm\infty} \frac{e^{\left(x_2^2+v^2\right)^{\frac{1+\varepsilon}{2}}}}{e^{\left(v^2\right)^{\frac{1+\varepsilon}{2}}}} \cdot \lim_{v\to\pm\infty} \frac{e^{\left(v^2\right)^{\frac{1+\varepsilon}{2}}}}{e^{v^2}}$$
$$= Ce^{x_2^2} \cdot \lim_{v\to\pm\infty} \frac{e^{\left(v^2\right)^{\frac{1+\varepsilon}{2}}}}{e^{v^2}} = 0 \quad \text{since}$$
$$\lim_{v\to\pm\infty} \frac{e^{\left(v^2\right)^{\frac{1+\varepsilon}{2}}}}{e^{v^2}} = \lim_{v\to\pm\infty} e^{\left[\left(v^2\right)^{\frac{1+\varepsilon}{2}} - v^2\right]} = \lim_{v\to\pm\infty} e^{\left\{\left(v^2\right)^{\frac{1+\varepsilon}{2}}\left[1-\left(v^2\right)^{\frac{1-\varepsilon}{2}}\right]\right\}} = 0.$$

The holomorphy of the function $\varphi(\zeta)$ follows from the form (19) of the function itself and the condition that the real part of z is positive. □

Below we represent $Q(\zeta)$ in the form

$$Q(\zeta) = Q_1(\zeta) \cdot Q_2(\zeta)$$

as follows:

Let the function $Q_1(\zeta)$ be equal to zero at all fractional points $z_{k,l}$ lying to the right of zero on the real axis, i.e., at the points

$$\alpha_{k,l} = l - \frac{k}{N}, \quad l = 1, 2, \ldots, \quad k = 1, 2, \ldots, N-1,$$

or, which is the same,

$$\alpha_j = \frac{j}{N}, \quad j = 1, 2, \ldots, \quad j \neq mN, \quad m \in \mathbb{N}.$$

We also require that the function $Q_1(\zeta)$ be equal to zero at the natural points not belonging to the sequence $\{q_n\}$. Thus, on the function $Q_1(\zeta)$, we impose the following conditions:

$$Q_1(\alpha_j) = 0, \quad j = 1, 2, \ldots, \quad j \neq mN, \quad m \in \mathbb{N}, \tag{20}$$

$$Q_1(\zeta) = 0, \quad \zeta \in \mathbb{N} \backslash \{q_n\}. \tag{21}$$

Denoting the set of indices by $J = \{j : j = 1, 2, \ldots, j \neq mN, m \in \{q_n\}\}$ and combining these two conditions, we have

$$Q_1(\alpha_j) = 0, \quad j \in J.$$

Thus, a function $Q_1(\zeta)$ satisfying conditions (20) and (21) can be chosen in the form of a canonical product

$$Q_1(\zeta) = \prod_{j \in J} \left(1 - \frac{\zeta}{\alpha_j}\right) e^{\frac{\zeta}{\alpha_j}}.$$

The convergence of the infinite product in the expression for $Q_1(\zeta)$ follows from the fact that $Q_1(\zeta)$ is a subproduct of the function $\dfrac{e^{\gamma N \zeta}}{\Gamma(-N\zeta + 1)}$, where γ is the Euler constant. Indeed, making the substitution $z = -N\zeta$ in the expression for

$$\frac{1}{\Gamma(z+1)} = e^{\gamma z} \prod_{j=1}^{\infty} \left(1 + \frac{z}{j}\right) e^{-\frac{z}{j}}, \quad \text{we obtain}$$

$$\frac{1}{\Gamma(-N\zeta + 1)} = e^{-\gamma N \zeta} \prod_{j=1}^{\infty} \left(1 - \frac{N\zeta}{j}\right) e^{\frac{N\zeta}{j}} = e^{-\gamma N \zeta} \prod_{j=1}^{\infty} \left(1 - \frac{\zeta}{j/N}\right) e^{\frac{\zeta}{j/N}}.$$

We choose the function $Q_2(\zeta)$ so that $Q_2(\zeta)$ at the points of the sequence $\{q_n\}$ takes the values

$$Q_2(q_n) = \frac{1}{Q_1(q_n) e^{q_n^2}}. \tag{22}$$

Assuming

$$\psi(\zeta) = \prod_{n=1}^{\infty} \left(1 - \frac{\zeta}{q_n}\right), \tag{23}$$

we formulate the following statement about the solution to the interpolation problem (22).

Lemma 2. *The solution to the interpolation problem* (22) *is an entire function*

$$Q_2(\zeta) = \sum_{n=1}^{\infty} \frac{1}{Q_1(q_n) e^{q_n^2}} \cdot \frac{\psi(\zeta)}{\psi'(q_n)(\zeta - q_n)}, \tag{24}$$

which is represented by a series uniformly converging in any bounded domain.

Proof. By theorem 4, the solution to the interpolation problem (22) is the Lagrange series (13), in which

$$a_n = q_n, \quad b_n = \frac{1}{Q_1(q_n)\, e^{q_n^2}}, \quad \psi_n = \arg a_n = 0$$

and the function $\psi(\zeta)$ is defined by (23). To prove the uniform convergence of series (24), we use the condition (15), in which we choose the exponent $\rho(r) = \rho$. It is necessary to verify that

$$\frac{1}{H(0)} \varlimsup_{n\to\infty} \frac{\ln|b_n|}{q_n^{\rho}} < 1. \tag{25}$$

We first check that $H(0) > 0$ for $0 < \rho < 1/2$. Since $\{a_n\}$ lie on one ray $\arg z = 0$, then, by (12),

$$H(0) = \frac{\pi\Delta}{\sin\pi\rho}\cos\pi\rho.$$

Furthermore,

$$\ln|b_n| = \ln\left|\frac{1}{Q_1(q_n)\, e^{q_n^2}}\right| = -\ln\left|Q_1(q_n)\, e^{q_n^2}\right| = -q_n^2 - \ln\left|\prod_{j\in J}\left(1-\frac{q_n}{\alpha_j}\right)e^{\frac{q_n}{\alpha_j}}\right|$$

$$= -q_n^2 - \sum_{j\in J}\ln\left|\left(1-\frac{q_n}{\alpha_j}\right)e^{\frac{q_n}{\alpha_j}}\right| = -q_n^2 - \sum_{j\in J}\left(\ln\left|1-\frac{q_n}{\alpha_j}\right| + \frac{q_n}{\alpha_j}\right).$$

We use the standard expansion

$$\ln(1-x) + x = -\sum_{p=2}^{\infty}\frac{x^p}{p}, \quad |x| < 1.$$

In our notation, we get

$$\ln\left|1-\frac{q_n}{\alpha_j}\right| + \frac{q_n}{\alpha_j} = -\sum_{p=2}^{\infty}\frac{1}{p}\left(\frac{q_n}{\alpha_j}\right)^p, \quad \left|\frac{q_n}{\alpha_j}\right| < 1.$$

The condition $q_n/\alpha_j < 1$ is obviously satisfied starting from some number $j > j_0$ (for fixed n). The choice of j_0 will be given below. Thus, we obtain

$$\ln|b_n| = -q_n^2 - \sum_{j\in J}\left(\ln\left|1-\frac{q_n}{\alpha_j}\right| + \frac{q_n}{\alpha_j}\right) = -q_n^2 + \sum_{\substack{j\in J\\ j>j_0}}\sum_{p=2}^{\infty}\frac{1}{p}\left(\frac{q_n}{\alpha_j}\right)^p$$

$$= -q_n^2 + \sum_{p=2}^{\infty}\sum_{\substack{j\in J\\ j>j_0}}\frac{1}{p}\left(\frac{q_n}{\alpha_j}\right)^p = -q_n^2 + \sum_{p=2}^{\infty}\frac{q_n^p}{p}\sum_{\substack{j\in J\\ j>j_0}}\frac{1}{\alpha_j^p}$$

$$\le -q_n^2 + \sum_{p=2}^{\infty}\frac{q_n^p}{p}\sum_{j=j_0+1}^{\infty}\frac{1}{\alpha_j^p} = -q_n^2 + \sum_{p=2}^{\infty}\frac{q_n^p}{p}\sum_{j=j_0+1}^{\infty}\frac{N^p}{j^p}$$

$$= -q_n^2 + \sum_{p=2}^{\infty}\frac{q_n^p N^p}{p}\sum_{j=j_0+1}^{\infty}\frac{1}{j^p} = -q_n^2 + \sum_{p=2}^{\infty}\frac{q_n^p N^p}{p}\left(\sum_{j=1}^{\infty}\frac{1}{j^p} - \sum_{j=1}^{j_0}\frac{1}{j^p}\right)$$

$$= -q_n^2 + \sum_{p=2}^{\infty}\frac{q_n^p N^p}{p}\left(\zeta(p) - \sum_{j=1}^{j_0}\frac{1}{j^p}\right) = -q_n^2 + \frac{q_n^2 N^2}{2}\left(\zeta(2) - \sum_{j=1}^{j_0}\frac{1}{j^2}\right)$$

$$+\sum_{p=3}^{\infty}\frac{q_n^p N^p}{p}\left(\zeta(p)-\sum_{j=1}^{j_0}\frac{1}{j^p}\right)$$
$$=\left[\frac{N^2}{2}\left(\zeta(2)-\sum_{j=1}^{j_0}\frac{1}{j^2}\right)-1\right]q_n^2+\sum_{p=3}^{\infty}\frac{q_n^p N^p}{p}\left(\zeta(p)-\sum_{j=1}^{j_0}\frac{1}{j^p}\right),$$

where $\zeta(p)$ is the Riemann zeta-function. The change of the order of summation in multiple series is valid due to the absolute convergence of each of the series.

If

$$\frac{N^2}{2}\left(\zeta(2)-\sum_{j=1}^{j_0}\frac{1}{j^2}\right)-1<0 \quad \text{then} \quad \zeta(2)-\sum_{j=1}^{j_0}\frac{1}{j^2}<\frac{2}{N^2}.$$

Since

$$\zeta(2)-\sum_{j=1}^{j_0}\frac{1}{j^2}<\zeta(2)-\sum_{j=1}^{j_0}\frac{1}{j(j+1)}=\frac{\pi^2}{6}-\sum_{j=1}^{j_0}\left(\frac{1}{j}-\frac{1}{j+1}\right)$$
$$=\frac{\pi^2}{6}-\left(1-\frac{1}{j_0+1}\right)<\frac{\pi^2}{6}+\frac{1}{j_0+1}<\frac{2}{N^2},$$

we have

$$\frac{1}{j_0+1}<\frac{2}{N^2},\quad j_0+1>\frac{N^2}{2},\quad j_0>\frac{N^2}{2}-1.$$

It is known (see, e.g., [2, Chap. IV, Theorem 4.11]) that for $s=\sigma+it$ *we have*

$$\zeta(s)=\sum_{n\le x}\frac{1}{n^s}-\frac{x^{1-s}}{1-s}+O\left(x^{-\sigma}\right)$$

uniformly for $\sigma\ge\sigma_0>0$, $|t|<2\pi x/C$ *whenever* C *is a given constant greater than* 1. Then

$$\zeta(p)-\sum_{j=1}^{j_0}\frac{1}{j^p}=-\frac{j_0^{1-p}}{1-p}+O\left(j_0^{-p}\right)\quad\text{or}\quad\zeta(p)-\sum_{j=1}^{j_0}\frac{1}{j^p}\le\frac{j_0^{1-p}}{p-1}+C_1 j_0^{-p},$$

where the constant C_1 does not depend on j_0.

We transform the second term in the expression for $\ln|b_n|$. We have

$$\sum_{p=3}^{\infty}\frac{q_n^p N^p}{p}\left(\zeta(p)-\sum_{j=1}^{j_0}\frac{1}{j^p}\right)\le\sum_{p=3}^{\infty}\frac{q_n^p N^p}{p}\left(\frac{j_0^{1-p}}{p-1}+C_1 j_0^{-p}\right)=\sum_{p=3}^{\infty}\frac{q_n^p N^p}{p}\frac{j_0^{1-p}}{p-1}$$
$$+\sum_{p=3}^{\infty}\frac{q_n^p N^p}{p}C_1 j_0^{-p}=j_0\sum_{p=3}^{\infty}\frac{1}{p(p-1)}\left(\frac{q_n}{j_0/N}\right)^p+C_1\sum_{p=3}^{\infty}\frac{1}{p}\left(\frac{q_n}{j_0/N}\right)^p$$
$$<j_0\sum_{p=3}^{\infty}\frac{1}{p(p-1)}+C_1\sum_{p=3}^{\infty}\left(\frac{q_n}{j_0/N}\right)^p=j_0\sum_{p=3}^{\infty}\frac{1}{p(p-1)}+C_1\frac{\left(\frac{q_n}{j_0/N}\right)^3}{1-\frac{q_n}{j_0/N}},$$

where we used the fact that

$$\frac{q_n}{j_0/N}=\frac{q_n}{\alpha_{j_0}}<1.$$

Thus,

$$\ln|b_n| < \left[\frac{N^2}{2}\left(\zeta(2) - \sum_{j=1}^{j_0}\frac{1}{j^2}\right) - 1\right] q_n^2 + j_0 \sum_{p=3}^{\infty} \frac{1}{p(p-1)} + C_1 \frac{\left(\frac{q_n}{j_0/N}\right)^3}{1 - \frac{q_n}{j_0/N}}.$$

It is easy to see that

$$\varlimsup_{n\to\infty} \frac{\ln|b_n|}{q_n^\rho} = -\infty \quad \text{for} \quad 0 < \rho < 1/2$$

and condition (25) is satisfied. □

Assuming the function $\varphi(\zeta)$ in the form (19) in Theorem 3 and taking into account property (20) of the function $Q_1(\zeta)$, we infer that $\varphi(\zeta)$ is equal to zero at all fractional points on the left-hand side of the formula of Theorem 3. We choose $x_1 = 0$ (in this case $x_2 > 0$). We have

$$\begin{aligned} &\frac{P(1)}{N}\Big(\varphi(1) + \varphi(2) + \ldots + \varphi(x_2 - 1) + \varphi(x_2)\Big) \\ &= \frac{P(1)}{N}\left(-\frac{1}{2}\varphi(0) + \frac{1}{2}\varphi(x_2)\right) + \int_0^{x_2}\varphi(z)\,dz \\ &+ \frac{1}{i}\int_0^{\infty}\Big(\varphi(-iy)\,F(f, 2\pi y) + \varphi(iy)\big[1 + F(f, -2\pi y)\big]\Big)dy \\ &-\frac{1}{i}\int_0^{\infty}\Big(\varphi(x_2 - iy)\,F(f, 2\pi y) + \varphi(x_2 + iy)\big[1 + F(f, -2\pi y)\big]\Big)dy. \end{aligned} \tag{26}$$

In view of (21), $\varphi(\zeta)$ is equal to zero at the natural points not belonging to the sequence $\{q_n\}$. At the points of the sequence (in view of (22)), we have

$$\varphi(q_n) = \frac{Q(q_n)\,e^{q_n^2}}{(z + q_n)^2} = \frac{Q_1(q_n)\,Q_2(q_n)\,e^{q_n^2}}{(z + q_n)^2} = \frac{1}{(z + q_n)^2}.$$

Thus, relation (26) can be written in a simpler form, and we obtain

Theorem 5. *The following equality holds:*

$$\begin{aligned} \frac{P(1)}{N}\sum_n \frac{1}{(z + q_n)^2} &= \frac{P(1)}{N}\left(-\frac{1}{2}\varphi(0) + \frac{1}{2}\varphi(x_2)\right) + \int_0^{x_2}\varphi(z)\,dz \\ &+ \frac{1}{i}\int_0^{\infty}\Big(\varphi(-iy)\,F(f, 2\pi y) + \varphi(iy)\big[1 + F(f, -2\pi y)\big]\Big)dy \\ &-\frac{1}{i}\int_0^{\infty}\Big(\varphi(x_2 - iy)\,F(f, 2\pi y) + \varphi(x_2 + iy)\big[1 + F(f, -2\pi y)\big]\Big)dy, \end{aligned}$$

where the summation is carried out over all points q_n *in the segment* $[1; x_2]$, x_2 *is a positive integer, the polynomial* $P(w)$ *is defined by* (8), *the function* $\varphi(\zeta)$ *is defined by* (19), *and the functions* $F(f, 2\pi y)$ *and* $1 + F(f, -2\pi y)$ *are defined by* (10) *and* (11) *respectively.*

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Vyacheslav Igorevich Kuzovatov
Siberian Federal University,
79, Svobodny ave.,
Krasnoyarsk, 660041, Russia
E-mail address: kuzovatov@yandex.ru

Alexander Mechislavovich Kytmanov
Siberian Federal University,
79, Svobodny ave.,
Krasnoyarsk, 660041, Russia
E-mail address: akytmanov@sfu-kras.ru