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ESTIMATES OF THE NORMS OF DERIVATIVES IN THE ONE- AND MULTIDIMENSIONAL CASES

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Abstract. We consider multiplicative inequalities of the Kolmogorov type for the norms of the intermediate derivatives of a function through the norms of the function itself and its highest derivative in various Lebesgue spaces for all kinds of one-dimensional areas. Some estimates of the constants in the inequalities in these spaces are established and the asymptotic behavior of such constants is given as the orders of both the highest and intermediate derivatives grow infinitely. In addition, we obtain an estimate for the norms of the mixed derivatives of a function in terms of the norms of the derivatives with respect to each variable separately in different Lebesgue spaces for the case of the multidimensional torus. The results obtained are of independent interest and also can be used in solving various problems of mathematical physics. This in particular applies to problems in which theorems substantially involve embeddings of the corresponding function spaces.

Keywords: space, domain, function, norm, inequality, estimate, asymptotics, infinity, intermediate, mixed, derivative

The theory of embeddings of the classical Sobolev spaces is devoted to algebraic relations between the parameters of the spaces adequate to the smoothness of functions belonging to these spaces.

In the Sobolev spaces of infinite order, all functions are infinitely differentiable, and hence the question of smoothness does not appear. In the above-mentioned theory, of importance are not simply relations between the parameters of the spaces under consideration but their asymptotic behavior. In this direction, mention the universal criterion for the embedding of a Sobolev space of infinite order established

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by Yu.A.Dubinskii in [1], expressed in terms of the norms of the embedding operators of the classical Sobolev spaces $W_p^{m+1} \subset W_p^m$ as $m \to \infty$.

In view of the absence at present of an efficient control of the behavior of the norms of the embedding operators as $m \to \infty$, of an undoubtful interest are easily verifiable algebraic embedding conditions expressed through the parameters of the spaces under consideration. Namely, of the spaces $W^\infty\{a_\alpha, p\}_{(G)}$ and $W^\infty\{b_\alpha, p\}_{(G)}$, where

$$(1) \qquad W^\infty\{a_\alpha, p\}_{(G)} \equiv \left\{ u(x) \in C^\infty(G) : \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha u(x)\|_p^p < \infty \right\}.$$

Here G is a domain in the ν-dimensional space, $\{a_\alpha\}$ and $\{b_\alpha\}$ are nonnegative number sequences of the form $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_\nu\}$, $\alpha_i \in \mathbb{N}$, $i = 1, 2, \ldots, \nu$, i.e., integer multi-indices, $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_\nu$, $u(x)$ is a function of the real variables $(x_1, x_2, \ldots, x_\nu) \in G$, $D^\alpha u(x) = D_1^{\alpha_1} \ldots D_\nu^{\alpha_\nu} u(x)$ are its derivatives of order α, $\| D^\alpha u(x)\|_p$ is the norm in the Lebesgue space $L_p(G)$.

For this purpose, we propose two methods, one of which is based on estimating the norms of the intermediate derivatives of a function defined in some one-dimensional domains and also on estimating the norms of the mixed derivatives of the functions defined in a multidimensional domain through the norms of its derivatives with respect to each of the variables separately. The second method is based on regularizing the sequences $\{a_\alpha\}$ and $\{b_\alpha\}$. In the present article, we expose the first of the methods.

Let us first dwell on the case of one-dimensional domains G and described the results obtained for estimating the growth rate of the constants c_{nk} in the multiplicative inequalities of the form

$$(2) \qquad \|f^{(k)}(x)\|_{L_p(G)} \leqslant c_{nk} \|f(x)\|_{L_p(G)}^{1-k/n} \|f^{(n)}(x)\|_{L_p(G)}^{k/n}$$

for each k $(0 \leqslant k \leqslant n)$, called Kolmogorov's inequalities (see [2]), as n and k tend to infinity. It should be noted here that the behavior of the indicated constants c_{nk} depends both on the form of the domain G (the whole number axis, a half-axis, a circle, a closed interval) and on the space L_p, $p \geqslant 1$, in which the norms of the function are considered but it still does not depend on the function itself.

Such inequalities were investigated by many famous mathematicians; a review of their works can be found in the articles of V.M. Tikhomirov [3], S. Mandelbrojt [4], and other authors. In the present article, we do not pretend to give an exhaustive review of the previous results but mention some publications essential for our study.

In A.N. Kolmogorov's paper, he obtained the following exact inequality for an n-times differentiable function on the whole axis for any k $(0 \leqslant k \leqslant n)$:

$$(3) \qquad M_k \leqslant M_0^{1-k/n} M_n^{k/n} \frac{t_{n-k}}{t_n^{1-k/n}},$$

where $M_j = \sup\limits_{x \in \mathbb{R}} |f^{(j)}(x)|$, $t_j = \frac{4}{\pi} \sum\limits_{m=0}^{\infty} \frac{(-1)^{m(j+1)}}{(2m+1)^{j+1}}$, $1 \leq \frac{t_{n-k}}{t_n^{1-k/n}} \leq 2$ under the condition $M_0 < \infty$, $M_n < \infty$. Here the less or equal sign in (3) cannot be replace by the less sign. As was shown by T Bang [5] and later A.A. Ligun [6], inequality (3) also holds for periodic functions $C_T(\mathbb{R})$ with period T.

Using an idea of I. Stein [7], we have managed to validate such an estimate for the k-derivative of a function in L_p for any p, $1 \leqslant p < \infty$, and in a rather easy way.

Proposition 1. *If* $u(x), u'(x), \dots, u^{(n)}(x) \in C_T(\mathbb{R}) \cap L_p$ *then, for any* p, $1 \leqslant p < \infty$, *we have the equality* (3) *(in which* $M_j = \left\| u^{(j)}(x) \right\|_p$ *) with the same constant* c_{nk}.

Доказательство. Let $1 \leqslant p < \infty$, $1 \leqslant k \leqslant n$. Put

$$h(x) = \begin{cases} \dfrac{|u^{(k)}(x)|^{p-1} \operatorname{sign} u^{(k)}(x)}{\| \, |u^{(k)}(x)|^{p-1} \|_{p'}} \text{ if } \; p > 1, \; p' = \dfrac{p}{p-1}; \\ \operatorname{sign} u^{(k)}(x) \;\text{ if }\; p = 1, \; x \in T. \end{cases}$$

Noticing that $\|h(x)\|_{p'} = 1$ and

$$\int_{-\pi}^{\pi} u^{(k)}(x) h(x) dx = \|u^{(k)}(x)\|_p, \tag{4}$$

introducing the function

$$f(x) = \int_{-\pi}^{\pi} u(x+y) h(y) dy$$

and reckoning with the hypotheses of Proposition 1, we get

$$f^{(k)}(x) = \int_{-\pi}^{\pi} u^{(k)}(x+y) h(y) dy.$$

With account taken of this equation and (4), we obtain the equality

$$f^{(k)}(0) = \int_{-\pi}^{\pi} u^{(k)}(y) h(y) dy = \|u^{(k)}(y)\|_p. \tag{5}$$

Inequality (3) for $p = \infty$ and equality (5) enable us to write the inequality

$$|f^{(k)}(0)| \leqslant c_{nk} \|f(x)\|_\infty^{1-k/n} \|f^{(n)}(x)\|_\infty^{k/n}. \tag{6}$$

Furthermore, using Hölder's inequality and the periodicity of the function, we obtain

$$\|f^{(k)}(x)\|_\infty = \max_{x \in T} \left| \int_{-\pi}^{\pi} u^{(k)}(x+y) h(y) dy \right| \leqslant \|u^{(k)}(y)\|_p \|h(y)\|_{p'} = \|u^{(k)}(y)\|_p. \tag{7}$$

Putting together relations (5), (6), and (7), we obtain inequalities (3) for all $p \in [1; \infty)$.

Thus, the proposition is proved. □

Remark. *If* $p = 2$ *then, for periodic functions,* $c_{nk} = 1$.

Indeed, using Parseval's identity

$$\|u(x)\|_2 = \left(\sum_{m=1}^{\infty} a_m^2(u) + b_m^2(u) \right)^{1/2},$$

where $a_m(u)$, $b_m(u)$ are the Fourier coefficients of the function $u(x)$, and Hölder's inequality, we obtain for all $k \leqslant n$:

$$\|u^{(k)}(x)\|_2^2 = \sum_{m=1}^{\infty} m^{2k}(a_m^2 + b_m^2) = \sum_{m=1}^{\infty} \left(m^{2n}(a_m^2 + b_m^2)\right)^{k/n} (a_m^2 + b_m^2)^{1-k/n}$$
$$\leqslant \left(\sum_{m=1}^{\infty} m^{2n}(a_m^2 + b_m^2)\right)^{k/n} \left(\sum_{m=1}^{\infty}(a_m^2 + b_m^2)\right)^{1-k/n}$$
$$= \|u(x)\|_2^{2(1-k/n)} \|u^{(n)}(x)\|_2^{2k/n}.$$

For the domain G coinciding with the half-axis $(0;\infty)$, for $p = \infty$, S. Mandelbrojt [4] proved inequality (2) with the constant $c_{nk} = \left(\frac{8en}{k}\right)^k$, $1 \leqslant k \leqslant n$. For an n times differentiable function on $(0;\infty)$, A.Gorny [8] showed that the constant c_{nk} can be replaced by $\left(\frac{e^2 n}{k}\right)^k$.

In 1967, S.B. Stechkin [9] obtained the following estimate for the constant c_{nk} in (2) for the half-axis for $p = \infty$:

$$c_{nk} \leqslant K \left(\frac{2n}{m}\right)^m, \quad m = \min(k,\ n-k). \tag{8}$$

Using an idea of I. Stein [7] and S.B.Stechkin's result, we validate this estimate for all $p \in [1;\infty)$.

Proposition 2. *Let $u(x), u'(x), \ldots, u^{(n)}(x) \in C(0;\infty) \cap L_p(0;\infty)$. Then, for any p, $1 \leqslant p < \infty$, we have inequality* (2) *with some constant c_{nk} satisfying* (8).

Доказательство. By analogy with Proposition 1, fox some $1 \leqslant k \leqslant n$ and put

$$h(x) = \begin{cases} \operatorname{sign} u^{(k)}(x) \dfrac{|u^{(k)}(x)|^{p-1}}{\|\,|u^{(k)}(x)|^{p-1}\|_{p'}}, & p > 1,\ x \in (0;\infty); \\ \operatorname{sign} u^{(k)}(x), \quad p = 1. \end{cases} \tag{9}$$

Introduce the function

$$f(x) = \int_0^{\infty} u(x+y)h(y)dy, \quad x \geqslant 0.$$

Obviously, $\|h(y)\|_{p'} = 1$ and

$$\left|\int_0^{\infty} u^{(k)}(x)h(x)dx\right| = \|u^{(k)}(x)\|_p. \tag{10}$$

By the hypotheses of this proposition and the definition of $f(x)$, we have

$$f^{(k)}(x) = \int_0^{\infty} u^{(k)}(x+y)h(y)dy. \tag{11}$$

In particular, from (11) for $x = 0$, with account taken of (10), we obtain

$$f^{(k)}(0) = \int_0^{\infty} u^{(k)}(y)h(y)dy = \|u^{(k)}(y)\|_p. \tag{12}$$

Using S.B. Stechkin's inequality [9] for $p = \infty$, write down (2) in the form

$$|f^{(k)}(0)| \leqslant c_{nk} \|f(x)\|_\infty^{1-k/n} \|f^{(n)}(x)\|_\infty^{k/n}, \tag{13}$$

where c_{nk} are defined by (8).

Relations (12), (13) and Hölder's inequality imply

$$\|f^{(k)}(x)\|_\infty = \max_{x\geqslant 0} \left| \int_0^\infty u^{(k)}(x+y)h(y)dy \right| \leqslant \max_{x\geqslant 0} \|u^{(k)}(x+y)\|_p \|h(y)\|_{p'} \tag{14}$$

$$\leqslant \|u^{(k)}(y)\|_p \|h(y)\|_{p'} = \|u^{(k)}(y)\|_p. \tag{15}$$

Combining (12), (13), (14), we obtain (2) for every $p \in [1;\infty)$ with the constant c_{nk} defined by (8).

Proposition 2 is proved. □

Remark. *The constant* c_{nk} *of the form* (8) *in* (2) *can be replaced by*

$$c_{nk} = K\left(\frac{2n}{k}\right)^k, \quad K > 0 \text{ is a constant.} \tag{16}$$

For this it suffices to demonstrate that

$$\left(2\frac{n}{k}\right)^k > \left(2\frac{n}{n-k}\right)^{n-k} \quad npu \quad k > \frac{n}{2} \tag{17}$$

since $\min(k,\ n-k) = k$ *for* $k < n/2$.

Indeed, putting $k = \alpha n$, $1/2 < \alpha < 1$, rewrite (17) as

$$\left(\frac{2}{\alpha}\right)^\alpha > \left(\frac{2}{1-\alpha}\right)^{1-\alpha},$$

, or, in equivalent form,

$$e^{(2\alpha-1)\ln 2 + (1-\alpha)\ln(1-\alpha)} > e^{\alpha \ln \alpha}. \tag{18}$$

Inequality (18) is equivalent to the inequality

$$\varphi(\alpha) \equiv (2\alpha - 1)\ln 2 + (1-\alpha)\ln(1-\alpha) - \alpha \ln \alpha > 0. \tag{19}$$

Inequality (19) is obvious because $\varphi(1/2) = 0$ and $\varphi'(\alpha) > 0$ for $1/2 < \alpha < 1$.

This makes it possible to prove (2) for the half-axis for any $\rho \in [1;\infty)$ with the constant of the form (16). The obtained estimate is more general than the analogous estimates in the works by S. Mandelbrojt [4], A.Ya. Ligun [6], V.I. Burenkov [10], N.P. Kuptsov [11], V.N. Gabushin [12].

In all these articles, it was shown that the constants c_{nk} in (2) grow with n. Stechkin and also we obtain a bound for the growth of c_{nk} not only as n grows but also as k grows in the case of the half-axis for any $p \geqslant 1$.

Let us now turn to the case of a multidimensional domain and expose the obtained result to estimating the norm of the mixed derivatives in terms of the norms of the derivatives with respect to each of the variables separately. We will focus in detail on the case of the multidimensional torus T^ν and prove the following:

Proposition 3. *If* $u(x_1, x_2, \ldots, x_\nu) \in C^\infty(T^\nu)$ *then the following inequalities hold:*

$$\|D^\beta u\|_2 \leqslant \left(\prod_{i=1}^{\nu} \|D_{x_i}^{\nu\beta_i} u\|_2\right)^{1/\nu}, \tag{20}$$

$$\|D^\beta u\|_p \leqslant A_p\biggl(\prod_{i=1}^{\nu} \|D_{x_i}^{\nu(\beta_i+1)}u\|_p\biggr)^{1/\nu}, \quad 1<p<\infty,\ p\neq 2, \tag{21}$$

where $\beta=(\beta_1,\beta_2,\dots,\beta_\nu)$ *is an integer multi-index.*

Доказательство. For transparency, the proof will be carried out for the two-dimensional torus T^2.

Representing $u(x,y)$ by the Fourier series

$$u(x,y)=\sum_{k,n=-\infty}^{\infty} c_{kn}e^{ikx}e^{iny},$$

where c_{kn} are the Fourier coefficients, we infer that

$$D_{xy}^{t+s}u(x,y)=\sum_{k,n=-\infty}^{\infty} c_{kn}(ik)^t(in)^s e^{ikx}e^{iny}. \tag{22}$$

Parseval's identity yields the equality

$$\|D_{xy}^{t+s}u(x,y)\|_2^2=\sum_{k,n=-\infty}^{\infty} c_{kn}^2k^{2t}n^{2s}. \tag{23}$$

Applying Cauchy's inequality and Parseval's identity to the right-hand side of (23), we obtain

$$\begin{aligned}\sum_{k,n=-\infty}^{\infty} c_{kn}^2k^{2t}n^{2s} &\leqslant \biggl(\sum_{k,n=-\infty}^{\infty} c_{kn}^2k^{4t}\biggr)^{1/2}\biggl(\sum_{k,n=-\infty}^{\infty} c_{kn}^2n^{4s}\biggr)^{1/2}\\ &= \|D_x^{2t}u(x,y)\|_2\|D_y^{2s}u(x,y)\|_2,\end{aligned}$$

which implies (20). For proving (21), we use the following theorem:

The Hardy–Littlewood Theorem. (see [13]) *If* $u(x,y)\in L_p(T^2)$, $1<p\leqslant 2$, *then its Fourier coefficients satisfy the inequality*

$$\biggl(\sum_{k,n=-\infty}^{\infty} |c_{kn}kn|^p(kn)^{-2}\biggr)^{1/p} \leqslant A_p\|u\|_p. \tag{24}$$

If $p\geqslant 2$ *and a sequence* $\{c_{kn}\}$ *is given for which the series*

$$\sum_{k,n=-\infty}^{\infty} |c_{kn}kn|^p(kn)^{-2}$$

converges then c_{kn} *are the Fourier coefficients of some function and* $(x,y)\in L_p(T^2)$*; moreover,*

$$\|u(x,y)\|_p \leqslant A_p\biggl(\sum_{k,n=-\infty}^{\infty} |c_{kn}kn|^p(kn)^{-2}\biggr)^{1/p}. \tag{25}$$

In inequalities (24) *and* (25), A_p *is a constant depending only on* p.

Using this theorem, prove (21) first for $p \geqslant 2$. Representing $D_{xy}^{t+s}u(x,u)$ by (22) and applying (25), we obtain

$$\|D_{xy}^{t+s}u\|_p \leqslant A_p\Big(\sum_{k,n=-\infty}^{\infty} |c_{kn}k^t n^s kn|^p (kn)^{-2}\Big)^{1/p} \tag{26}$$

$$= A_p\Big(\sum_{k,n=-\infty}^{\infty} (|c_{kn}||k|^t|n|^s|k|^{1-(2/p)}|n|^{1-(2/p)})^p\Big)^{1/p}. \tag{27}$$

Making use of two known inequalities for any $1 \leqslant q \leqslant p < \infty$ (see [14]):

$$\Big(\sum_{k=-\infty}^{\infty} |c_k|^p\Big)^{1/p} \leqslant \Big(\sum_{k=-\infty}^{\infty} |c_k|^q\Big)^{1/q}, \tag{28}$$

$$\|u(x,y)\|_{L_q(T^2)} \leqslant K\|u\|_{L_p(T^2)}. \tag{29}$$

Applying (28), Cauchy's inequality, Parseval's identity, and (29) to the right-hand side of (26), we infer

$$\|D_{xy}^{t+s}u\|_p \leqslant A_p\Big(\sum_{k,n=-\infty}^{\infty} (|c_{kn}||k|^t|n|^s|k|^{1-(2/p)}|n|^{1-(2/p)})^p\Big)^{1/p}$$
$$\leqslant A_p\Big(\sum_{k,n=-\infty}^{\infty} (|c_{kn}||k|^{t+1-(2/p)}|n|^{s+1-(2/p)})^2\Big)^{1/2}$$
$$\leqslant A_p\bigg(\Big(\sum_{k,n=-\infty}^{\infty} c_{kn}^2|k|^{4t+4-(8/p)}\Big)^{1/2}\Big(\sum_{k,n=-\infty}^{\infty} c_{kn}^2|n|^{4s+4-(8/p)}\Big)^{1/2}\bigg)^{1/2}$$
$$\leqslant A_p\big(\|D_x^{2t+2}u\|_2\|D_y^{2s+2}u\|_2\big)^{1/2} \leqslant A_pK\big(\|D_x^{2t+2}u\|_p\|D_y^{2s+2}u\|_p\big)^{1/2}.$$

This implies (21) for $p \geqslant 2$.

We have the following chain of inequalities for $p \in (1;2)$:

$$\|D_{xy}^{t+s}u\|_p \leqslant K\|D_{xy}^{t+s}u\|_2 = K\Big(\sum_{k,n=-\infty}^{\infty} c_{kn}^2k^{2t}n^{2s}\Big)^{1/2}$$
$$= K\Big(\sum_{k,n=-\infty}^{\infty} |c_{kn}k^{2t+1}n^{-1}||c_{kn}n^{2s+1}k^{-1}|\Big)^{1/2}$$
$$\leqslant K\bigg(\Big(\sum_{k,n=-\infty}^{\infty} c_{kn}^2k^{4t+2}n^{-2}\Big)^{1/2}\Big(\sum_{k,n=-\infty}^{\infty} c_{kn}^2n^{4s+2}k^{-2}\Big)^{1/2}\bigg)^{1/2}$$
$$\leqslant K\bigg(\Big(\sum_{k,n=-\infty}^{\infty} |c_{kn}k^{2t+1}n^{-1}|^p\Big)^{1/p}\Big(\sum_{k,n=-\infty}^{\infty} |c_{kn}n^{2s+1}k^{-1}|^p\Big)^{1/p}\bigg)^{1/2}$$
$$\leqslant K\bigg(\Big(\sum_{k,n=-\infty}^{\infty} |c_{kn}k^{2t+2}|^p|kn|^{p-2}\Big)^{1/p}\Big(\sum_{k,n=-\infty}^{\infty} |c_{kn}n^{2s+2}|^p|kn|^{p-2}\Big)^{1/p}\bigg)^{1/2}$$
$$\leqslant KA_p\big(\|D_x^{2t+2}u\|_p\|D_y^{2s+2}u\|_p\big)^{1/2},$$

which leads to (21). Proposition 3 is proved. □

The inequalities obtained in the article are of independent interest and are also substantial in the theory of comparison of Sobolev spaces of infinite order. Note that the definition of these spaces involve the number sequences $\{a_\alpha\}$ and $\{b_\alpha\}$ that regulate the growth of the norms of the functions and their derivatives in the Lebesgue spaces $L_p(G)$. Therefore, in the embedding theorems, of importance are relations between these number sequences and the behavior of the constants $\{c_{mk}\}$ in inequalities of the type (2). Explain this on an example. To separate the main operator of two differential operators of infinite order, one must compare the Sobolev spaces of infinite order that are the domains of these operators. This makes it possible to establish the solvability of boundary value problems for infinite-order nonlinear differential equations containing subordinate terms.

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