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ON ALGEBRAIC ISOMORPHISMS OF RATIONAL COHOMOLOGY OF A KÜNNEMAN COMPACTIFICATION OF THE NÉRON MINIMAL MODEL

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Abstract. It is proved that the Grothendieck standard conjecture of Lefschetz type holds for rational cohomology of degree 2 or 3 of a Künnemann compactification of the Néron minimal model of an absolutely simple principally polarized Abelian variety over the field of rational functions of a smooth projective curve under certain restrictions on the ring of endomorphisms of the Abelian variety.

Keywords: Abelian variety, Néron minimal model, Künnemann compactification, Grothendieck standard conjecture of Lefschetz type.

Introduction

Let H be an ample divisor on a smooth complex projective d-dimensional variety X. Then, for any natural number $i \leq d$, the map

$$L^{d-i} : H^i(X,\mathbb{Q}) \xrightarrow{\smile \operatorname{cl}_X(H)^{\smile\, d-i}} H^{2d-i}(X,\mathbb{Q})$$

is an isomorphism by the strong Lefschetz theorem. The Grothendieck standard conjecture $B(X)$ of Lefschetz type [1] asserts that there exists an algebraic $\mathbb{Q}$-cycle Z on the Cartesian product $X \times X$ which determines the inverse *algebraic* isomorphism

$$H^{2d-i}(X,\mathbb{Q}) \xrightarrow[\sim]{x \mapsto \operatorname{pr}_{2*}(\operatorname{pr}_1^* x \smile \operatorname{cl}_{X\times X}(Z))} H^i(X,\mathbb{Q}).$$

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Denote by ${}^{\mathrm{c}}\Lambda$ the dual operator for L of the classical Hodge theory. It is well known that the conjecture $B(X)$ is equivalent to the algebraicity of the operator ${}^{\mathrm{c}}\Lambda$ [2, Proposition 2.3].

There is an abstract form of the standard conjecture for étale cohomology of smooth projective varieties over arbitrary fields [1]. From now on we consider only varieties over fields of characteristic zero. Provided this condition the standard conjecture $B(X)$ is equivalent to the coincidence of the numerical and homological equivalences of algebraic cycles on the Cartesian product $X\times X$ [3, Formula (1.11)]; besides, in accordance with [4, Proposition 1.7] the conjecture $B(X)$ is equivalent to the semi-simplicity of the $\mathbb{Q}$-algebra

$$\mathcal{A}(X)=\mathrm{cl}_{X\times X}(\mathrm{CH}_*(X\times X))\otimes_{\mathbb{Z}}\mathbb{Q}$$

of algebraic self-correspondences on the variety X with the bilinear composition law [2, Formula 1.3.1]

$$g\circ f=\mathrm{pr}_{13*}(\mathrm{pr}_{12}^*(f)\smile\mathrm{pr}_{23}^*(g)).$$

In abstract case, $B(X)\ \Rightarrow\ C(X)$, where the standard conjecture $C(X)$ of Künneth type asserts the algebraicity of Künneth components of the class of the diagonal $\Delta_X\hookrightarrow X\times X$ [2, Lemma 2.4]. The conjecture $B(X)$ is compatible with Cartesian product [2, Corollary 2.5], hyperplane section [2, Theorem 2.13] and specialization (with possible change of characteristic) [2, Introduction]. In characteristic zero it is compatible with monoidal transformations along smooth centres [5, Theorem 4.3].

By definition, a d-dimensional elliptic variety is birationally equivalent to a variety containing a smooth family of elliptic curves parameterized by some affine variety of dimension $d-1$.

It is known that the standard conjecture $B(X)$ is true for all smooth complex projective curves, surfaces, Abelian varieties [6] and threefolds of Kodaira dimension $\varkappa(X)<3$ [7] (in particular, it holds for all complex elliptic threefolds and for compactifications of Néron minimal models of Abelian surfaces over fields of algebraic functions of one variable with the field of constants $\mathbb{C}$). Besides, $B(X)$ holds for Hilbert schemes of points on surfaces [8, Corollary 7.5], for hyperkähler varieties deformation equivalent to Hilbert schemes of points of $K3$ surfaces [9], for the fibre product $X_1\times_C X_2$ of two projective non-isotrivial smooth families $\pi_k:X_k\to C$ $(k=1,2)$ of regular surfaces with geometric genus 1 over a smooth projective curve C under the assumption that ranks of lattices of transcendental cycles on generic geometric fibres X_{ks} $(k=1,2)$ are different prime odd numbers [10], [11].

If S is a $K3$ or Abelian surface, H an ample linear bundle on S and X the Gieseker - Maruyama - Simpson moduli space of H-stable torsion-free sheaves of rank r on S with fixed Chern classes $\mathrm{c}_1,\mathrm{c}_2$, then the standard conjecture of Lefschetz type holds for X under the assumption that X is projective [8, Theorem 7.8, Corollary 7.9].

Besides the standard conjecture holds for the Altman - Kleiman compactification X of the relative Jacobian of a family $\mathcal{C}\to\mathbb{P}^2$ of hyperelliptic curves of genus 2 with weak degeneracies under the condition that the canonical projection $X\to\mathbb{P}^2$ is a Lagrangian fibration [12].

The conjecture $B(X)$ holds for every smooth projective compactification X of the Néron minimal model of an Abelian scheme of relative dimension 3 over an affine curve provided that the generic scheme fibre of the Abelian scheme has reductions of multiplicative type at all infinite places [12]. Besides, it holds for a 4-dimensional smooth projective complex variety, fibred over a smooth projective curve, if every degenerated fibre is a union of smooth irreducible components of multiplicity 1 with normal crossings, for generic geometric fibre $X_{\overline{\eta}}$ the standard conjecture $B(X_{\overline{\eta}})$ holds, there exists at least one degenerated fibre X_δ, for irreducible components V_i of every degenerated fibre $X_\delta = V_1 + \cdots + V_m$ the rings of rational cohomology $H^*(V_i, \mathbb{Q})$ and $H^*(V_i \cap V_j, \mathbb{Q})$ are generated by classes of algebraic cycles [13].

As it was shown by Charles [14, Theorem 1], an algebraic isomorphism

$$H^{2d-2}(X,\mathbb{Q}) \stackrel{\sim}{\to} H^2(X,\mathbb{Q})$$

(inverse to the Lefschetz isomorphism) exists iff, for some smooth quasi-projective variety S and for an appropriate algebraic cycle $Z \in \mathrm{CH}^2(X \times S)$ of codimension 2, there exists a point $s \in S$ such that the canonical map $\phi_{Z,s} : \wedge^2\Theta_{S,s} \to H^2(X, \mathcal{O}_X)$ is surjective (where $\Theta_{S,s}$ is the tangent space to the variety S at the point s).

Let $\mathcal{M} \to C$ be the Néron minimal model of the Abelian variety $\mathcal{M}_\eta$ over the field $\kappa(\eta)$ of rational functions of a smooth complex projective curve C. Suppose that at any place $s \in C$ the reduction of the Abelian variety $\mathcal{M}_\eta$ is stable in Grothendieck's sense. In this case the connected component $\mathcal{M}_s^0$ of the neutral element of the algebraic group $\mathcal{M}_s$ is an extension of an Abelian variety by a linear torus whose dimension r_s is called *the toric* (*reductive*) rank at the place s [15, Section 2.1.12].

Let R be a Dedekind domain with the fraction field K and let A_η be an Abelian variety over $\eta = \operatorname{Spec} K$ such that all reductions are stable in Grothendieck's sense. As it was shown by Künnemann [16, Section 5.8], in this case there exists a finite extension K' of the field K such that the Abelian variety $A_\eta \otimes_K K'$ has (not necessarily unique) a flat projective regular model P' over the integral closure R' of the ring R in the field K'; this model P' has strict semi-stable reductions over each localization of the ring R' (in particular, every special fibre P'_s is a union of smooth divisors of multiplicity 1 with normal crossings [17, Section 1.9]), and the scheme P' contains the Néron minimal model $\mathcal{A}'$ of the variety $A_\eta \otimes_K K'$ in the case when all residue fields of the scheme $\operatorname{Spec} R'$ are perfect [17, Section 4.4, Theorem 4.6].

Since the coordinate ring $\mathbb{C}[C']$ of any smooth affine curve C' over the field $\mathbb{C}$ is a Dedekind domain, then after the base change determined by an appropriate ramified covering $\widetilde{C} \to C$, we may assume by the results of Künnemann cited above that, for the Néron minimal model $\mathcal{M} \to C$, there exists a smooth compactification X of the variety $\mathcal{M}$ which is flat and projective over the curve C such that the following conditions hold:

(i) the model X/C has strictly semi-stable reductions (in particular, all fibres of the structure morphism $\pi : X \to C$ are unions of smooth irreducible components of multiplicity 1 with normal crossings);

(ii) the variety X contains the variety $\mathcal{M}$ as an open dense subscheme;

(iii) the restriction $\pi|_{\mathcal{M}} : \mathcal{M} \to C$ coincides with the structure morphism of the Néron model;

(iv) the connected component $\mathcal{M}_s^0$ of the neutral element of any fibre $\mathcal{M}_s$ $(s \in C)$ is an extension of an Abelian variety by a linear torus of dimension r_s;

(v) C-group law $\mathcal{M}^0 \times_C \mathcal{M}^0 \to \mathcal{M}^0$ can be expanded to a group C-action $\mathcal{M}^0 \times_C X \to X$.

We call such compactifications of the Néron model by *Künnemann compactifications.*

By definition, the Abelian variety $\mathcal{M}_\eta$ has a *trivial* trace if, for any finite ramified covering $\widetilde{C} \to C$, the group scheme $\mathcal{M} \times_C \widetilde{C} \to \widetilde{C}$ has no non-trivial *constant* Abelian subscheme.

Let $\mathbb{N}^+ = \{1, 2, 3, \dots\}$ be the set of all non-zero natural numbers. In this article we prove the following main result:

Theorem. *Let $\mathcal{M} \to C$ be the Néron minimal model of an absolutely simple $(d-1)$-dimensional principally polarized Abelian variety $\mathcal{M}_\eta$ over the field $\kappa(\eta)$ of rational functions of a smooth complex projective curve C.*

Assume that the trace of the Abelian variety $\mathcal{M}_\eta$ is trivial and the centre $Z_\mathbb{Q}$ of the division $\mathbb{Q}$-algebra $E_\mathbb{Q} \stackrel{\text{def}}{=} \operatorname{End}_{\overline{\kappa(\eta)}}(\mathcal{M}_\eta \otimes_{\kappa(\eta)} \overline{\kappa(\eta)}) \otimes_\mathbb{Z} \mathbb{Q}$ is a totally real field of degree e over $\mathbb{Q}$.

If one of the following conditions holds:

(i) $\operatorname{End}_{\overline{\kappa(\eta)}}(\mathcal{M}_\eta \otimes_{\kappa(\eta)} \overline{\kappa(\eta)}) = \mathbb{Z}$ *and, for any embedding of fields $\kappa(\eta) \hookrightarrow \mathbb{C}$, the complexification* $\operatorname{Lie} \operatorname{Hg}(\mathcal{M}_\eta \otimes_{\kappa(\eta)} \mathbb{C}) \otimes_\mathbb{Q} \mathbb{C}$ *of the Lie algebra of the Hodge group of the Abelian variety $\mathcal{M}_\eta \otimes_{\kappa(\eta)} \mathbb{C}$ is a simple Lie algebra of type C_{d-1}* (*this condition automatically holds if*

$$d - 1 \notin \operatorname{Ex}(1) \stackrel{\text{def}}{=} \left\{ 4^l, \frac{1}{2}\binom{4l+2}{2l+1}^{2m-1}, 2^{8lm+4l-4m-3}, 4^l(m+1)^{2l+1} \mid l, m \in \mathbb{N}^+ \right\}$$
$$= \{4, 10, 16, 32, 64, 108, 126, 256, 500, 512, 864, 1024, 1372, 1716, 2048, \dots\});$$

(ii) *the Abelian variety $\mathcal{M}_\eta \otimes_{\kappa(\eta)} \overline{\kappa(\eta)}$ belongs to type* I *of Albert's classification and $\frac{d-1}{e}$ is an odd integer*;

(iii) *the Abelian variety $\mathcal{M}_\eta \otimes_{\kappa(\eta)} \overline{\kappa(\eta)}$ belongs to type* II *of Albert's classification and $\frac{d-1}{e}$ is not divisible by* 4;

(iv) *the Abelian variety $\mathcal{M}_\eta \otimes_{\kappa(\eta)} \overline{\kappa(\eta)}$ belongs to type* III *of Albert's classification, $\frac{d-1}{e}$ is not divisible by* 4 *and* $(\forall r \in \mathbb{N}^+)$ $\frac{d-1}{e} \neq \binom{4r}{2r}$,

then there exists a finite ramified covering $\widetilde{C} \to C$ such that, for any Künnemann compactification $\widetilde{X}$ of the Néron minimal model of the Abelian variety $\mathcal{M}_\eta \otimes_{\kappa(\eta)} \kappa(\widetilde{\eta})$, there exist algebraic isomorphisms

$$H^{2d-2}(\widetilde{X}, \mathbb{Q}) \stackrel{\sim}{\to} H^2(\widetilde{X}, \mathbb{Q}), \qquad H^{2d-3}(\widetilde{X}, \mathbb{Q}) \stackrel{\sim}{\to} H^3(\widetilde{X}, \mathbb{Q}).$$

Remark. In the case under consideration, there exist algebraic isomorphisms

$$H^{2d-2}(\widetilde{X}, \mathbb{Q}) \stackrel{\sim}{\to} H^2(\widetilde{X}, \mathbb{Q}), \qquad H^{2d-3}(\widetilde{X}, \mathbb{Q}) \stackrel{\sim}{\to} H^3(\widetilde{X}, \mathbb{Q})$$

which are inverse to the Lefschetz isomorphisms in degrees 2 or 3 of rational cohomology of the variety $\widetilde{X}$ [14, Lemma 6].

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§ 1. Some remarks on Poincaré classes, global monodromy and cohomology of local systems

1.1. It is well known that the Hodge decomposition

$$V_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{C} = \oplus_{p+q=n} V_\mathbb{C}^{p,q}$$

of the Hodge $\mathbb{Q}$-substructure $V_{\mathbb{Q}} \hookrightarrow H^n(X,\mathbb{Q})$ yields the action $h: U^1 \to \mathrm{GL}(V_{\mathbb{R}})$ of the group $U^1 \stackrel{\mathrm{def}}{=} \{e^{i\theta} \mid \theta \in \mathbb{R}\}$ on the real space $V_{\mathbb{R}} \stackrel{\mathrm{def}}{=} V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$, such that $h(e^{i\theta})(v^{p,q}) = e^{i\theta(p-q)} \cdot v^{p,q}$ for any element $v^{p,q} \in V_{\mathbb{C}}^{p,q}$ ([18], Section 2.1.5). By definition, the Hodge group $\mathrm{Hg}(V_{\mathbb{Q}})$ is the smallest algebraic $\mathbb{Q}$-subgroup of $\mathrm{GL}(V_{\mathbb{Q}})$ whose group of $\mathbb{R}$-points contains the group $h(U^1)$ ([19], Definition B51). It is known that the group $\mathrm{Hg}(V_{\mathbb{Q}})$ is a connected reductive group, and in the case $(r-l)n = 2p$ the space of invariants $[V_{\mathbb{Q}}^{\otimes r} \otimes_{\mathbb{Q}} (V_{\mathbb{Q}}^{\vee})^{\otimes l}]^{\mathrm{Hg}(V_{\mathbb{Q}})}$ coincides with *the space of Hodge cycles* $[V_{\mathbb{Q}}^{\otimes r} \otimes_{\mathbb{Q}} (V_{\mathbb{Q}}^{\vee})^{\otimes l}] \cap [V_{\mathbb{Q}}^{\otimes r} \otimes_{\mathbb{Q}} (V_{\mathbb{Q}}^{\vee})^{\otimes l}]_{\mathbb{C}}^{p,p}$ ([19], Corollary B55).

A polarization of the Hodge $\mathbb{Q}$-substructure $V_{\mathbb{Q}} \hookrightarrow H^n(X,\mathbb{Q})$ is a morphism of rational Hodge structures $\psi_{V_{\mathbb{Q}}} : V_{\mathbb{Q}} \otimes_{\mathbb{Q}} V_{\mathbb{Q}} \to \mathbb{Q}(-n)$ such that the real bilinear form $(u,v) \mapsto (2\pi i)^n\, \psi_{V_{\mathbb{Q}}}(u, h(i)v)$ is symmetric and positive-definite on $V_{\mathbb{R}}$ ([18], Section 2.1.14).

Denote by $\psi^0_{V_{\mathbb{Q}}}$ the composite $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} V_{\mathbb{Q}} \xrightarrow{\psi_{V_{\mathbb{Q}}}} \mathbb{Q}(-n) \xrightarrow[\sim]{(2\pi i)^n} \mathbb{Q}$. Clearly, the bilinear form $\psi^0_{V_{\mathbb{Q}}}$ is non-degenerate; besides, this property holds for the restriction $\psi^0_{V_{\mathbb{Q}}}|_{W_{\mathbb{Q}}}$ of the form $\psi^0_{V_{\mathbb{Q}}}$ to any non-trivial Hodge $\mathbb{Q}$-substructure $W_{\mathbb{Q}} \subset V_{\mathbb{Q}}$, because the form $(u,v) \mapsto (2\pi i)^n\, \psi_{V_{\mathbb{Q}}}(u, h(i)v)$ is positive-definite on $V_{\mathbb{R}}$ and, consequently, it's restriction is positive-definite on $W_{\mathbb{R}}$. It is well-known that the classical rational Hodge structure $H^n(X,\mathbb{Q})$ is polarizable. Therefore every Hodge $\mathbb{Q}$-substructure $V_{\mathbb{Q}} \subset H^n(X,\mathbb{Q})$ is polarizable too.

It is known that the bilinear form $\psi_{V_{\mathbb{Q}}\,\mathbb{R}}$ is invariant with respect to the action of the group U^1 ([18], Section 2.1.16). On the other hand, the group U^1 trivially acts on the structure $\mathbb{Q}(-n)_{\mathbb{R}}$, therefore the bilinear form $\psi^0_{V_{\mathbb{Q}}\,\mathbb{R}}$ also is U^1-invariant. Thus the form $\psi^0_{V_{\mathbb{Q}}}$ is invariant with respect to the canonical action of the Hodge group $\mathrm{Hg}(V_{\mathbb{Q}})$ in the space $V_{\mathbb{Q}}$. In particular, there exists an embedding of *connected* algebraic $\mathbb{Q}$-groups $\mathrm{Hg}(V_{\mathbb{Q}}) \hookrightarrow [\mathrm{Aut}(\psi^0_{V_{\mathbb{Q}}})]^0$, where $[\mathrm{Aut}(\psi^0_{V_{\mathbb{Q}}})]^0$ is the connected component of unity of the $\mathbb{Q}$-group $\mathrm{Aut}(\psi^0_{V_{\mathbb{Q}}})$.

Consider the diagonal action of the group $[\mathrm{Aut}(\psi^0_{V_{\mathbb{Q}}})]^0$ in the space $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} V_{\mathbb{Q}}$, determined by the formula $\mu(x \otimes y) = \mu(x) \otimes \mu(y)$. It is clear that elements of the subspace $[V_{\mathbb{Q}} \otimes_{\mathbb{Q}} V_{\mathbb{Q}}]^{[\mathrm{Aut}(\psi^0_{V_{\mathbb{Q}}})]^0} \subset [V_{\mathbb{Q}} \otimes_{\mathbb{Q}} V_{\mathbb{Q}}]^{\mathrm{Hg}(V_{\mathbb{Q}})}$ are Hodge cycles. Assume that the number n is odd or n is even and $\dim_{\mathbb{Q}} V_{\mathbb{Q}} \neq 2$. Then the standard representation of the group $[\mathrm{Aut}(\psi^0_{V_{\mathbb{Q}}})]^0$ in the space $V_{\mathbb{Q}}$ is an absolutely irreducible *orthogonal* or *symplectic* representation ([18], Section 2.1.16) *of a semi-simple* $\mathbb{Q}$-group ([20], Ch. I, § 6, n^0 7, Proposition 9) and, according to the Schur lemma, the 1-dimensional $\mathbb{Q}$-space $[V_{\mathbb{Q}} \otimes_{\mathbb{Q}} V_{\mathbb{Q}}]^{[\mathrm{Aut}(\psi^0_{V_{\mathbb{Q}}})]^0}$ is generated by some element $\wp_0(V_{\mathbb{Q}})$ (which is determined uniquely up to a non-zero scalar multiple). We call the element $\wp_0(V_{\mathbb{Q}})$ *the Poincaré class* of the polarizable rational Hodge structure $(V_{\mathbb{Q}}, \psi_{V_{\mathbb{Q}}})$.

In particular, if the number n is odd or n is even and $\dim_{\mathbb{Q}} H^n(X,\mathbb{Q}) \neq 2$, then there exists the Poincaré class $\wp_0(H^n(X,\mathbb{Q}))$ associated with some polarization form $\psi_{H^n(X,\mathbb{Q})}$ on the classical rational Hodge structure $H^n(X,\mathbb{Q})$, so that the Hodge cycle $\wp_0(H^n(X,\mathbb{Q}))$, considered as (non necessarily algebraic) correspondence, yields an isomorphism of $[\mathrm{Aut}(\psi^0_{H^n(X,\mathbb{Q})})]^0$-modules $H^n(X,\mathbb{Q})^{\vee} \xrightarrow{\sim} H^n(X,\mathbb{Q})$.

1.2. By the strong Lefschetz theorem and by the Poincaré duality the bilinear form

$$\Phi : H^n(X,\mathbb{Q}) \times H^n(X,\mathbb{Q}) \xrightarrow{x \times y \mapsto x \smile y \smile \mathrm{cl}_X(H)^{\smile\, d-n}} H^{2d}(X,\mathbb{Q}) = \mathbb{Q}(-d) \xrightarrow[\sim]{(2\pi i)^d} \mathbb{Q}$$

is non-degenerate [2, Section 1.2A]. Let $<>: H^{2d}(X,\mathbb{Q}) \tilde{\to} \mathbb{Q}$ be the orientation isomorphism. Since $\mathrm{cl}_X(H) \in H^2(X,\mathbb{Q})^{\mathrm{Hg}(H^2(X,\mathbb{Q}))} = \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \overset{\mathrm{def}}{=} \mathrm{NS}_{\mathbb{Q}}(X)$ by the Lefschetz theorem on divisors, the group U^1 acts trivially on the subset $\mathrm{NS}_{\mathbb{Q}}(X) \hookrightarrow H^{1,1}(X,\mathbb{C})$, so that we have (with a trivial action of U^1 on $\mathbb{R}$):

$$\forall \sigma \in U^1 \qquad \Phi_{\mathbb{R}}(x,y) = [\Phi_{\mathbb{R}}(x,y)]^\sigma = < x \smile \mathrm{cl}_X(H)^{d-n} y >^\sigma =$$
$$< x^\sigma \smile \mathrm{cl}_X(H)^{d-n} \smile y^\sigma > = \Phi_{\mathbb{R}}(x^\sigma, y^\sigma).$$

Therefore the form $\Phi_{\mathbb{R}}$ is U^1-invariant, so there exists a canonical embedding

$$\mathrm{Hg}(H^n(X,\mathbb{Q})) \hookrightarrow \mathrm{Aut}(\Phi)^0 = \begin{cases} \mathrm{Sp}(\Phi), & \text{for odd} \quad n, \\ \mathrm{SO}(\Phi), & \text{for even} \quad n, \end{cases}$$

and Φ is a $\mathrm{Hg}(H^n(X,\mathbb{Q}))$-invariant form.

If the number n is odd or n is even and $\dim_{\mathbb{Q}} H^n(X,\mathbb{Q}) \neq 2$, then the 1-dimensional $\mathbb{Q}$-space $[H^n(X,\mathbb{Q}) \otimes H^n(X,\mathbb{Q})]^{\mathrm{Aut}(\Phi)^0}$ of invariants of the diagonal action of the group $\mathrm{Aut}(\Phi)^0$ is generated by a Hodge cycle $\wp(H^n(X,\mathbb{Q}))$, which is again called *the Poincaré class.* It determines an isomorphism of $[\mathrm{Aut}(\Phi)]^0$-modules $H^n(X,\mathbb{Q})^\vee \tilde{\to} H^n(X,\mathbb{Q})$. The Poincaré duality theorem yields an identification of the Weil cohomology $H^n(X,\mathbb{Q})^\vee = H^{2d-n}(X,\mathbb{Q})$ ([2], Section 1.2A), therefore the isomorphism under consideration is the composite ([2], Section 1.3)

$$H^{2d-n}(X,\mathbb{Q}) \xrightarrow{\mathrm{pr}_1^*} H^{2d-n}(X,\mathbb{Q}) \otimes_{\mathbb{Q}} H^0(X,\mathbb{Q})$$
$$\xrightarrow{\smile \wp(H^n(X,\mathbb{Q}))} H^{2d}(X,\mathbb{Q}) \otimes_{\mathbb{Q}} H^n(X,\mathbb{Q}) \xrightarrow{\mathrm{pr}_{2*}} H^n(X,\mathbb{Q}).$$

In contrast to the situation of Section 1.1, a restriction of the form Φ to a non-trivial rational Hodge substructure $V_{\mathbb{Q}} \subset H^n(X,\mathbb{Q})$ may be degenerated. A simple example is suggested by a referee: if X_s is a smooth fibre of a morphism $\pi : X \to C$ of the variety X onto a smooth projective curve C, then in virtue of the equality $\mathrm{cl}_X(X_s) \smile \mathrm{cl}_X(X_s) = 0$ the restriction of the form $\Phi : H^2(X,\mathbb{Q}) \times H^2(X,\mathbb{Q}) \to \mathbb{Q}$ to the non-trivial rational Hodge substructure $\mathbb{Q} \cdot \mathrm{cl}_X(X_s) \subset H^2(X,\mathbb{Q})$ is trivial.

Nevertheless, if a restriction of the form Φ to a non-trivial rational Hodge substructure $V_{\mathbb{Q}} \subset H^n(X,\mathbb{Q})$ is non-degenerate, then there is a decomposition of Hodge $\mathbb{Q}$-structures [21, Ch. IX, § 4, n^0 1, Corollary of Propositon 1]

$$H^n(X,\mathbb{Q}) = V_{\mathbb{Q}} \oplus V_{\mathbb{Q}}^\perp,$$

where $V_{\mathbb{Q}}^\perp$ is an orthogonal complement of the $\mathbb{Q}$-space $V_{\mathbb{Q}}$ with respect to the form Φ and the Poincaré classes

$$\wp(V_{\mathbb{Q}}) \in [V_{\mathbb{Q}} \otimes_{\mathbb{Q}} V_{\mathbb{Q}}]^{[\mathrm{Aut}(\Phi|_{V_{\mathbb{Q}}})]^0}, \qquad \wp(V_{\mathbb{Q}}^\perp) \in [V_{\mathbb{Q}}^\perp \otimes_{\mathbb{Q}} V_{\mathbb{Q}}^\perp]^{[\mathrm{Aut}(\Phi|_{V_{\mathbb{Q}}^\perp})]^0}$$

are well defined under the assumption that the number n is odd or n is even and $\dim_{\mathbb{Q}} V_{\mathbb{Q}} \neq 2$, $\dim_{\mathbb{Q}} V_{\mathbb{Q}}^\perp \neq 2$. Moreover, in this situation the Poincaré classes $\wp(V_{\mathbb{Q}})$, $\wp(V_{\mathbb{Q}}^\perp)$ are *Hodge cycles.*

1.3. We may assume that

$$\{s \in C \mid \text{ the fibre } \mathcal{M}_s \text{ is non-compact}\} = \Delta \overset{\mathrm{def}}{=} \{\delta \in C \mid \mathrm{Sing}(X_\delta) \neq \varnothing\}.$$

Set $C' = C \setminus \Delta$. It is evident that the structure morphism $\pi : X \to C$ is smooth over C'. Let $C' \overset{j}{\hookrightarrow} C$ be the canonical embedding, $X' = X \setminus \pi^{-1}(\Delta)$, $\pi' = \pi|_{X'} : X' \to C'$.

Considering, if necessary, a ramified covering $\widetilde{C} \to C$ and a projective Künnemann model $\widetilde{X} \to \widetilde{C}$ of the corresponding Néron model $\widetilde{\mathcal{M}} \to \widetilde{C}$ of the generic scheme fibre of the canonical projection $X \times_C \widetilde{C} \to \widetilde{C}$, we may assume in virtue of [16, Section 5.8]; [17, Section 4.4, Theorem 4.6] that any singular fibre X_δ is a union of smooth irreducible components of multiplicity 1 with normal crossings. One may also assume that there is a *countable* subset $\Delta_{\text{countable}} \subset C'$ such that, for any point $s \in C' \setminus \Delta_{\text{countable}}$, the closure G of the image of the monodromy representation $\pi_1(C', s) \to \mathrm{GL}(H^1(X_s, \mathbb{Q}))$ in the Zariski topology of the algebraic group $\mathrm{GL}(H^1(X_s, \mathbb{Q}))$ is a connected semi-simple [18, Corollary 4.2.9] *normal* [22, Theorem 7.3] subgroup of the Hodge group

$$\mathrm{Hg}(X_s) \stackrel{\text{def}}{=} \mathrm{Hg}(H^1(X_s, \mathbb{Q}))$$

of the Abelian variety X_s. In accordance with Mumford's result, the reductive Hodge group $\mathrm{Hg}(X_s)$ is commutative (and, consequently, it is a linear $\mathbb{Q}$-torus) if and only if X_s is an Abelian variety of CM-type [23]. Therefore it follows from the triviality of the trace of the Abelian variety X_η that the generic scheme fibre X_η can not be an Abelian variety of CM-type (because the group G is a *non-trivial* [18, Sections (4.1.3.3), 4.4.3] connected *semi-simple* group). By the same reasons the variety X_s can not be an Abelian variety of CM-type for any point $s \in C' \setminus \Delta_{\text{countable}}$. We may also assume that local monodromies (Picard - Lefschetz transformations) *are unipotent* and $\mathrm{End}_{\kappa(\eta)}(X_\eta) = \mathrm{End}_{\overline{\kappa(\eta)}}(X_\eta \otimes_{\kappa(\eta)} \overline{\kappa(\eta)})$.

Consider the normalization $f : Z \to \pi^{-1}(\Delta)$ of the scheme $\pi^{-1}(\Delta)$. Then Z is a disjoint union of smooth irreducible components of the divisor $\pi^{-1}(\Delta)$. Since $f : Z \to \pi^{-1}(\Delta)$ is a resolution of singularities of the subscheme $i_\Delta : \pi^{-1}(\Delta) \hookrightarrow X$, there is a canonical exact sequence of mixed Hodge $\mathbb{Q}$-structures [24, Corollary (8.2.8)]:

$$H^{n-2}(Z, \mathbb{Q}) \xrightarrow{(i_\Delta f)_*} H^n(X, \mathbb{Q}) \xrightarrow{\varphi_n} H^n(X', \mathbb{Q}),$$

where $(i_\Delta f)_*$ is a morphism of bidegree $(1,1)$ of pure Hodge structures and φ_n is the restriction map. In particular,

$$(i_\Delta f)_* H^{n-2}(Z, \mathbb{Q}) = \mathrm{Ker}[H^n(X, \mathbb{Q}) \xrightarrow{\varphi_n} H^n(X', \mathbb{Q})]. \tag{1.1}$$

By the conditions (i) - (iv) of the theorem, for any embedding of fields $\kappa(\eta) \hookrightarrow \mathbb{C}$, the Hodge group $\mathrm{Hg}(X_\eta \otimes_{\kappa(\eta)} \mathbb{C})$ is a $\mathbb{Q}$-simple algebraic group [25, § 4, Deduction of Theorem 4.1 from Lemmas 1 - 3]. Therefore

$$G = \mathrm{Hg}(X_\eta \otimes_{\kappa(\eta)} \mathbb{C}) = \mathrm{Hg}(X_s) \quad \forall\, s \in C' \setminus \Delta_{\text{countable}} \tag{1.2}$$

by Proposition 4.1 in [25]. It is well known that the canonical representation of the Lie algebra $\mathrm{Lie}\,\mathrm{Hg}(X_s) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ in the space $H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is determined by *minuscule weights* [26], [22, Theorem 0.5.1] in Bourbaki's sense [20].

1.4. Note that if $\mathrm{End}_{\overline{\kappa(\eta)}}(\mathcal{M}_\eta \otimes_{\kappa(\eta)} \overline{\kappa(\eta)}) = \mathbb{Z}$ and $d - 1 \notin \mathrm{Ex}(1)$ then, for any embedding of fields $\kappa(\eta) \hookrightarrow \mathbb{C}$, the complexification $\mathrm{Lie}\,\mathrm{Hg}(\mathcal{M}_\eta \otimes_{\kappa(\eta)} \mathbb{C}) \otimes_{\mathbb{Q}} \mathbb{C}$ of the Lie algebra of the Hodge group of the Abelian variety $\mathcal{M}_\eta \otimes_{\kappa(\eta)} \mathbb{C}$ is a simple Lie algebra of type C_{d-1} [27, Theorem 1.1].

1.5. It follows from the conditions (i) - (iv) of the theorem and from the triviality of the trace that there is a canonical isomorphism [25, § 4, Corollary of Theorem (Hom – End), Theorem 4.1]

$$\mathrm{End}_{C'}(X') \xrightarrow{\sim} \mathrm{End}_{C'}(R^1\pi'_*\mathbb{Z}). \tag{1.3}$$

A choice of a point $s \in C' \setminus \Delta_{\text{countable}}$ determines the canonical embeddings

$$\mathrm{Im}[\pi_1(C', s) \to \mathrm{GL}(H^1(X_s, \mathbb{Q}))] \subset G \subset \mathrm{Hg}(X_s).$$

Therefore it follows from (1.3) and from the well known equality [19, Lemma B.60]

$$\mathrm{End}_{\mathrm{Hg}(X_s)} H^1(X_s, \mathbb{Q}) = \mathrm{End}_{\mathbb{C}}(X_s) \otimes_{\mathbb{Z}} \mathbb{Q}$$

that there are canonical maps

$$\begin{aligned} &\mathrm{End}_{\kappa(\eta)}(X_\eta) \otimes_{\mathbb{Z}} \mathbb{Q} \tilde{\to} \mathrm{End}_{\pi_1(C',s)} H^1(X_s, \mathbb{Q}) \\ &\hookleftarrow \mathrm{End}_{\mathrm{Hg}(X_s)} H^1(X_s, \mathbb{Q}) = \mathrm{End}_{\mathbb{C}}(X_s) \otimes_{\mathbb{Z}} \mathbb{Q}. \end{aligned} \tag{1.4}$$

The restriction map $\mathrm{End}_{\kappa(\eta)}(X_\eta) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{End}_{\mathbb{C}}(X_s) \otimes_{\mathbb{Z}} \mathbb{Q}$ is injective, hence it follows from (1.4) that there exists a canonical isomorphism

$$\mathrm{End}_{\kappa(\eta)}(X_\eta) \otimes_{\mathbb{Z}} \mathbb{Q} \tilde{\to} \mathrm{End}_{\mathbb{C}}(X_s) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{1.5}$$

In particular, the Abelian variety X_s is simple.

§ 2. A construction of an algebraic isomorphism of rational cohomology of degrees $d-2$ and 2

2.1. From now on we denote by

$$K_{nX} \stackrel{\text{def}}{=} \mathrm{Ker}[H^n(X, \mathbb{Q}) \to H^0(C, R^n\pi_*\mathbb{Q})]$$

the kernel of the edge map of the Leray spectral sequence $E_2^{p,q}(\pi)$ of the structure morphism $\pi : X \to C$. Besides, for any irreducible smooth projective variety W, we denote by $<>: H^{2\dim_{\mathbb{C}} W}(W, \mathbb{Q}) \tilde{\to} \mathbb{Q}$ the orientation isomorphism of Weil cohomology [2, Section 1.2.A] determined by a choice of an element $\sqrt{-1} \in \mathbb{C}$.

By the assumption of the theorem, the generic scheme fibre $\mathcal{M}_\eta$ is a principally polarized Abelian variety; hence, for every point $s \in C'$, the Abelian variety X_s has a principal polarization determined by some ample divisor H_s on the variety X_s. Taking into account arguments of [3, § 4], we may assume that there exists a rigid Poincaré bundle $\mathcal{P}'$ on the Abelian scheme $X' \times_{C'} X'$, inducing the Poincaré bundle $\mathcal{P}'_s$ on the Cartesian product $X_s \times X_s$ for every point $s \in C'$.

One may assume that

$$\mathrm{c}_1(\mathcal{P}'_s) = \wp(H^1(X_s, \mathbb{Q})) \in H^2(X_s \times X_s, \mathbb{Q}) \cap H^{1,1}(X_s \times X_s, \mathbb{C}),$$

where $\wp(H^1(X_s, \mathbb{Q}))$ is the Poincaré class in the sense of Section 1.2; it is *algebraic* by the Lefschetz theorem on divisors and it is determined uniquely (up to a non-zero scalar multiple) by the corresponding bilinear form

$$\Phi_s : H^1(X_s, \mathbb{Q}) \times H^1(X_s, \mathbb{Q}) \xrightarrow{x \times y \mapsto <x \smile \mathrm{cl}_{X_s}(H_s)^{\smile d-2} \smile y>} \mathbb{Q}.$$

Besides, for the group G (defined in Section 1.3) and for arbitrary point $s \in C' \setminus \Delta_{\text{countable}}$, there are embeddings

$$G \hookrightarrow \mathrm{Hg}(H^1(X_s, \mathbb{Q})) \hookrightarrow \mathrm{Sp}(H^1(X_s, \mathbb{Q}), \Phi_s).$$

Since the Poincaré class $\wp(H^1(X_s, \mathbb{Q}))$ is a generator of the 1-dimensional subspace

$$\begin{aligned} [H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(X_s, \mathbb{Q})]^{\mathrm{Sp}(H^1(X_s,\mathbb{Q}),\Phi_s)} &\hookrightarrow [H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(X_s, \mathbb{Q})]^G \\ &= [H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(X_s, \mathbb{Q})]^{\pi_1(C',s)} \end{aligned}$$

under the diagonal action $\pi_1(C', s)$ on $H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(X_s, \mathbb{Q})$, then the Poincaré class $\wp(H^1(X_s, \mathbb{Q}))$ determines the section

$$\Lambda'_{1,1} \in H^0(C', R^1\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q}) \xrightarrow{\sim} [H^1(X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(X_s, \mathbb{Q})]^{\pi_1(C',s)}$$

of type $(1,1)$ of the local system of Hodge structures $R^1\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q}$ inducing the Poincaré class $\wp(H^1(X_s, \mathbb{Q}))$ for every point $s \in C'$.

Consider the canonical diagram of the fibre product

$$\begin{array}{ccc} X' \times_{C'} X' & \xrightarrow{p'_1} & X' \\ \downarrow^{p'_2} & \searrow^{\tau'} & \downarrow^{\pi'} \\ X' & \xrightarrow{\pi'} & C'. \end{array}$$

As it was explained in Section 1.2, the *algebraic* Poincaré class $\wp(H^1(X_s, \mathbb{Q}))$ determines an algebraic isomorphism $H^{2d-3}(X_s, \mathbb{Q}) \xrightarrow{\sim} H^1(X_s, \mathbb{Q})$. Therefore the section $\Lambda'_{1,1}$ yields the isomorphism of local systems $R^{2d-3}\pi'_*\mathbb{Q} \xrightarrow{\sim} R^1\pi'_*\mathbb{Q}$ determined by the composite of morphisms of sheaves

$$\begin{aligned} R^{2d-3}\pi'_*\mathbb{Q} \xrightarrow{(p'_1)^*} R^{2d-3}\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} \pi'_*\mathbb{Q} &\xrightarrow{\smile \Lambda'_{1,1}} R^{2d-2}\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q} \\ &\xrightarrow{(p'_2)_*} R^1\pi'_*\mathbb{Q}. \end{aligned} \tag{2.1}$$

Let

$$\begin{array}{ccc} X \times_C X & \xrightarrow{p_1} & X \\ \downarrow^{p_2} & \searrow^{\tau} & \downarrow^{\pi} \\ X & \xrightarrow{\pi} & C \end{array}$$

be the canonical diagram of the fibre product, $\iota : X \times_C X \hookrightarrow X \times X$ the canonical embedding, $\sigma : Y \to X \times_C X$ a resolution of singularities of the variety $X \times_C X$. One may assume that σ induces an isomorphism over C'. In particular, Y can be considered as a smooth projective compactification of the fibre product $X' \times_{C'} X'$. Besides, we may assume in virtue of Hironaka's results that $(\tau\sigma)^{-1}(\Delta)$ is a union of smooth divisors (of some positive multiplicities) with normal crossings.

By Deligne's theorem, the canonical morphism $H^2(Y, \mathbb{Q}) \to H^0(C', R^2\tau'_*\mathbb{Q})$ is a surjective morphism of Hodge $\mathbb{Q}$-structures [18, Theorem 4.1.1, Proof of Corollary 4.1.2]. Since $\Lambda'_{1,1} \in H^0(C', R^1\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q}) \subset H^0(C', R^2\tau'_*\mathbb{Q})$ is an element of Hodge type $(1,1)$, then from Lefschetz' theorem on divisors it follows that there exists an algebraic $\mathbb{Q}$-cycle $D^{(1)}$ on the variety Y such that the image of the class $\mathrm{cl}_Y(D^{(1)}) \in H^2(Y, \mathbb{Q}) \cap H^{1,1}(Y, \mathbb{C})$ with respect to the canonical surjective morphism $H^2(Y, \mathbb{Q}) \to H^0(C', R^2\tau'_*\mathbb{Q})$ coincides with the section $\Lambda'_{1,1}$.

For any open subset $U \hookrightarrow C^{\mathrm{an}}$, there is a canonical proper morphism $p_k\sigma : (\tau\sigma)^{-1}(U) \to \pi^{-1}(U)$ inducing canonical maps [28, Ch. II, Section 4.16]

$$\begin{aligned} (p_k\sigma)^* &: H^q(\pi^{-1}(U), \mathbb{Q}) \to H^q((\tau\sigma)^{-1}(U), \mathbb{Q}), \\ (p_k\sigma)^*_{\mathrm{c}} &: H^q_{\mathrm{c}}(\pi^{-1}(U), \mathbb{Q}) \to H^q_{\mathrm{c}}((\tau\sigma)^{-1}(U), \mathbb{Q}), \end{aligned}$$

and, by the Poincaré duality theorem [29, Ch. VI, § 11, Corollary 11.2], there is a canonical map

$$(p_k\sigma)_* = [(p_k\sigma)^*_{\mathrm{c}}]^\vee : H^{2d-2+i}((\tau\sigma)^{-1}(U), \mathbb{Q}) \to H^i(\pi^{-1}(U), \mathbb{Q}).$$

Consequently, there are canonical morphisms of sheaves

$$(p_k\sigma)^* : R^q\pi_*\mathbb{Q} \to R^q(\tau\sigma)_*\mathbb{Q},$$
$$(p_k\sigma)_* : R^{2d-2+i}(\tau\sigma)_*\mathbb{Q} \to R^i\pi_*\mathbb{Q},$$

and

$$(p_k\sigma)^*|_{C'} = (p'_k)^*, \qquad (p_k\sigma)_*|_{C'} = (p'_k)_*.$$

Therefore the compatibility of the $\smile$-products with the Leray spectral sequence

$$E_2^{p,q}(\tau\sigma) = H^p(C, R^q(\tau\sigma)_*\mathbb{Q})$$

([30], Vol. II, Ch. 4, Lemma 4.13) allows us to expand (2.1) to a sequence of morphisms of sheaves

$$\text{(2.2)} \quad R^{2d-3}\pi_*\mathbb{Q} \xrightarrow{(p_1\sigma)^*} R^{2d-3}(\tau\sigma)_*\mathbb{Q} \xrightarrow{\smile \mathrm{cl}_Y(D^{(1)})} R^{2d-1}(\tau\sigma)_*\mathbb{Q} \xrightarrow{(p_2\sigma)_*} R^1\pi_*\mathbb{Q},$$

whose composite is an isomorphism outside the finite set Δ.

2.2. It follows from the existence of the natural embedding

$$R^2\pi'_*\mathbb{Q} = \wedge^2 R^1\pi'_*\mathbb{Q} \hookrightarrow R^1\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q}$$

and from (1.3) that $H^0(C', R^2\pi'_*\mathbb{Q})$ is a rational Hodge structure of type $(1,1)$ because a polarization on X determines an isomorphism of families of Hodge structures [18, Section 4.2.3]

$$[R^1\pi'_*\mathbb{Q}]^\vee \xrightarrow{\sim} R^1\pi'_*\mathbb{Q}(1),$$

the ring $\mathrm{End}_{C'}(X') \otimes_{\mathbb{Z}} \mathbb{Q}$ coincides with the component of type $(0,0)$ of the Hodge $\mathbb{Q}$-structure $\mathrm{End}_{C'}(R^1\pi'_*\mathbb{Q})$ [18, Section 4.4.6] and there are morphisms of rational Hodge structures

$$H^0(C', R^2\pi'_*\mathbb{Q}) \hookrightarrow H^0(C', R^1\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q})$$
$$\xrightarrow{\sim} H^0(C', [R^1\pi'_*\mathbb{Q}]^\vee \otimes_{\mathbb{Q}} R^1\pi'_*\mathbb{Q})(-1) = \mathrm{End}_{C'}(R^1\pi'_*\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}(-1)$$
$$\xrightarrow{\sim} \mathrm{End}_{C'}(X') \otimes_{\mathbb{Z}} \mathbb{Q}(-1).$$

Consequently, if $\Delta = \varnothing$, then the $\mathbb{Q}$-space

$$[H^0(C, R^2\pi_*\mathbb{Q}) \otimes_{\mathbb{Q}} H^0(C, R^2\pi_*\mathbb{Q})] \cap H^{2,2}(X \times X, \mathbb{C})$$

is generated by classes of algebraic cycles, therefore by [3, Theorem 10.1] there exists an *algebraic* isomorphism $H^{2d-2}(X,\mathbb{Q}) \xrightarrow{\sim} H^2(X,\mathbb{Q})$. On the other hand, $H^0(C, R^3\pi_*\mathbb{Q}) = 0$ in virtue of Lemma 3.2 below, therefore by [3, Theorem 10.1] there exists an *algebraic* isomorphism $H^{2d-3}(X,\mathbb{Q}) \xrightarrow{\sim} H^3(X,\mathbb{Q})$.

2.3. From now on we assume that $\Delta \neq \varnothing$. Then there exists a *singular* fibre, whose components are smooth varieties of multiplicity 1 with normal crossings, therefore $\mathrm{rank}\,\mathrm{NS}(X) \geq 3$ [13, Formula (2.24)].

The sequence (2.2) yields a sequence of canonical maps of cohomology

$$H^1(C, R^{2d-3}\pi_*\mathbb{Q}) \xrightarrow{[(p_1\sigma)^*]_1} H^1(C, R^{2d-3}(\tau\sigma)_*\mathbb{Q})$$
$$\text{(2.3)} \qquad \xrightarrow{[\smile \mathrm{cl}_Y(D^{(1)})]_1} H^1(C, R^{2d-1}(\tau\sigma)_*\mathbb{Q}) \xrightarrow{[(p_2\sigma)_*]_1} H^1(C, R^1\pi_*\mathbb{Q}).$$

By the functoriality properties of cohomology [31, Ch. II, Theorem 3.11] the composite of these maps coincides with the canonical map

$$\text{(2.4)} \qquad H^1(C, R^{2d-3}\pi_*\mathbb{Q}) \xrightarrow{[x \mapsto (p_2\sigma)_*((p_1\sigma)^*x \smile \mathrm{cl}_Y(D^{(1)}))]_1} H^1(C, R^1\pi_*\mathbb{Q}),$$

corresponding to the morphism of sheaves

$$R^{2d-3}\pi_*\mathbb{Q} \xrightarrow{x\mapsto (p_2\sigma)_*((p_1\sigma)^*x\smile \mathrm{cl}_Y(D^{(1)}))} R^1\pi_*\mathbb{Q}. \tag{2.5}$$

Since the kernel and the cokernel of the map (2.5) are concentrated on Δ, their higher cohomology vanish, therefore the map (2.4) is surjective.

The composite of morphisms of sheaves (2.1) yields an isomorphism of local systems $R^{2d-3}\pi'_*\mathbb{Q} \widetilde{\to} R^1\pi'_*\mathbb{Q}$, therefore there is an isomorphism of sheaves

$$j_*R^{2d-3}\pi'_*\mathbb{Q} \widetilde{\to} j_*R^1\pi'_*\mathbb{Q}, \tag{2.6}$$

where $j : C' \hookrightarrow C$ is the canonical embedding. Finitely, by the theorem on local invariant cycles ([32, Section (3.7)]; [33, Proposition (15.12)]) the canonical map $R^p\pi_*\mathbb{Q} \to j_*R^p\pi'_*\mathbb{Q}$ is surjective with the kernel concentrated on a finite set Δ. Consequently, there is a canonical isomorphism

$$H^1(C, R^p\pi_*\mathbb{Q}) \widetilde{\to} H^1(C, j_*R^p\pi'_*\mathbb{Q}).$$

Therefore by the surjectivity of the map (2.4) it follows from (2.6) that there exists an isomorphism of bidegree $(2-d, 2-d)$ of mixed Hodge structures

$$H^1(C, R^{2d-3}\pi_*\mathbb{Q}) \xrightarrow[\sim]{[x\mapsto (p_2\sigma)_*((p_1\sigma)^*x\smile \mathrm{cl}_Y(D^{(1)}))]_1} H^1(C, R^1\pi_*\mathbb{Q}). \tag{2.7}$$

2.4. The Leray spectral sequences $E_2^{p,q}(\pi) = H^p(C, R^q\pi_*\mathbb{Q})$ and $E_2^{p,q}(\tau\sigma) = H^p(C, R^q(\tau\sigma)_*\mathbb{Q})$ are degenerated: $E_2^{p,q} = E_\infty^{p,q}$ [33, Corollary (15.15)]. Therefore in notations of Section 2.1, for any natural number n, there are exact sequences of Hodge $\mathbb{Q}$-structures ([13], Formula (2.4))

$$0 \to H^2(C, R^{n-2}\pi_*\mathbb{Q}) \to K_{nX} \xrightarrow{\alpha_{nX}} H^1(C, R^{n-1}\pi_*\mathbb{Q}) \to 0, \tag{2.8}$$

$$0 \to H^2(C, R^{n-2}(\tau\sigma)_*\mathbb{Q}) \to K_{nY} \xrightarrow{\alpha_{nY}} H^1(C, R^{n-1}(\tau\sigma)_*\mathbb{Q}) \to 0. \tag{2.9}$$

Let $\mathrm{NS}_\mathbb{Q}(X) = \mathrm{NS}(X) \otimes_\mathbb{Z} \mathbb{Q}$ and let $T^2_\mathbb{Q}(X)$ be the sum of all irreducible Hodge $\mathbb{Q}$-substructures of dimension greater than 1 in $H^2(X, \mathbb{Q})$. By the Lefschetz theorem on divisors and by the strong Lefschetz theorem we have *canonical* decompositions

$$H^2(X, \mathbb{Q}) = T^2_\mathbb{Q}(X) \oplus \mathrm{NS}_\mathbb{Q}(X);$$

$$H^{2d-2}(X, \mathbb{Q}) = \left[T^2_\mathbb{Q}(X) \smile \mathrm{cl}_X(H)^{\smile d-2}\right] \oplus \left[\mathrm{NS}_\mathbb{Q}(X) \smile \mathrm{cl}_X(H)^{\smile d-2}\right], \tag{2.10}$$

where $\mathrm{NS}_\mathbb{Q}(X) = H^2(X, \mathbb{Q}) \cap H^{1,1}(X, \mathbb{C})$ and $\mathrm{NS}_\mathbb{Q}(X) \smile \mathrm{cl}_X(H)^{\smile d-2}$ are identified with spaces of algebraic cohomology classes, $T^2_\mathbb{Q}(X)$ and $T^2_\mathbb{Q}(X) \smile \mathrm{cl}_X(H)^{\smile d-2}$ are identified with spaces of transcendental cohomology classes.

2.5. Lemma. *Rational Hodge structure $H^0(C, R^2\pi_*\mathbb{Q})$ has type* $(1, 1)$.

Proof. One may assume that $\Delta \neq \varnothing$ and, consequently, $H^2(C', \mathbb{Q}) = 0$, because the affine curve C' has cohomological dimension 1 ([29], Ch. VI, § 7, Theorem 7.2). Let $D^*(\delta)$ be a small open punctured disc on the curve C with a centre at the point $\delta \in \Delta$. The Leray spectral sequence for the embedding $j : C' \subset C$ yields an exact sequence of mixed Hodge structures ([33], P. 457, Corollary (13.10), Remark (14.5))

$$0 \to H^1(C, j_*R^1\pi'_*\mathbb{Q}) \to H^1(C', R^1\pi'_*\mathbb{Q}) \to H^0(C, R^1j_*R^1\pi'_*\mathbb{Q})$$
$$\to H^2(C, j_*R^1\pi'_*\mathbb{Q}),$$

where

$$H^0(C, R^1 j_* R^1 \pi'_* \mathbb{Q})$$
$$= \oplus_{\delta\in\Delta} H^1(D^*(\delta), R^1\pi'_*\mathbb{Q}) \widetilde{\to} \oplus_{\delta\in\Delta} H^1(X_s,\mathbb{Q})/N_\delta H^1(X_s,\mathbb{Q}),$$

the space $H^1(X_s, \mathbb{Q})$ $(s \in C')$ has the limit mixed Hodge structure associated with the local monodromy γ around the point $\delta \in C$ (the Picard - Lefschetz transformation) and $N_\delta = \log\gamma$. By the theorem on local invariant cycles this sequence takes the form

$$0 \to H^1(C, R^1\pi_*\mathbb{Q}) \to H^1(C', R^1\pi'_*\mathbb{Q}) \to \oplus_{\delta\in\Delta} H^1(X_s,\mathbb{Q})/N_\delta H^1(X_s,\mathbb{Q}).$$

On the other hand, for the case $n = 2$, the exact sequence (2.8) yields canonical isomorphisms of rational Hodge structures

$$H^1(C, R^1\pi_*\mathbb{Q}) \widetilde{\to} \operatorname{Ker}[H^2(X,\mathbb{Q}) \to H^0(C, R^2\pi_*\mathbb{Q})]/H^2(C,\mathbb{Q})$$
$$\widetilde{\to} \operatorname{Ker}[H^2(X,\mathbb{Q})/H^2(C,\mathbb{Q}) \to H^0(C, R^2\pi_*\mathbb{Q})],$$

therefore there is a canonical exact sequence of Hodge $\mathbb{Q}$-structures

$$0 \to H^1(C, R^1\pi_*\mathbb{Q}) \to H^2(X,\mathbb{Q})/H^2(C,\mathbb{Q}) \to H^0(C, R^2\pi_*\mathbb{Q}) \to 0.$$

By similar reasons, the degenerated Leray spectral sequence

$$E_2^{p,q}(\pi') = H^p(C', R^q\pi'_*\mathbb{Q})$$

yields the canonical exact sequence of mixed Hodge $\mathbb{Q}$-structures

$$0 \to H^1(C', R^1\pi'_*\mathbb{Q}) \to H^2(X',\mathbb{Q}) \to H^0(C', R^2\pi'_*\mathbb{Q}) \to 0,$$

such that the canonical morphism of rational Hodge structures

$$H^2(X,\mathbb{Q}) \to H^0(C', R^2\pi'_*\mathbb{Q})$$

is surjective ([18], Theorem 4.1.1). As a result, denoting by

$$\overline{\varphi_2} : H^2(X,\mathbb{Q})/H^2(C,\mathbb{Q}) \to H^2(X',\mathbb{Q})$$

the canonical map induced by the restriction map $\varphi_2 : H^2(X,\mathbb{Q}) \to H^2(X',\mathbb{Q})$, and taking into account the evident equality $\operatorname{Im}(\overline{\varphi_2}) = \operatorname{Im}(\varphi_2)$ and the commutativity of the diagram of morphisms

$$\begin{array}{ccc} X' & \subset & X \\ \downarrow^{\pi'} & & \downarrow^{\pi} \\ C' & \subset & C, \end{array}$$

we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & H^1(C, R^1\pi_*\mathbb{Q}) & \to & H^2(X,\mathbb{Q})/H^2(C,\mathbb{Q}) & \to & H^0(C, R^2\pi_*\mathbb{Q}) & \to & 0 \\ & & \bigcap & & \downarrow^{\overline{\varphi_2}} & & \downarrow^{\psi_2} & & \\ 0 & \to & H^1(C', R^1\pi'_*\mathbb{Q}) & \to & H^2(X',\mathbb{Q}) & \to & H^0(C', R^2\pi'_*\mathbb{Q}) & \to & 0. \end{array}$$

Evidently this diagram yields the commutative diagram with exact rows

$$\begin{array}{ccccc} 0 \to & H^1(C,R^1\pi_*\mathbb{Q}) & \to H^2(X,\mathbb{Q})/H^2(C,\mathbb{Q}) \to & H^0(C,R^2\pi_*\mathbb{Q}) & \to 0 \\ & \bigcap & \downarrow^{\overline{\varphi_2}} & \downarrow^{\psi_2} & \\ 0 \to & H^1(C',R^1\pi'_*\mathbb{Q}) \cap \operatorname{Im}(\varphi_2) \to & \operatorname{Im}(\varphi_2) & \to H^0(C',R^2\pi'_*\mathbb{Q}) & \to 0 \end{array}$$

and the corresponding exact sequence of Hodge $\mathbb{Q}$-structures ([13], Section 2.6) of the snake-like diagram ([34], § 1, Proposition 2)

$$0 \to \operatorname{Ker}(\overline{\varphi_2}) \to \operatorname{Ker}(\psi_2) \to \frac{H^1(C', R^1\pi'_*\mathbb{Q}) \cap \operatorname{Im}(\varphi_2)}{H^1(C, R^1\pi_*\mathbb{Q})} \to 0.$$

Note that $\mathrm{Ker}(\varphi_2)$ is generated by classes of divisors in virtue of Corollary (8.2.8) in [24]. Therefore the Hodge $\mathbb{Q}$-structure $\mathrm{Ker}(\overline{\varphi_2})$ has type $(1,1)$. It is well known that the weight filtration of the limit mixed Hodge structure $H^1(X_s,\mathbb{Q})$ has the form $0 \subset W_0 \subset W_1 \subset W_2 = H^1(X_s,\mathbb{Q})$, and the map $N_\delta : W_2/W_1 \overset{\sim}{\to} W_0$ is an isomorphism of bidegree $(-1,-1)$ ([35], Lemma (6.4), Theorem (6.16)). Clearly the Hodge structure W_2/W_1 is of type $(1,1)$, because W_0 is a pure rational Hodge structure of type $(0,0)$. On the other hand, W_1/W_0 is a pure rational Hodge structure of type $(1,0)+(0,1)$, therefore the exact sequence of morphisms of mixed Hodge structures

$$0 \to W_1/W_0 \to W_2/W_0 \to W_2/W_1 \to 0$$

shows that $H^1(X_s,\mathbb{Q})/N_\delta H^1(X_s,\mathbb{Q}) = W_2/N_\delta(W_2) = W_2/W_0$ is a mixed Hodge structure of type $(1,1)+(1,0)+(0,1)$ ([18], Sections 2.3.6 - 2.3.7). It is easy to see that there exists a canonical embedding of mixed Hodge structures

$$\frac{H^1(C',R^1\pi'_*\mathbb{Q}) \cap \mathrm{Im}(\varphi_2)}{H^1(C,R^1\pi_*\mathbb{Q})} \hookrightarrow \bigoplus_{\delta\in\Delta} \frac{H^1(X_s,\mathbb{Q})}{N_\delta H^1(X_s,\mathbb{Q})}.$$

Thus the rational Hodge structure $\frac{H^1(C',R^1\pi'_*\mathbb{Q}) \cap \mathrm{Im}(\varphi_2)}{H^1(C,R^1\pi_*\mathbb{Q})}$ of type $(2,0)+(1,1)+(0,2)$ in fact has type $(1,1)$. Therefore the Hodge $\mathbb{Q}$-structure $\mathrm{Ker}(\psi_2)$ is of type $(1,1)$. Consequently, the Hodge $\mathbb{Q}$-structure $H^0(C,R^2\pi_*\mathbb{Q})$ is of type $(1,1)$, because the map ψ_2 is surjective and, by the result of Section 2.2, the rational Hodge structure $H^0(C',R^2\pi'_*\mathbb{Q})$ has type $(1,1)$. Lemma is proved.

2.6. The space $H^2(C,\mathbb{Q})$ is a 1-dimensional Hodge substructure of type $(1,1)$ generated by the class $\mathrm{cl}_X(X_s)$ of a fibre, therefore it easily follows from Lemma 2.5, from the exactness of the sequence (2.8) of rational Hodge structures for $n=2$, from Lefschetz' theorem on divisors and from the surjectivity of the canonical edge morphism $H^2(X,\mathbb{Q}) \to H^0(C,R^2\pi_*\mathbb{Q})$ of rational Hodge structures [33, Corollary (15.14)] that the sum of all irreducible Hodge $\mathbb{Q}$-substructures of dimension greater than 1 in the space $H^1(C,R^1\pi_*\mathbb{Q})$ is canonically identified with the space $T^2_{\mathbb{Q}}(X)$. In particular, there is a canonical embedding of rational Hodge structures

$$T^2_{\mathbb{Q}}(X) \hookrightarrow H^1(C,R^1\pi_*\mathbb{Q}). \tag{2.11}$$

According to the strong Lefschetz theorem on fibres of a smooth morphism π' and to the theorem on local invariant cycles we have the equalities

$$H^1(C,R^{2d-3}\pi_*\mathbb{Q}) = H^1(C,j_*R^{2d-3}\pi'_*\mathbb{Q})$$
$$= H^1(C,j_*R^1\pi'_*\mathbb{Q}) \smile \mathrm{cl}_X(H)^{\smile d-2} = H^1(C,R^1\pi_*\mathbb{Q}) \smile \mathrm{cl}_X(H)^{\smile d-2},$$

therefore by (2.7) and (2.11) the space $T^2_{\mathbb{Q}}(X) \smile \mathrm{cl}_X(H)^{\smile d-2}$ is canonically identified with the space of transcendental classes of the rational Hodge structure $H^1(C,R^{2d-3}\pi_*\mathbb{Q})$ and, in particular, there exists a canonical embedding

$$T^2_{\mathbb{Q}}(X) \smile \mathrm{cl}_X(H)^{\smile d-2} \hookrightarrow H^1(C,R^{2d-3}\pi_*\mathbb{Q}). \tag{2.12}$$

Finally, the surjectivity of canonical edge morphism of rational Hodge structures $H^{2d-2}(X,\mathbb{Q}) \to H^0(C,R^{2d-2}\pi_*\mathbb{Q})$ [33, Corollary (15.14)], formulae (2.8) for the case $n = 2d-2$, (2.10) and (2.12) show that the rational Hodge structure $H^0(C,R^{2d-2}\pi_*\mathbb{Q})$ is generated by images of *algebraic* cohomology classes, so the formula (2.8) yields the existence of the canonical embedding

$$T^2_{\mathbb{Q}}(X) \smile \mathrm{cl}_X(H)^{\smile d-2} \hookrightarrow K_{(2d-2)X}. \tag{2.13}$$

According to the theorem on local invariant cycles the canonical map

$$R^{2d-3}(\tau\sigma)_*\mathbb{Q} \to j_*R^{2d-3}(\tau\sigma)'_*\mathbb{Q} = j_*R^{2d-3}\tau'_*\mathbb{Q}$$

is surjective and it determines a canonical isomorphism

$$\begin{aligned} H^1(C, R^{2d-3}(\tau\sigma)_*\mathbb{Q}) &\xrightarrow{\sim} H^1(C, j_*R^{2d-3}\tau'_*\mathbb{Q}) \\ &= \oplus_{a+b=2d-3} H^1(C, j_*R^a\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} j_*R^b\pi'_*\mathbb{Q}). \end{aligned}$$

Consequently, the map

$$H^1(C, R^{2d-3}\pi_*\mathbb{Q}) \xrightarrow{[(p_1\sigma)^*]_1} H^1(C, R^{2d-3}(\tau\sigma)_*\mathbb{Q})$$

corresponds to the canonical embedding

$$\begin{aligned} &H^1(C, j_*R^{2d-3}\pi'_*\mathbb{Q}) \\ &\overset{x\mapsto x\oplus[0]}{\hookrightarrow} H^1(C, j_*R^{2d-3}\pi'_*\mathbb{Q}) \oplus [H^1(C, j_*R^{2d-4}\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} j_*R^1\pi'_*\mathbb{Q}) \\ &\oplus H^1(C, j_*R^{2d-5}\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} j_*R^2\pi'_*\mathbb{Q}) \oplus \ldots \oplus H^1(C, j_*\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} j_*R^{2d-3}\pi'_*\mathbb{Q})]. \end{aligned} \tag{2.14}$$

2.7. Admitting some freedom in notation, we denote by

$$(p_k\sigma)^* : H^n(X, \mathbb{Q}) \to H^n(Y, \mathbb{Q})$$

a canonical *injective* map *of Weil cohomology* [2, Section 1.2.A, Proposition 1.2.4] (determined by a *surjective* morphism $p_k\sigma : Y \to X$), which is different from the morphism of sheaves $(p_k\sigma)^* : R^n\pi_*\mathbb{Q} \xrightarrow{(p_k\sigma)^*} R^n(\tau\sigma)_*\mathbb{Q}$ defined above.

For any point $s \in C$, denote by $\iota_{X_s/X} : X_s \hookrightarrow X$ and by $\iota_{Y_s/Y} : Y_s \hookrightarrow Y$ the canonical embeddings. Set $p_{1s} = p_1|_{X_s\times X_s}$, $\quad \sigma_s = \sigma|_{Y_s}$.

The morphism π is proper, therefore the fibre of the sheaf $R^n\pi_*\mathbb{Q}$ over a point $s \in C$ coincides with the space $H^n(X_s, \mathbb{Q})$ [28, Ch. II, § 4, Remark 4.17.1].

On the other hand, the Leray spectral sequence is functorial, therefore the restriction homomorphism $\iota^*_{X_s/X} : H^n(X, \mathbb{Q}) \to H^n(X_s, \mathbb{Q})$ transforms the Leray spectral sequence $E_2^{p,q}(\pi) = H^p(C, R^q\pi_*\mathbb{Q})$ into the Leray spectral sequence corresponding to the morphism $X_s \to s$; consequently, the homomorphism $\iota^*_{X_s/X}$ coincides with the composite ([36], Ch. 9, the beginning of § 5)

$$H^n(X, \mathbb{Q}) = F^0H^n(X, \mathbb{Q}) \to E_\infty^{0,n}(\pi) \to E_2^{0,n}(\pi) = H^0(C, R^n\pi_*\mathbb{Q}) \to H^n(X_s, \mathbb{Q}).$$

Thus the map $\iota^*_{X_s/X}$ is the composite of canonical maps

$$H^n(X, \mathbb{Q}) \to H^0(C, R^n\pi_*\mathbb{Q}) \hookrightarrow \prod_{s\in C} H^n(X_s, \mathbb{Q}) \to H^n(X_s, \mathbb{Q})$$

and, consequently, for all elements $\omega \in \mathrm{Ker}[H^n(X, \mathbb{Q}) \to H^0(C, R^n\pi_*\mathbb{Q})]$, we have the equality $\iota^*_{X_s/X}(\omega) = 0$. On the other hand, the commutative diagram of morphisms

$$\begin{array}{ccc} X & \xleftarrow{p_1\sigma} & Y \\ \uparrow{\scriptstyle \iota_{X_s/X}} & & \uparrow{\scriptstyle \iota_{Y_s/Y}} \\ X_s & \xleftarrow{p_{1s}\sigma_s} & Y_s \end{array}$$

yields the commutative diagram of canonical maps

$$\begin{array}{ccc} H^n(X,\mathbb{Q}) & \xrightarrow{(p_1\sigma)^*} & H^n(Y,\mathbb{Q}) \\ \downarrow{\iota^*_{X_s/X}} & & \downarrow{\iota^*_{Y_s/Y}} \\ H^n(X_s,\mathbb{Q}) & \xrightarrow{(p_{1s}\sigma_s)^*} & H^n(Y_s,\mathbb{Q}), \end{array}$$

so that, for any point $s \in C$, it follows from the equality $\iota^*_{X_s}(\omega) = 0$ that

$$\iota^*_{Y_s/Y}(p_1\sigma)^*(\omega) = 0.$$

The morphism $\tau\sigma$ is proper, therefore the canonical restriction map

$$\iota^*_{Y_s/Y} : H^n(Y,\mathbb{Q}) \to H^n(Y_s,\mathbb{Q})$$

is the composite of canonical maps

$$H^n(Y,\mathbb{Q}) \to H^0(C, R^n(\tau\sigma)_*\mathbb{Q}) \hookrightarrow \prod_{s\in C} H^n(Y_s,\mathbb{Q}) \to H^n(Y_s,\mathbb{Q}).$$

Thus, it follows from the equalities $\iota^*_{Y_s/Y}(p_1\sigma)^*(\omega) = 0$ $(s \in C)$ that

$$(p_1\sigma)^*(\omega) \in \mathrm{Ker}[H^n(Y,\mathbb{Q}) \to H^0(C, R^n(\tau\sigma)_*\mathbb{Q}).$$

As a result, we obtain the following inclusion

$$(p_1\sigma)^* K_{nX} \hookrightarrow K_{nY}. \tag{2.15}$$

2.8. The functoriality of constructions under consideration and (2.8) - (2.15) yield a commutative diagram of canonical morphisms of rational Hodge structures

$$\begin{array}{ccc} K_{nX} & \xrightarrow{(p_k\sigma)^*|_{K_{nX}}} & K_{nY} \\ \downarrow{\alpha_{nX}} & & \downarrow{\alpha_{nY}} \\ H^1(C, R^{n-1}\pi_*\mathbb{Q}) & \xrightarrow{[(p_k\sigma)^*]_1} & H^1(C, R^{n-1}(\tau\sigma)_*\mathbb{Q}) \\ \downarrow{\simeq} & & \downarrow{\simeq} \\ H^1(C, j_*R^{n-1}\pi'_*\mathbb{Q}) & & H^1(C, j_*R^{n-1}\tau'_*\mathbb{Q}). \end{array} \tag{2.16}$$

It is well known that a linear map of finite-dimensional linear $\mathbb{Q}$-spaces $u : E \to F$ is surjective iff the conjugate map $u^\vee \stackrel{\mathrm{def}}{=} {}^t u : F^\vee \to E^\vee$ of dual spaces is injective [37, Ch. II, § 4, Section 9, Corollary of Theorem 3]; on the other hand, it follows from [37, Ch. II, § 4, Section 9, Theorem 4] that if the map u is injective, then the conjugate map ${}^t u : F^\vee \to E^\vee$ is surjective.

It is evident that the canonical map of cohomology

$$H^1(C, R^{2d-1}(\tau\sigma)_*\mathbb{Q}) \xrightarrow{[(p_k\sigma)_*]_1} H^1(C, R^1\pi_*\mathbb{Q})$$

coincides, by the theorem on local invariant cycles, with the map

$$H^1(C, j_*R^{2d-1}(\tau'\sigma')_*\mathbb{Q}) \to H^1(C, j_*R^1\pi'_*\mathbb{Q}),$$

induced by the map of sheaves $R^{2d-1}(\tau'\sigma')_*\mathbb{Q} = R^{2d-1}\tau'_*\mathbb{Q} \xrightarrow{(p'_k)_*} R^1\pi'_*\mathbb{Q}$.

It follows from (2.8) and from the equalities $R^{2d}\pi_*\mathbb{Q} = R^{2d-1}\pi_*\mathbb{Q} = 0$ that

$$H^2(C, R^{2d-2}\pi_*\mathbb{Q}) = H^{2d}(X,\mathbb{Q}) = \mathbb{Q}(-d).$$

On the other hand, the choice of an element $\sqrt{-1} \in \mathbb{C}$ determines the orientation isomorphism $H^{2d}(X,\mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}$ of Weil cohomology [2, Section 1.2.A] and the non-degenerate pairing

$$H^{2d-2}(X,\mathbb{Q}) \times H^2(X,\mathbb{Q}) \xrightarrow{x\times x' \mapsto x \smile x'} H^{2d}(X,\mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}$$

of the Poincaré duality. Besides, in accordance with the theorem on local invariant cycles, there is a non-degenerate canonical pairing [33, Proposition (10.5)]

$$H^1(C, j_*R^{2d-3}\pi'_*\mathbb{Q}) \times H^1(C, j_*R^1\pi'_*\mathbb{Q}) \xrightarrow{x\times x' \mapsto x\smile x'} H^2(C, j_*R^{2d-2}\pi'_*\mathbb{Q})$$
$$= H^2(C, R^{2d-2}\pi_*\mathbb{Q}) = H^{2d}(X,\mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}.$$

By the similar reasons there exist non-degenerate pairings

$$H^{2d}(Y,\mathbb{Q}) \times H^{2d-2}(Y,\mathbb{Q}) \xrightarrow{y\times y' \mapsto y\smile y'} H^{4d-2}(Y,\mathbb{Q}) \xrightarrow{\sim} \mathbb{Q};$$
$$H^1(C, j_*R^{2d-1}(\tau\sigma)'_*\mathbb{Q}) \times H^1(C, j_*R^{2d-3}(\tau\sigma)'_*\mathbb{Q})$$
$$\xrightarrow{y\times y' \mapsto y\smile y'} H^2(C, j_*R^{4d-4}(\tau\sigma)'_*\mathbb{Q}) = H^2(C, R^{4d-4}(\tau\sigma)_*\mathbb{Q})$$
$$= H^{4d-2}(Y,\mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}.$$

Therefore the map $H^1(C, R^{2d-1}(\tau\sigma)_*\mathbb{Q}) \xrightarrow{[(p_k\sigma)_*]_1} H^1(C, R^1\pi_*\mathbb{Q})$ coincides with the surjective map which is dual to the canonical injection

$$H^1(C, R^{2d-3}\pi_*\mathbb{Q}) \xrightarrow{[(p_k\sigma)^*]_1} H^1(C, R^{2d-3}(\tau\sigma)_*\mathbb{Q})$$

determined in the diagram (2.16) for $n = 2d-2$. In virtue of the surjectivity of maps $\alpha_{(2d-2)X}$ and $\alpha_{(2d-2)Y}$ in (2.8), (2.9), the commutative diagram dual to the diagram (2.16) in the case $n = 2d-2$, takes the form

$$\text{(2.17)}\qquad \begin{array}{ccc} K^\vee_{(2d-2)X} & \xleftarrow{\left((p_k\sigma)^*|_{K_{(2d-2)X}}\right)^\vee} & K^\vee_{(2d-2)Y} \\ \cup\uparrow^{[\alpha_{(2d-2)X}]^\vee} & & \cup\uparrow^{[\alpha_{(2d-2)Y}]^\vee} \\ H^1(C, R^1\pi_*\mathbb{Q}) & \xleftarrow{[(p_k\sigma)_*]_1} & H^1(C, R^{2d-1}(\tau\sigma)_*\mathbb{Q}) \\ \downarrow^{\simeq} & & \downarrow^{\simeq} \\ H^1(C, j_*R^1\pi'_*\mathbb{Q}) & & H^1(C, j_*R^{2d-1}\tau'_*\mathbb{Q}). \end{array}$$

2.9. Since the choice of an element $\sqrt{-1} \in \mathbb{C}$ identifies cohomology with coefficients in the Hodge - Tate structure $\mathbb{Q}(n) = \mathbb{Q}(1)^{\otimes n}$ and Weil cohomology, then the canonical embedding of Hodge $\mathbb{Q}$-structures $K_{nX} \hookrightarrow H^n(X,\mathbb{Q})$ and the Poincaré duality [2, Section 1.2.A] yield the canonical exact sequence of rational Hodge structures $0 \to G^{2d-n}_{X/C} \to H^{2d-n}(X,\mathbb{Q}) \to K^\vee_{nX} \to 0$ and the identification $K^\vee_{nX} = H^{2d-n}(X,\mathbb{Q})/G^{2d-n}_{X/C}$. Finally, the diagram dual to the commutative diagram

$$\begin{array}{ccc} K_{(2d-2)X} & \hookrightarrow & H^{2d-2}(X,\mathbb{Q}) \\ \cap\downarrow^{(p_2\sigma)^*|_{K_{(2d-2)X}}} & & \cap\downarrow^{(p_2\sigma)^*} \\ K_{(2d-2)Y} & \hookrightarrow & H^{2d-2}(Y,\mathbb{Q}), \end{array}$$

yields (in obvious notations) the commutative diagram

$$\begin{array}{ccccc} H^2(X,\mathbb{Q})/G^2_{X/C} & = & K^\vee_{(2d-2)X} & \leftarrow & H^2(X,\mathbb{Q}) \\ \uparrow^{((p_2\sigma)^*|_{K_{(2d-2)X}})^\vee} & & \uparrow^{((p_2\sigma)^*|_{K_{(2d-2)X}})^\vee} & & \uparrow^{(p_2\sigma)_*} \\ H^{2d}(Y,\mathbb{Q})/G^{2d}_{Y/C} & = & K^\vee_{(2d-2)Y} & \leftarrow & H^{2d}(Y,\mathbb{Q}). \end{array}$$

Therefore, for all $y \in H^{2d}(Y,\mathbb{Q})$, we have:

$$\left((p_2\sigma)^*|_{K_{(2d-2)X}}\right)^\vee (y + G^{2d}_{Y/C}) = (p_2\sigma)_*(y) + G^2_{X/C}.$$

Consequently, the diagram (2.17) yields the commutative diagram of morphisms of Hodge structures

$$
\begin{array}{ccc}
H^2(X,\mathbb{Q})/G^2_{X/C} & \xleftarrow{y+G^{2d}_{Y/C}\mapsto (p_2\sigma)_*(y)+G^2_{X/C}} & H^{2d}(Y,\mathbb{Q})/G^{2d}_{Y/C} \\
\cup\uparrow^{\beta_{2X}} & & \cup\uparrow^{\beta_{(2d)Y}} \\
H^1(C,R^1\pi_*\mathbb{Q}) & \xleftarrow{[(p_2\sigma)_*]_1} & H^1(C,R^{2d-1}(\tau\sigma)_*\mathbb{Q}).
\end{array}
\tag{2.18}
$$

In accordance with (2.13) and (2.15) there is a commutative diagram of morphisms of Hodge structures

(2.19)

$$
\begin{array}{ccc}
(p_1\sigma)^*(T^2_{\mathbb{Q}}(X)\smile \mathrm{cl}_X(H)^{\smile\, d-2}) & \xrightarrow{\smile\, \mathrm{cl}_Y(D^{(1)})} & (p_1\sigma)^*(T^2_{\mathbb{Q}}(X)\smile \mathrm{cl}_X(H)^{\smile\, d-2})\smile \mathrm{cl}_Y(D^{(1)}) \\
\cap & & \cap \\
K_{(2d-2)Y} & \xrightarrow{\smile\, \mathrm{cl}_Y(D^{(1)})} & K_{(2d-2)Y}\smile \mathrm{cl}_Y(D^{(1)}) \\
\downarrow^{\alpha_{(2d-2)Y}} & & \downarrow^{\alpha_{(2d-2)Y}\,\smile\, \mathrm{cl}_Y(D^{(1)})} \\
H^1(C,R^{2d-3}(\tau\sigma)_*\mathbb{Q}) & \xrightarrow{[\smile\, \mathrm{cl}_Y(D^{(1)})]_1} & H^1(C,R^{2d-1}(\tau\sigma)_*\mathbb{Q}) \\
\downarrow^{\simeq} & & \downarrow^{\simeq} \\
H^1(C,j_*R^{2d-3}\tau'_*\mathbb{Q}) & & H^1(C,j_*R^{2d-1}\tau'_*\mathbb{Q}).
\end{array}
$$

Since restrictions of elements $w\in K_{(2d-2)Y}$ and $w \smile \mathrm{cl}_Y(D^{(1)})$ to the fibre Y_s are trivial for all $s\in C$, then

$$K_{(2d-2)Y} \smile \mathrm{cl}_Y(D^{(1)}) \subset K_{(2d)Y},$$

therefore the diagram (2.19) yields the commutative diagram of morphisms of Hodge structures

(2.20)

$$
\begin{array}{ccc}
(p_1\sigma)^*(T^2_{\mathbb{Q}}(X)\smile \mathrm{cl}_X(H)^{\smile\, d-2}) & \xrightarrow{\smile\, \mathrm{cl}_Y(D^{(1)})} & (p_1\sigma)^*(T^2_{\mathbb{Q}}(X)\smile \mathrm{cl}_X(H)^{\smile\, d-2})\smile \mathrm{cl}_Y(D^{(1)}) \\
\cap & & \cap \\
K_{(2d-2)Y} & \xrightarrow{\smile\, \mathrm{cl}_Y(D^{(1)})} & K_{(2d)Y} \\
\downarrow^{\alpha_{(2d-2)Y}} & & \downarrow^{\alpha_{(2d)Y}} \\
H^1(C,R^{2d-3}(\tau\sigma)_*\mathbb{Q}) & \xrightarrow{[\smile\, \mathrm{cl}_Y(D^{(1)})]_1} & H^1(C,R^{2d-1}(\tau\sigma)_*\mathbb{Q}) \\
\downarrow^{\simeq} & & \downarrow^{\simeq} \\
H^1(C,j_*R^{2d-3}\tau'_*\mathbb{Q}) & & H^1(C,j_*R^{2d-1}\tau'_*\mathbb{Q}).
\end{array}
$$

On the other hand, it follows from (2.3) - (2.4) and (2.11) - (2.12) that the restriction of the isomorphism (2.7) to the subspace $T^2_{\mathbb{Q}}(X) \smile \mathrm{cl}_X(H)^{\smile\, d-2} \hookrightarrow H^1(C,R^{2d-3}\pi_*\mathbb{Q})$ (coinciding with the sum of all irreducible Hodge $\mathbb{Q}$-substructures greater than 1 in the Hodge $\mathbb{Q}$-structure $H^1(C,R^{2d-3}\pi_*\mathbb{Q})$) yields the equality

$$[(p_2\sigma)_*]_1\circ[\smile\, \mathrm{cl}_Y(D^{(1)})]_1\circ[(p_1\sigma)^*]_1(T^2_{\mathbb{Q}}(X) \smile \mathrm{cl}_X(H)^{\smile\, d-2}) = T^2_{\mathbb{Q}}(X).$$

Therefore (2.16) in the case $n = 2d-2$, the diagrams (2.20), (2.17) and the injectivity of maps β_{2X}, $\beta_{(2d)Y}$ in the diagram (2.18) show that

$$(p_2\sigma)_*\Big((p_1\sigma)^*(T^2_{\mathbb{Q}}(X) \smile \mathrm{cl}_X(H)^{\smile\, d-2}) \smile \mathrm{cl}_Y(D^{(1)})\Big) = T^2_{\mathbb{Q}}(X). \tag{2.21}$$

Let $\mathrm{pr}_k : X\times X\to X$ be the canonical projection. Then $\mathrm{pr}_k\,\iota\sigma = p_k\sigma$. Since pr_k and $\iota\sigma$ are morphisms of smooth projective varieties, then by the projection formula [2, Section 1.2.A] for all $x \in T^2_{\mathbb{Q}}(X) \smile \mathrm{cl}_X(H)^{\smile\, d-2}$ we have:

$$(p_2\sigma)_*\left((p_1\sigma)^*x \smile \mathrm{cl}_Y(D^{(1)})\right) = (\mathrm{pr}_2\,\iota\sigma)_*\left((\mathrm{pr}_1\,\iota\sigma)^*x \smile \mathrm{cl}_Y(D^{(1)})\right) =$$
$$\mathrm{pr}_{2*}(\iota\sigma)_*\left((\iota\sigma)^*\,\mathrm{pr}_1^*\,x \smile \mathrm{cl}_Y(D^{(1)})\right) = \mathrm{pr}_{2*}\left(\mathrm{pr}_1^*\,x \smile (\iota\sigma)_*\,\mathrm{cl}_Y(D^{(1)})\right).$$

Therefore it follows from (2.21) that *an algebraic* class

$$u \stackrel{\text{def}}{=} (\iota\sigma)_*\,\mathrm{cl}_Y(D^{(1)}) \in H^4(X \times X, \mathbb{Q})$$

determines an *algebraic* isomorphism of type $(2-d, 2-d)$ of rational Hodge structures

$$T^2_{\mathbb{Q}}(X) \smile \mathrm{cl}_X(H)^{\smile\, d-2} \stackrel{\sim}{\to} T^2_{\mathbb{Q}}(X). \tag{2.22}$$

2.10. By the Lefschetz theorem on divisors we have the relations

$$\mathrm{NS}_{\mathbb{Q}}(X) = H^2(X,\mathbb{Q})^{\mathrm{Hg}(H^2(X,\mathbb{Q}))} = H^2(X,\mathbb{Q}) \cap H^{1,1}(X,\mathbb{C});$$
$$T^2_{\mathbb{Q}}(X)^{\mathrm{Hg}(H^2(X,\mathbb{Q}))} = 0.$$

It is evident that $u = u_{0,4} + u_{1,3} + u_{2,2} + u_{3,1} + u_{4,0}$, where $u_{i,4-i} \in [H^i(X,\mathbb{Q}) \otimes_{\mathbb{Q}} H^{4-i}(X,\mathbb{Q})] \cap H^{2,2}(X \times X, \mathbb{C})$, hence by [19, Corollary B.55] we have:

$$u_{2,2} \in [T^2_{\mathbb{Q}}(X) \oplus \mathrm{NS}_{\mathbb{Q}}(X)]^{\otimes 2} \cap H^{2,2}(X \times X, \mathbb{C}) =$$
$$[T^2_{\mathbb{Q}}(X) \otimes_{\mathbb{Q}} T^2_{\mathbb{Q}}(X)]^{\mathrm{Hg}(H^2(X,\mathbb{Q}))} \oplus [\mathrm{NS}_{\mathbb{Q}}(X) \otimes_{\mathbb{Q}} \mathrm{NS}_{\mathbb{Q}}(X)]^{\mathrm{Hg}(H^2(X,\mathbb{Q}))} \oplus$$
$$[T^2_{\mathbb{Q}}(X) \otimes_{\mathbb{Q}} \mathrm{NS}_{\mathbb{Q}}(X)]^{\mathrm{Hg}(H^2(X,\mathbb{Q}))} \oplus [\mathrm{NS}_{\mathbb{Q}}(X) \otimes_{\mathbb{Q}} T^2_{\mathbb{Q}}(X)]^{\mathrm{Hg}(H^2(X,\mathbb{Q}))} =$$
$$[T^2_{\mathbb{Q}}(X) \otimes_{\mathbb{Q}} T^2_{\mathbb{Q}}(X)]^{\mathrm{Hg}(H^2(X,\mathbb{Q}))} \oplus [\mathrm{NS}_{\mathbb{Q}}(X) \otimes_{\mathbb{Q}} \mathrm{NS}_{\mathbb{Q}}(X)].$$

In particular, $u_{2,2} = t_{2,2} + n_{2,2}$, where $t_{2,2} \in T^2_{\mathbb{Q}}(X) \otimes_{\mathbb{Q}} T^2_{\mathbb{Q}}(X)$ and $n_{2,2} \in \mathrm{NS}_{\mathbb{Q}}(X) \otimes_{\mathbb{Q}} \mathrm{NS}_{\mathbb{Q}}(X)$.

It is evident that

$$T^2_{\mathbb{Q}}(X) \smile \mathrm{NS}_{\mathbb{Q}}(X)^{\smile\, d-1} = 0, \tag{2.23}$$

because otherwise this space coincides with a sum of certain irreducible Hodge $\mathbb{Q}$-structures of dimension > 1, contrary to the equality $\dim_{\mathbb{Q}} H^{2d}(X,\mathbb{Q}) = 1$.

By the results of Section 1.2 and (2.23), the restriction of the non-degenerate bilinear form

$$\Phi : H^2(X,\mathbb{Q}) \times H^2(X,\mathbb{Q}) \xrightarrow{x \times y \mapsto x \smile y \smile \mathrm{cl}_X(H)^{\smile\, d-2}} H^{2d}(X,\mathbb{Q}) = \mathbb{Q}(-d) \xrightarrow[\sim]{(2\pi i)^d} \mathbb{Q}$$

to the subspace $\mathrm{NS}_{\mathbb{Q}}(X) \subset H^2(X,\mathbb{Q})$ is non-degenerate. Hence it follows from the inequality $\dim_{\mathbb{Q}} \mathrm{NS}_{\mathbb{Q}}(X) \geq 3$ that there exists the Poincaré class $\wp(\mathrm{NS}_{\mathbb{Q}}(X))$ as a generator of the 1-dimensional space

$$[\mathrm{NS}_{\mathbb{Q}}(X) \otimes_{\mathbb{Q}} \mathrm{NS}_{\mathbb{Q}}(X)]^{\mathrm{SO}(\mathrm{NS}_{\mathbb{Q}}(X), \Phi|_{\mathrm{NS}_{\mathbb{Q}}(X)})}.$$

Correspondences $u_{0,4}, u_{1,3}, u_{3,1}, u_{4,0}$ annihilate the space $H^{2d-2}(X,\mathbb{Q})$ in virtue of [38, Formula (1.2)]. Therefore by (2.23) the correspondence $-n_{2,2} + \wp(\mathrm{NS}_{\mathbb{Q}}(X))$ annihilates the subspace

$$T^2_{\mathbb{Q}}(X) \smile \mathrm{cl}_X(H)^{\smile\, d-2} \subset H^{2d-2}(X,\mathbb{Q}),$$

and the algebraic class $u - n_{2,2} + \wp(\mathrm{NS}_{\mathbb{Q}}(X))$ determines the isomorphism (2.22). Finally, the correspondence $t_{2,2}$ annihilates the subspace

$$\mathrm{NS}_{\mathbb{Q}}(X) \smile \mathrm{cl}_X(H)^{\smile\, d-2} \subset H^{2d-2}(X,\mathbb{Q})$$

in virtue of (2.23).

On the other hand, the correspondence $\wp(\mathrm{NS}_{\mathbb{Q}}(X))$ determines an algebraic isomorphism

$$\mathrm{NS}_{\mathbb{Q}}(X) \smile \mathrm{cl}_X(H)^{\smile d-2} \overset{\sim}{\to} \mathrm{NS}_{\mathbb{Q}}(X);$$

this result is an easy consequence of (2.23), because in virtue of the results of Section 1.2 and arguments of Section 3.5 below applied to the decomposition $H^2(X,\mathbb{Q}) = T^2_{\mathbb{Q}}(X) \oplus \mathrm{NS}_{\mathbb{Q}}(X)$ of rational Hodge structures, to the symmetric forms

$$\Psi^{\perp} = \Phi|_{T^2_{\mathbb{Q}}(X)}, \qquad \Psi = \Phi|_{\mathrm{NS}_{\mathbb{Q}}(X)}$$

and to the canonical embedding of algebraic $\mathbb{Q}$-groups

$$\mathrm{SO}\,(\mathrm{NS}_{\mathbb{Q}}(X),\Psi) \times \ \mathrm{SO}\left(T^2_{\mathbb{Q}}(X),\Psi^{\perp}\right) \ \hookrightarrow \ \mathrm{SO}(H^2(X,\mathbb{Q}),\Phi),$$

the Poincaré class $\wp(H^2(X,\mathbb{Q}))$ belongs to the space

$$T^2_{\mathbb{Q}}(X) \otimes_{\mathbb{Q}} T^2_{\mathbb{Q}}(X) \ \oplus \ [\mathrm{NS}_{\mathbb{Q}}(X) \otimes_{\mathbb{Q}} \mathrm{NS}_{\mathbb{Q}}(X)]^{\mathrm{SO}(\mathrm{NS}_{\mathbb{Q}}(X),\Phi|_{\mathrm{NS}_{\mathbb{Q}}(X)})}$$

and it determines an isomorphism $H^{2d-2}(X,\mathbb{Q}) \overset{\sim}{\to} H^2(X,\mathbb{Q})$, which is the composite of maps

$$H^{2d-2}(X,\mathbb{Q}) \xrightarrow{\mathrm{pr}_1^*} H^{2d-2}(X,\mathbb{Q}) \otimes_{\mathbb{Q}} H^0(X,\mathbb{Q})$$
$$\xrightarrow{\smile \wp(H^2(X,\mathbb{Q}))} H^{2d}(X,\mathbb{Q}) \otimes_{\mathbb{Q}} H^2(X,\mathbb{Q}) \xrightarrow{\mathrm{pr}_{2*}} H^2(X,\mathbb{Q}).$$

Therefore it follows from (2.10) and (2.22) that the algebraic class

$$u - n_{2,2} + \wp(\mathrm{NS}_{\mathbb{Q}}(X)) = u_{0,4} + u_{1,3} + t_{2,2} + u_{3,1} + u_{4,0} + \wp(\mathrm{NS}_{\mathbb{Q}}(X))$$

determines an algebraic isomorphism $H^{2d-2}(X,\mathbb{Q}) \overset{\sim}{\to} H^2(X,\mathbb{Q})$.

§ 3. Some isomorphisms and canonical decompositions of rational Hodge structures

3.1. First of all we are going to construct an algebraic isomorphism

$$H^1(C, R^{2d-4}\pi_*\mathbb{Q}) \overset{\sim}{\to} H^1(C, R^2\pi_*\mathbb{Q}).$$

By [2, Lemma 2A12, Remark 2A13] the algebraic correspondence $\mathrm{c}_1(\mathcal{P}'_s)^{\smile 2}$ yields an algebraic isomorphism $H^{2d-4}(X_s,\mathbb{Q}) \overset{\sim}{\to} H^2(X_s,\mathbb{Q})$. Using arguments of Sections 2.1 and 2.3, we may assume that the element ${\Lambda'_{1,1}}^{\smile 2}$ yields the isomorphism of local systems $R^{2d-4}\pi'_*\mathbb{Q} \overset{\sim}{\to} R^2\pi'_*\mathbb{Q}$ determined by the composite of morphisms of sheaves

$$R^{2d-4}\pi'_*\mathbb{Q} \xrightarrow{(p'_1)^*} R^{2d-4}\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} \pi'_*\mathbb{Q}$$
$$\xrightarrow{\smile {\Lambda'_{1,1}}^{\smile 2}} R^{2d-2}\pi'_*\mathbb{Q} \otimes_{\mathbb{Q}} R^2\pi'_*\mathbb{Q} \xrightarrow{(p'_2)_*} R^2\pi'_*\mathbb{Q}. \tag{3.1}$$

We may also expand (3.1) to a sequence of morphisms of sheaves

$$R^{2d-4}\pi_*\mathbb{Q} \xrightarrow{(p_1\sigma)^*} R^{2d-4}(\tau\sigma)_*\mathbb{Q} \xrightarrow{\smile \mathrm{cl}_Y(D^{(1)})^{\smile 2}} R^{2d-2}(\tau\sigma)_*\mathbb{Q} \xrightarrow{(p_2\sigma)_*} R^2\pi_*\mathbb{Q},$$

whose composite is an isomorphism outside the finite set Δ. Therefore there exists an isomorphism of bidegree $(3-d, 3-d)$ of mixed Hodge structures

$$H^1(C, R^{2d-4}\pi_*\mathbb{Q}) \xrightarrow[\sim]{[x \mapsto (p_2\sigma)_*((p_1\sigma)^* x \smile \mathrm{cl}_Y(D^{(1)})^{\smile 2})]_1} H^1(C, R^2\pi_*\mathbb{Q}). \tag{3.2}$$

3.2. Lemma. *For any odd natural number n, there are the equalities*

$$H^0(C', R^n\tau'_*\mathbb{Q}) = H^0(C', R^n\pi'_*\mathbb{Q}) = H^2(C, R^n(\tau\sigma)_*\mathbb{Q}) = H^2(C, R^n\pi_*\mathbb{Q}) = 0.$$

Proof. For a point $s \in C' \setminus \Delta_{\text{countable}}$ one has by (1.2)

$$H^0(C', R^n\tau'_*\mathbb{Q}) \tilde{\to} H^n(X_s \times X_s, \mathbb{Q})^G = H^n(X_s \times X_s, \mathbb{Q})^{\mathrm{Hg}(X_s)} = 0$$

because in notations of Section 1.1, for any $\theta \in \mathbb{R}$, an element $e^{i\theta} \in U^1$ acts on the Hodge component $H^{p,n-p}(X_s \times X_s, \mathbb{C})$ as a multiplication by an element $e^{ip\theta}e^{-i(n-p)\theta} = e^{i(2p-n)\theta}$, therefore the group $\mathrm{Hg}(X_s) \otimes_{\mathbb{Q}} \mathbb{C} = G \otimes_{\mathbb{Q}} \mathbb{C}$ acts on $H^n(X_s \times X_s, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$ without non-zero fixed points. Similarly one has the equality $H^0(C', R^n\pi'_*\mathbb{Q}) = 0$. It remains to note, in virtue of the theorem on local invariant cycles and [33, Proposition (10.5)], that

$$\begin{aligned} H^2(C, R^n(\tau\sigma)_*\mathbb{Q}) &= H^2(C, j_*R^n(\tau'\sigma')_*\mathbb{Q}) \\ &\tilde{\to} H^0(C', R^n(\tau'\sigma')_*\mathbb{Q})^\vee = H^0(C', R^n\tau'_*\mathbb{Q})^\vee; \end{aligned}$$
$$H^2(C, R^n\pi_*\mathbb{Q}) = H^2(C, j_*R^n\pi'_*\mathbb{Q}) \tilde{\to} H^0(C', R^n\pi'_*\mathbb{Q})^\vee.$$

3.3. It follows from Lemma 3.2 and from (2.8) - (2.9) that, for any odd natural number n, one has canonical identifications of rational Hodge structures

$$K_{nX} \xrightarrow[\sim]{\alpha_{nX}} H^1(C, R^{n-1}\pi_*\mathbb{Q}); \tag{3.3}$$

$$K_{nY} \xrightarrow[\sim]{\alpha_{nY}} H^1(C, R^{n-1}(\tau\sigma)_*\mathbb{Q}). \tag{3.4}$$

Besides, there is a non-degenerate canonical pairing [33, Proposition (10.5)]

$$\begin{aligned} H^1(C, j_*R^{2d-4}\pi'_*\mathbb{Q}) \times H^1(C, j_*R^2\pi'_*\mathbb{Q}) &\xrightarrow{x\times x' \mapsto x \smile x'} H^2(C, j_*R^{2d-2}\pi'_*\mathbb{Q}) \\ &= H^2(C, R^{2d-2}\pi_*\mathbb{Q}) = H^{2d}(X, \mathbb{Q}) \tilde{\to} \mathbb{Q}, \end{aligned}$$

identifying (in accordance with the theorem on local invariant cycles) the space $H^1(C, R^{2d-4}\pi_*\mathbb{Q})^\vee$ with $H^1(C, R^2\pi_*\mathbb{Q})$. By the similar reason we have the identification $H^1(C, R^{2d-4}(\tau\sigma)_*\mathbb{Q})^\vee = H^1(C, R^{2d}(\tau\sigma)_*\mathbb{Q})$. Hence from (3.3) and (3.4) we obtain the canonical isomorphisms

$$K^\vee_{(2d-3)X} \xleftarrow[\sim]{[\alpha_{(2d-3)X}]^\vee} H^1(C, R^{2d-4}\pi_*\mathbb{Q})^\vee = H^1(C, R^2\pi_*\mathbb{Q}) \xrightarrow[\sim]{[\alpha_{3X}]^{-1}} K_{3X}; \tag{3.5}$$

$$\begin{aligned} K^\vee_{(2d-3)Y} &\xleftarrow[\sim]{[\alpha_{(2d-3)Y}]^\vee} H^1(C, R^{2d-4}(\tau\sigma)_*\mathbb{Q})^\vee \\ &= H^1(C, R^{2d}(\tau\sigma)_*\mathbb{Q}) \xrightarrow[\sim]{[\alpha_{(2d+1)Y}]^{-1}} K_{(2d+1)Y}. \end{aligned} \tag{3.6}$$

On the other hand, acting by the algorithm of Section 2.9, we see that the Poincaré duality yields the identifications

$$K^\vee_{(2d-3)X} = H^3(X, \mathbb{Q})/G^3_{X/C}, \qquad K^\vee_{(2d-3)Y} = H^{2d+1}(Y, \mathbb{Q})/G^{2d+1}_{Y/C}$$

and the equality $\left((p_2\sigma)^*|_{K_{(2d-3)X}}\right)^\vee (y + G^{2d+1}_{Y/C}) = (p_2\sigma)_*(y) + G^3_{X/C}$ for any element $y \in H^{2d+1}(Y, \mathbb{Q})$. It follows from (3.5) - (3.6) that exact sequences of rational Hodge structures

$$0 \to G^3_{X/C} \to H^3(X, \mathbb{Q}) \to H^3(X, \mathbb{Q})/G^3_{X/C} \to 0,$$
$$0 \to G^{2d+1}_{Y/C} \to H^{2d+1}(Y, \mathbb{Q}) \to H^{2d+1}(Y, \mathbb{Q})/G^{2d+1}_{Y/C} \to 0$$

have *canonical splittings*, so that we obtain a commutative diagram

$$\begin{array}{ccc} H^3(X,\mathbb{Q})/G^3_{X/C} & \xleftarrow{y+G^{2d+1}_{Y/C}\mapsto (p_2\sigma)_*(y)+G^3_{X/C}} & H^{2d+1}(Y,\mathbb{Q})/G^{2d+1}_{Y/C} \\ \| & & \| \\ K_{3X} & \xleftarrow{\left((p_2\sigma)^*|K_{(2d-3)X}\right)^\vee=(p_2\sigma)_*|K_{(2d+1)Y}} & K_{(2d+1)Y} \\ \simeq\uparrow[\alpha_{3X}]^{-1} & & \simeq\uparrow[\alpha_{(2d+1)Y}]^{-1} \\ H^1(C,R^2\pi_*\mathbb{Q}) & \xleftarrow{[(p_2\sigma)_*]_1} & H^1(C,R^{2d}(\tau\sigma)_*\mathbb{Q}). \end{array} \tag{3.7}$$

Finally, there is a commutative diagram

$$\begin{array}{ccc} K_{(2d-3)Y} & \xrightarrow{\smile\,\mathrm{cl}_Y(D^{(1)})^{\smile 2}} & K_{(2d+1)Y} \\ \simeq\downarrow\alpha_{(2d-3)Y} & & \simeq\downarrow\alpha_{(2d+1)Y} \\ H^1(C,R^{2d-4}(\tau\sigma)_*\mathbb{Q}) & \xrightarrow{[\smile\,\mathrm{cl}_Y(D^{(1)})^{\smile 2}]_1} & H^1(C,R^{2d}(\tau\sigma)_*\mathbb{Q}). \end{array} \tag{3.8}$$

For $x \in H^{2d-3}(X,\mathbb{Q})$, the projection formula yields:

$$\begin{aligned}(p_2\sigma)_*\left((p_1\sigma)^*x \smile [\mathrm{cl}_Y(D^{(1)})]^{\smile 2}\right) &= [\mathrm{pr}_2\,\iota\sigma]_*\left([\mathrm{pr}_1\,\iota\sigma]^*x \smile [\mathrm{cl}_Y(D^{(1)})]^{\smile 2}\right)\\ &= \mathrm{pr}_{2*}(\iota\sigma)_*\left((\iota\sigma)^*\,\mathrm{pr}_1^*\,x \smile [\mathrm{cl}_Y(D^{(1)})]^{\smile 2}\right)\\ &= \mathrm{pr}_{2*}\left(\mathrm{pr}_1^*\,x \smile (\iota\sigma)_*\left[[\mathrm{cl}_Y(D^{(1)})]^{\smile 2}\right]\right).\end{aligned}$$

Therefore diagrams (2.16) for $n = 2d-3$, (3.7) - (3.8) show that an algebraic class

$$(\iota\sigma)_*\left[[\mathrm{cl}_Y(D^{(1)})]^{\smile 2}\right] \in H^6(X\times X,\mathbb{Q})$$

determines the isomorphism (3.2), which in virtue of the identification (3.3) takes the form

$$K_{(2d-3)X} \xrightarrow[\sim]{x\mapsto \mathrm{pr}_{2*}\left(\mathrm{pr}_1^*\,x\smile(\iota\sigma)_*\left[[\mathrm{cl}_Y(D^{(1)})]^{\smile 2}\right]\right)} K_{3X}. \tag{3.9}$$

3.4. It follows from the theorem on local invariant cycles and from the strong Lefschetz theorem for fibres of the smooth morphism π' that the $\smile$-multiplication by the class $\mathrm{cl}_X(H)^{\smile d-3}$ yields the equalities

$$\begin{aligned} &\mathrm{cl}_X(H)^{\smile d-3} \smile H^1(C,R^2\pi_*\mathbb{Q})\\ &= \mathrm{cl}_X(H)^{\smile d-3} \smile H^1(C,j_*R^2\pi'_*\mathbb{Q}) = H^1(C,j_*R^{2d-4}\pi'_*\mathbb{Q}) = H^1(C,R^{2d-4}\pi_*\mathbb{Q}),\end{aligned}$$

therefore from (3.3) we get the equality

$$K_{(2d-3)X} = \mathrm{cl}_X(H)^{\smile d-3} \smile K_{3X}. \tag{3.10}$$

Taking into account (3.3), the theorem on locally invariant cycles and the non-degeneracy of the canonical pairing [33, Proposition (10.5)]

$$\begin{aligned} &H^1(C,j_*R^{2d-4}\pi'_*\mathbb{Q}) \times H^1(C,j_*R^2\pi'_*\mathbb{Q})\\ &\xrightarrow{x\times x'\mapsto x\smile x'} H^2(C,j_*R^{2d-2}\pi_*\mathbb{Q}) = H^{2d}(X,\mathbb{Q})\end{aligned}$$

we see, that the canonical pairing $K_{(2d-3)X} \times K_{3X} \xrightarrow{x\times x'\mapsto x\smile x'} H^{2d}(X,\mathbb{Q})$ is non-degenerate. Therefore the restriction of the *non-degenerate* bilinear form

$$\Phi : H^3(X,\mathbb{Q})\times H^3(X,\mathbb{Q}) \xrightarrow{x\times y\mapsto x\smile y\smile \mathrm{cl}_X(H)^{\smile d-3}} H^{2d}(X,\mathbb{Q}) = \mathbb{Q}(-d) \xrightarrow[\sim]{(2\pi i)^d} \mathbb{Q}.$$

to the subspace $K_{3X} \subset H^3(X,\mathbb{Q})$ is non-degenerate. Hence, as it was noticed in Section 1.2, there exists the decomposition of rational Hodge structures

$$H^3(X,\mathbb{Q}) = K_{3X} \oplus K_{3X}^{\perp},$$

where $K_{3X}^{\perp}$ is the orthogonal complement of the subspace $K_{3X} \hookrightarrow H^3(X,\mathbb{Q})$ with respect to the *non-degenerate* bilinear form Φ.

It follows from (3.10) that the subspace

$$\begin{aligned} K_{3X}^{\perp} &= \{x \in H^3(X,\mathbb{Q}) \mid x \smile y \smile \mathrm{cl}_X(H)^{\smile d-3} = 0 \quad \forall y \in K_{3X}\} \\ &= \{x \in H^3(X,\mathbb{Q}) \mid x \smile y = 0 \quad \forall y \in K_{(2d-3)X}\} \end{aligned} \tag{3.11}$$

does not depend on the choice of a divisor H. In particular, the decomposition of rational Hodge structures

$$H^3(X,\mathbb{Q}) = K_{3X} \oplus K_{3X}^{\perp} \tag{3.12}$$

is *canonical.*

Therefore an exact sequence of rational Hodge structures

$$0 \to K_{3X} \to H^3(X,\mathbb{Q}) \to H^0(C, R^3\pi_*\mathbb{Q}) \to 0 \tag{3.13}$$

allows us *canonically* identify the Hodge $\mathbb{Q}$-structure $H^0(C, R^3\pi_*\mathbb{Q})$ with the Hodge substructure $K_{3X}^{\perp} \hookrightarrow H^3(X,\mathbb{Q})$.

Besides, it follows from the strong Lefschetz theorem that the subspace

$$\begin{aligned} &\mathrm{cl}_X(H)^{\smile d-3} \smile K_{3X}^{\perp} \\ &= \mathrm{cl}_X(H)^{\smile d-3} \smile \{x \in H^3(X,\mathbb{Q}) \mid x \smile y \smile \mathrm{cl}_X(H)^{\smile d-3} = 0 \quad \forall y \in K_{3X}\} \\ &= \{x \in H^{2d-3}(X,\mathbb{Q}) \mid x \smile y = 0 \quad \forall y \in K_{3X}\} \end{aligned}$$

does not depend on the choice of a divisor H. Therefore, in accordance with (3.10), (3.12) and the strong Lefschetz theorem, an exact sequence of rational Hodge structures

$$0 \to K_{(2d-3)X} \to H^{2d-3}(X,\mathbb{Q}) \to H^0(C, R^{2d-3}\pi_*\mathbb{Q}) \to 0$$

allows us *canonically* identify the rational Hodge structure $H^0(C, R^{2d-3}\pi_*\mathbb{Q})$ with the Hodge substructure

$$\mathrm{cl}_X(H)^{\smile d-3} \smile K_{3X}^{\perp} = \{x \in H^{2d-3}(X,\mathbb{Q}) \mid K_{3X} \smile x = 0\} \stackrel{\mathrm{def}}{=} K_{(2d-3)X}^{\perp}. \tag{3.14}$$

In particular, there is the *canonical* decomposition of rational Hodge structures

$$H^{2d-3}(X,\mathbb{Q}) = K_{(2d-3)X} \oplus K_{(2d-3)X}^{\perp}. \tag{3.15}$$

In virtue of (3.10) - (3.11) we have:

$$K_{(2d-3)X} \smile K_{3X}^{\perp} = \mathrm{cl}_X(H)^{\smile d-3} \smile K_{3X} \smile K_{3X}^{\perp} = 0. \tag{3.16}$$

3.5. The non-degeneracy of the skew-symmetric form $\Psi \stackrel{\mathrm{def}}{=} \Phi|_{K_{3X}}$ implies the non-degeneracy of the form $\Psi^{\perp} \stackrel{\mathrm{def}}{=} \Phi|_{K_{3X}^{\perp}}$ [21, Ch. IX, § 4, n^0 1, Corollary of Propositon 1], therefore the decomposition (3.12) determines a canonical embedding of algebraic groups

$$\mathrm{Sp}\,(K_{3X},\Psi) \times \mathrm{Sp}\left(K_{3X}^{\perp},\Psi^{\perp}\right) \hookrightarrow \mathrm{Sp}(H^3(X,\mathbb{Q}),\Phi)),$$

which in turn yields the inclusion

$$\mathbb{Q}\cdot\wp(H^3(X,\mathbb{Q}))\ \subset\ \mathbb{Q}\cdot\wp(K_{3X})+\mathbb{Q}\cdot\wp(K_{3X}^{\perp}),$$

because for elements

$$x_i\in K_{3X}\ (i=1,2),\quad x_i'\in K_{3X}^{\perp}\ (i=1,2),\quad \sigma\in\mathrm{Sp}\,(K_{3X},\Psi),\quad \tau\in\mathrm{Sp}\left(K_{3X}^{\perp},\Psi^{\perp}\right)$$

the action of an element $\sigma\times\tau\in\mathrm{Sp}(H^3(X,\mathbb{Q}),\Phi)$ is given by the formulae

$$\begin{aligned}(\sigma\times\tau)(x_1\otimes x_2)&=\sigma x_1\otimes\sigma x_2=\sigma(x_1\otimes x_2),\\ (\sigma\times\tau)(x_1'\otimes x_2')&=\tau x_1'\otimes\tau x_2'=\tau(x_1'\otimes x_2'),\\ (\sigma\times\tau)(x_1\otimes x_1')&=\sigma x_1\otimes\tau x_1',\\ (\sigma\times\tau)(x_1'\otimes x_1)&=\tau x_1'\otimes\sigma x_1\end{aligned}$$

and, in particular, invariants of tensor products

$$K_{3X}\otimes K_{3X}^{\perp},\qquad K_{3X}^{\perp}\otimes K_{3X}$$

with respect to actions of the algebraic group $\mathrm{Sp}\,(K_{3X},\Psi)\,\times\,\mathrm{Sp}\left(K_{3X}^{\perp},\Psi^{\perp}\right)$ are trivial in virtue of the irreducibility of the standard representations of symplectic groups. On the other hand, it was noticed in Section 1.2 that the correspondence $\wp(H^3(X,\mathbb{Q}))$ determines the isomorphism

$$H^{2d-3}(X,\mathbb{Q})\ \xrightarrow[\sim]{x\mapsto \mathrm{pr}_{2*}(\mathrm{pr}_1^*(x)\smile\wp(H^3(X,\mathbb{Q})))}\ H^3(X,\mathbb{Q}).\tag{3.17}$$

Therefore there is the inclusion

$$\mathbb{Q}^{\times}\cdot\wp(H^3(X,\mathbb{Q}))\ \subset\ \mathbb{Q}^{\times}\cdot\wp(K_{3X})+\mathbb{Q}^{\times}\cdot\wp(K_{3X}^{\perp})$$

and, consequently, one may assume that

$$\wp(H^3(X,\mathbb{Q}))=\wp(K_{3X})+\wp(K_{3X}^{\perp}).\tag{3.18}$$

It follows from (3.14) that the correspondence $\wp(K_{3X})$ annihilates the subspace $K_{(2d-3)X}^{\perp}\subset H^{2d-3}(X,\mathbb{Q})$ in the decomposition (3.15). Therefore the restriction of the isomorphism (3.17) to the subspace $K_{(2d-3)X}^{\perp}\ \subset\ H^{2d-3}(X,\mathbb{Q})$ determines the isomorphism

$$K_{(2d-3)X}^{\perp}\ \xrightarrow[\sim]{x\mapsto \mathrm{pr}_{2*}(\mathrm{pr}_1^*(x)\smile\wp(K_{3X}^{\perp}))}\ K_{3X}^{\perp}.\tag{3.19}$$

3.6. By Lemma 3.2 we have the equalities $H^0(C',R^3\pi'_*\mathbb{Q})=H^2(C,R^1\pi_*\mathbb{Q})=0$. On the other hand, the theorem on local invariant cycles and the Leray spectral sequence for the embedding $j\,:\,C'\,\subset\,C$ yields an embedding of mixed Hodge structures ([33], P. 457, Corollary (13.10), Remark (14.5))

$$H^1(C,R^2\pi_*\mathbb{Q})\ \hookrightarrow\ H^1(C',R^2\pi'_*\mathbb{Q}).$$

Consequently, taking into account the equality $H^2(C',R^1\pi'_*\mathbb{Q})=0$ in virtue of ([29, Ch. VI, § 7, Theorem 7.2]), (1.1), (3.3) and arguments of Section 2.5, we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccccc}0\to & H^1(C,R^2\pi_*\mathbb{Q}) & \to & H^3(X,\mathbb{Q}) & \to & H^0(C,R^3\pi_*\mathbb{Q}) & \to 0\\ & \bigcap & & \downarrow^{\varphi_3} & & \downarrow^{\psi_3} & \\ 0\to & H^1(C',R^2\pi'_*\mathbb{Q})\cap\mathrm{Im}(\varphi_3) & \to & \mathrm{Im}(\varphi_3) & \to & 0 & \to 0\end{array}$$

and the corresponding exact sequence of Hodge $\mathbb{Q}$-structures ([13], Section 2.6) of the snake-like diagram ([34], § 1, Proposition 2)

$$(3.20)\quad 0 \to (i_\Delta f)_* H^1(Z,\mathbb{Q}) \to H^0(C, R^3\pi_*\mathbb{Q}) \to \frac{H^1(C', R^2\pi'_*\mathbb{Q}) \cap \operatorname{Im}(\varphi_3)}{H^1(C, R^2\pi_*\mathbb{Q})} \to 0.$$

3.7. Lemma. *There is the equality* $K_{3X}^{\perp} = (i_\Delta f)_* H^1(Z,\mathbb{Q})$.

Proof. First of all we are going to check that

$$(3.21)\qquad (i_\Delta f)_* H^1(Z,\mathbb{Q}) \smile K_{(2d-3)X} = 0.$$

Irreducible components of a smooth variety Z are naturally identified with irreducible components $X_{\delta i}$ of the divisor $\pi^{-1}(\Delta) = \sum_{\delta\in\Delta} X_\delta$. Denote by $\iota_{X_{\delta i}/X} : X_{\delta i} \hookrightarrow X$ the canonical embedding. From the commutativity of the diagram

$$(3.22)\qquad \begin{array}{ccc} X_{\delta i} & \xrightarrow{\iota_{X_{\delta i}/X}} & X \\ \| & & \uparrow{\scriptstyle i_\Delta f} \\ X_{\delta i} & \hookrightarrow & Z \end{array}$$

of canonical morphisms it follows the equality

$$(3.23)\qquad (i_\Delta f)_*|_{H^1(X_{\delta i},\mathbb{Q})} = \iota_{X_{\delta i}/X*},$$

therefore in order to prove the formula (3.21) it suffices to verify that

$$\iota_{X_{\delta i}/X*} H^1(X_{\delta i},\mathbb{Q}) \smile K_{(2d-3)X} = 0.$$

By definition, the operator $\iota_{X_{\delta i}/X*}$ is conjugate to the operator $\iota^*_{X_{\delta i}/X}$ ([2], Section 1.2.A), therefore

$$< \iota_{X_{\delta i}/X*} H^1(X_{\delta i},\mathbb{Q}) \smile K_{(2d-3)X} >=< H^1(X_{\delta i},\mathbb{Q}) \smile \iota^*_{X_{\delta i}/X} K_{(2d-3)X} >$$

and, consequently, it suffices to verify the equality

$$(3.24)\qquad \iota^*_{X_{\delta i}/X} K_{(2d-3)X} = 0.$$

Since $\iota^*_{X_{\delta i}/X} = \iota^*_{X_{\delta i}/X_\delta}\, \iota^*_{X_\delta/X}$ and $\iota^*_{X_\delta/X} K_{(2d-3)X} = 0$ by arguments of Section 2.7, we see that the formulae (3.24) and (3.21) are true.

There is a canonical exact sequence of rational Hodge structures [33, P. 473]

$$0 \to \mathcal{A}_3 \to H^0(C, R^3\pi_*\mathbb{Q}) \to H^0(C, j_*R^3\pi'_*\mathbb{Q}) \to 0,$$

so $H^0(C, R^3\pi_*\mathbb{Q}) = \mathcal{A}_3$ in virtue of Lemma 3.2 and the equality

$$H^0(C, j_*R^3\pi'_*\mathbb{Q}) = H^0(C', R^3\pi'_*\mathbb{Q}).$$

Denote by $D(\delta) \subset C$ a small open disc with the centre at the point $\delta \in \Delta$. Set $X_{D(\delta)} = X \times_C D(\delta)$. Degenerated fibres of the morphism π are unions of *smooth* $(d-1)$-dimensional varieties of multiplicity 1 with normal crossings, therefore in accordance with [33, the 3d row from bottom on P. 473] we have:

$$\mathcal{A}_3 = \oplus_{\delta\in\Delta} \operatorname{Ker}[H^3(X_\delta,\mathbb{Q}) \to R^3\pi'_*\mathbb{Q}],$$

where, in virtue of the theorem on local invariant cycles [33, Proposition (15.12)], for any point s of the punctured disc $D^*(\delta)$ with a centre at δ, the composite

$$H^3(X_\delta,\mathbb{Q}) \xrightarrow{\sim} H^3(X_{D(\delta)},\mathbb{Q}) \to H^3(X_s,\mathbb{Q})$$

of the isomorphism of deformation retraction (determined by the Clemens map of the variety $X_{D(\delta)}$ onto the degenerated fibre X_δ compatible with the radial retraction $D(\delta) \to \{\delta\}$ ([32]; [39, Ch. 5, Section 1.2, Section 3.3]) and of the restriction has as the image the space of cohomology classes invariant under the local monodromy [33, proof of Proposition (15.12)]. In other words, there is a surjective composite $H^3(X_\delta,\mathbb{Q}) \twoheadrightarrow H^3(X_{D(\delta)},\mathbb{Q}) \to H^3(X_s,\mathbb{Q})^{\pi_1(D^*(\delta),s)}$. Thus

$$H^0(C, R^3\pi_*\mathbb{Q}) = \oplus_{\delta\in\Delta}\operatorname{Ker}[H^3(X_\delta,\mathbb{Q}) \to H^3(X_s,\mathbb{Q})^{\pi_1(D^*(\delta),s)}]. \tag{3.25}$$

It follows from (3.21), (3.11) - (3.13) and (3.25) that

$$\begin{aligned}(i_\Delta f)_*\, H^1(Z,\mathbb{Q}) \subset K_{3X}^{\perp} &\twoheadrightarrow H^0(C,R^3\pi_*\mathbb{Q})\\ &= \oplus_{\delta\in\Delta}\operatorname{Ker}[H^3(X_\delta,\mathbb{Q}) \to H^3(X_s,\mathbb{Q})^{\pi_1(D^*(\delta),s)}].\end{aligned} \tag{3.26}$$

Let Z_δ be the normalization of the divisor $\pi^{-1}(\delta) = X_\delta$. It follows from (3.26) that for the proof of the lemma it suffices to prove the equality

$$(i_\Delta f)_*\, H^1(Z_\delta,\mathbb{Q}) = \operatorname{Ker}[H^3(X_\delta,\mathbb{Q}) \to H^3(X_s,\mathbb{Q})^{\pi_1(D^*(\delta),s)}]$$

for any point $\delta\in\Delta$.

It is well known that there is a long exact sequence of cohomology with compact supports [29, Ch. III, § 1, Remark 1.30]

$$\cdots \to H_c^{2d-3}(X_{D(\delta)}\setminus \pi^{-1}(\delta),\mathbb{Q}) \to H_c^{2d-3}(X_{D(\delta)},\mathbb{Q}) \to H_c^{2d-3}(\pi^{-1}(\delta),\mathbb{Q}) \to \ldots,$$

therefore the Poincaré duality [29, Ch. VI, § 11, Corollary 11.2] yields the exact sequence [24, Corollary (8.2.8)]

$$H^1(Z_\delta,\mathbb{Q}) \xrightarrow{(i_\Delta f)_*|_{H^1(Z_\delta,\mathbb{Q})}} H^3(X_{D(\delta)},\mathbb{Q}) \to H^3(X_{D^*(\delta)},\mathbb{Q}). \tag{3.27}$$

Thus the Clemens theory ([32]; [39, Ch. 5, Section 1.2, Section 3.3]), the theorem on local invariant cycles [33, Proof of Proposition (15.12)] and (3.27) yield the equalities

$$\begin{aligned}(i_\Delta f)_*\, H^1(Z_\delta,\mathbb{Q}) &= \operatorname{Ker}[H^3(X_{D(\delta)},\mathbb{Q}) \to H^3(X_{D^*(\delta)},\mathbb{Q})]\\ &= \operatorname{Ker}[H^3(X_\delta,\mathbb{Q}) \to H^3(X_s,\mathbb{Q})^{\pi_1(D^*(\delta),s)}].\end{aligned}$$

Lemma is proved.

3.8. Lemma. *The Poincaré class $\wp(K_{3X}^{\perp})$ is algebraic.*

Proof. According to Lemma 3.7 we have:

$$\wp\left(K_{3X}^{\perp}\right) = \wp\left((i_\Delta f)_* H^1(Z,\mathbb{Q})\right).$$

It follows directly from the definition of the Poincaré class in Section 1.2 that

$$\wp\left((i_\Delta f)_* H^1(Z,\mathbb{Q})\right) \in H^6(X\times X,\mathbb{Q}) \cap H^6(X\times X,\mathbb{C})^{3,3} \stackrel{\text{def}}{=} H^6(X\times X,\mathbb{Q})^{3,3}.$$

On the other hand, the morphism of pure Hodge $\mathbb{Q}$-structures $(i_\Delta f)_*$ has bidegree $(1,1)$ [30, Vol. I, P. 179] and it induces a surjection of pure Hodge structures $H^1(Z,\mathbb{Q}) \to (i_\Delta f)_*\, H^1(Z,\mathbb{Q})$. According to the definition of the Hodge group of rational Hodge structure [19, Definition B.51], this surjection yields a canonical surjection of Hodge groups $\mathrm{Hg}(H^1(Z,\mathbb{Q})) \to \mathrm{Hg}((i_\Delta f)_*\, H^1(Z,\mathbb{Q}))$, so that the $\mathbb{Q}$-space $(i_\Delta f)_*\, H^1(Z,\mathbb{Q})$ has a natural structure of a $\mathrm{Hg}(H^1(Z,\mathbb{Q}))$-module.

In the situation of Section 1.1, there are polarizations $\psi_{H^1(X_{\delta i},\mathbb{Q})}$ and bilinear skew-symmetric $\mathrm{Hg}(H^1(X_{\delta i},\mathbb{Q}))$-invariant non-degenerate forms $\psi^0_{H^1(X_{\delta i},\mathbb{Q})}$ such that the form

$$\psi^0_{H^1(Z,\mathbb{Q})} \stackrel{\text{def}}{=} \sum_{\delta\in\Delta,\ i} \psi^0_{H^1(X_{\delta i},\mathbb{Q})}$$

is a $\mathrm{Hg}(H^1(Z,\mathbb{Q}))$-invariant bilinear skew-symmetric non-degenerate form.

Let $K^{\perp} \subset H^1(Z,\mathbb{Q})$ be the orthogonal complement of the $\mathbb{Q}$-space

$$K \overset{\text{def}}{=} \operatorname{Ker}[H^1(Z,\mathbb{Q}) \to (i_\Delta f)_* \, H^1(Z,\mathbb{Q})]$$

with respect to the form $\psi^0_{H^1(Z,\mathbb{Q})}$. We have the decomposition of $\mathrm{Hg}(H^1(Z,\mathbb{Q}))$-modules $H^1(Z,\mathbb{Q}) = K \oplus K^{\perp}$ and the isomorphism of $\mathrm{Hg}(H^1(Z,\mathbb{Q}))$-modules $K^{\perp} \,\tilde{\to}\, [(i_\Delta f)_* \, H^1(Z,\mathbb{Q})](1)$.

By the properties of Hodge groups discussed in Section 1.1, we have:

$$\begin{aligned}
&[(i_\Delta f)_* \, H^1(Z,\mathbb{Q}) \otimes_{\mathbb{Q}} (i_\Delta f)_* \, H^1(Z,\mathbb{Q})]^{3,3} \\
&= [(i_\Delta f)_* \, H^1(Z,\mathbb{Q}) \otimes_{\mathbb{Q}} (i_\Delta f)_* \, H^1(Z,\mathbb{Q})]^{\mathrm{Hg}((i_\Delta f)_* \, H^1(Z,\mathbb{Q}))} \\
&= [(i_\Delta f)_* \, H^1(Z,\mathbb{Q}) \otimes_{\mathbb{Q}} (i_\Delta f)_* \, H^1(Z,\mathbb{Q})]^{\mathrm{Hg}(H^1(Z,\mathbb{Q}))} \\
&= [[(i_\Delta f)_* \otimes_{\mathbb{Q}} (i_\Delta f)_*] \left(K^{\perp} \otimes_{\mathbb{Q}} K^{\perp}\right)]^{\mathrm{Hg}(H^1(Z,\mathbb{Q}))} \\
&= [(i_\Delta f)_* \otimes_{\mathbb{Q}} (i_\Delta f)_*] \left([K^{\perp} \otimes_{\mathbb{Q}} K^{\perp}]^{\mathrm{Hg}(H^1(Z,\mathbb{Q}))}\right) \\
&\hookrightarrow [(i_\Delta f)_* \otimes_{\mathbb{Q}} (i_\Delta f)_*] \left([H^1(Z,\mathbb{Q}) \otimes_{\mathbb{Q}} H^1(Z,\mathbb{Q})]^{\mathrm{Hg}(H^1(Z,\mathbb{Q}))}\right) \\
&= [(i_\Delta f)_* \otimes_{\mathbb{Q}} (i_\Delta f)_*] \left([H^1(Z,\mathbb{Q}) \otimes_{\mathbb{Q}} H^1(Z,\mathbb{Q})] \cap H^{1,1}(Z \times Z,\mathbb{C})\right).
\end{aligned}$$

Taking into account that, by Lefschetz' theorem on divisors, there is an inclusion

$$[H^1(Z,\mathbb{Q}) \otimes_{\mathbb{Q}} H^1(Z,\mathbb{Q})] \cap H^{1,1}(Z \times Z,\mathbb{C}) \hookrightarrow \mathrm{NS}_{\mathbb{Q}}(Z \times Z),$$

we see that the Poincaré class

$$\wp\left((i_\Delta f)_* H^1(Z,\mathbb{Q})\right) \in [(i_\Delta f)_* \, H^1(Z,\mathbb{Q}) \otimes_{\mathbb{Q}} (i_\Delta f)_* \, H^1(Z,\mathbb{Q})]^{3,3}$$

is algebraic. Lemma is proved.

3.9. Denote by $\Delta^{\text{multiplicative}}$ the set of all places $\delta \in \Delta$ such that the Abelian variety X_η has a totally degenerated reduction of multiplicative type at the place δ (in other words, the toric rank r_δ equals $d-1$). Since, for such places δ, the fibre X_δ is a union of smooth $(d-1)$-dimensional varieties $X_{\delta i}$ of multiplicity 1 with normal crossings, and the variety $X_{\delta i}$ is the closure of the torus Gm^{d-1} in the Zariski topology of the fibre X_δ, then $H^1(X_{\delta i},\mathbb{Q}) = 0$ and, consequently, $H^1(Z_\delta,\mathbb{Q}) = 0$. In particular, Lemma 3.7 yields the equality

$$K^{\perp}_{3X} = \sum_{\delta \in \Delta \setminus \Delta^{\text{multiplicative}}} (i_\Delta f)_* \, H^1(Z_\delta,\mathbb{Q}).$$

Therefore it follows from (3.3) and (3.12) that there exists the canonical decomposition of rational Hodge structures

$$H^3(X,\mathbb{Q}) = H^1(C, R^2\pi_*\mathbb{Q}) \bigoplus \sum_{\delta \in \Delta \setminus \Delta^{\text{multiplicative}}} (i_\Delta f)_* \, H^1(Z_\delta,\mathbb{Q}). \tag{3.28}$$

§ 4. A proof of the theorem

4.1. For any point $\delta \in \Delta = \{\delta \in C \mid r_\delta > 0\}$ set

$$m_\delta \overset{\text{def}}{=} \operatorname{Card}(\mathcal{M}_\delta/\mathcal{M}^0_\delta),$$
$$m \overset{\text{def}}{=} \prod_{\delta \in \Delta} m_\delta.$$

Fix a prime number p, *which does not divide* the number m. Denote by $p^{m!}_{X/C} : X - \to X$ a *rational* map, coinciding on the generic scheme fibre X_η of the structure morphism $\pi : X \to C$ with the isogeny of the multiplication by the number $p^{m!}$.

In virtue of the universal property of the Néron model [15, (1.1.2)] there is the canonical isomorphism

$$\mathrm{End}_C(\mathcal{M}) \widetilde{\to} \mathrm{End}_{\kappa(\eta)}(X_\eta). \tag{4.1}$$

Consider a commutative diagram

$$\begin{array}{ccc} \widetilde{X} & & \\ \downarrow^{\sigma} & \searrow^{\nu} & \\ X & -\xrightarrow{p^{m!}_{X/C}} & X \end{array} \tag{4.2}$$

of a resolution of indeterminacies of the rational map $p^{m!}_{X/C}$. By Hironaka's results and by (4.1) we may assume that the morphism σ is the composite of monoidal transformations with non-singular centres, and $\sigma|_{\sigma^{-1}(\mathcal{M})} : \sigma^{-1}(\mathcal{M}) \to \mathcal{M}$ is the identity morphism.

Let

$$[p^{m!}_{X/C}]^* : H^*(X, \mathbb{Q}) \xrightarrow{x \mapsto \sigma_* \nu^*(x)} H^*(X, \mathbb{Q})$$

be the linear operator determined by the diagram (4.2).

4.2. Lemma. *The linear operator* $[p^{m!}_{X/C}]^* : H^3(X, \mathbb{Q}) \to H^3(X, \mathbb{Q})$ *preserves the decomposition*

$$H^3(X, \mathbb{Q}) = H^1(C, R^2\pi_*\mathbb{Q}) \bigoplus \sum_{\delta \in \Delta \setminus \Delta^{\mathrm{multiplicative}}} (i_\Delta f)_* H^1(Z_\delta, \mathbb{Q}),$$

and it induces the multiplication by the number $p^{2m!}$ *in the space*

$$H^1(C, R^2\pi_*\mathbb{Q}) = \mathrm{Ker}[H^3(X, \mathbb{Q}) \to H^0(C, R^3\pi_*\mathbb{Q})]$$

and the multiplication by the number $p^{m!}$ *in the space*

$$\sum_{\delta \in \Delta \setminus \Delta^{\mathrm{multiplicative}}} (i_\Delta f)_* H^1(Z_\delta, \mathbb{Q}).$$

Proof. In virtue of the theorem on local invariant cycles there is the canonical identification

$$H^1(C, R^2\pi_*\mathbb{Q}) = H^1(C, j_*R^2\pi'_*\mathbb{Q}); \tag{4.3}$$

similarly, for the structure morphism $\widetilde{\pi} : \widetilde{X} \to C$, we obtain the canonical identification

$$H^1(C, R^2\widetilde{\pi}_*\mathbb{Q}) = H^1(C, j_*R^2\widetilde{\pi}'_*\mathbb{Q}). \tag{4.4}$$

The commutative diagram of rational maps (4.2) yields the commutative diagram of C'-morphisms

$$\begin{array}{ccc} \widetilde{X}' & & \\ \simeq\downarrow^{\sigma'} & \searrow^{\nu'} & \\ X' & \xrightarrow{p^{m!}_{X'/C'}} & X', \end{array}$$

so that, for any open subset $U \subset C'$, one can determine the commutative diagram

$$\begin{array}{ccc} H^2((\widetilde{\pi}')^{-1}(U),\mathbb{Q}) & & \\ \simeq\uparrow^{\sigma'^*} & \nwarrow^{\nu'^*} & \\ H^2((\pi')^{-1}(U),\mathbb{Q}) & \xleftarrow{[p^{m!}_{X'/C'}]^*} & H^2((\pi')^{-1}(U),\mathbb{Q}), \end{array}$$

which in turn yields the commutative diagram of local systems

$$\begin{array}{ccc} R^2\,\widetilde{\pi}'_*\mathbb{Q} & & \\ \simeq\uparrow^{\sigma'^*} & \nwarrow^{\nu'^*} & \\ R^2\pi'_*\mathbb{Q} & \xleftarrow{[p^{m!}_{X'/C'}]^*} & R^2\pi'_*\mathbb{Q}. \end{array} \tag{4.5}$$

For any fibre X_s of the Abelian scheme $\pi' : X' \to C'$, the isogeny of the multiplication by the number $p^{m!}$ induces the multiplication by $p^{m!}$ in the space $H^1(X_s,\mathbb{Q})$ [2, Lemma 2A3, Section 2A11]. Consequently, the canonical map in the bottom row of the diagram (4.5) is the multiplication by the number $p^{2m!}$ in virtue of the equality $R^2\pi'_*\mathbb{Q} = \wedge^2 R^1\pi'_*\mathbb{Q}$, so that

$$\nu'^* = p^{2m!}\,\sigma'^*. \tag{4.6}$$

It is evident that the isomorphism $\sigma'^* : R^2\pi'_*\mathbb{Q} \xrightarrow{\sim} R^2\,\widetilde{\pi}'_*\mathbb{Q}$ determines the isomorphism

$$j_*R^2\pi'_*\mathbb{Q} \xrightarrow{\sim} j_*R^2\,\widetilde{\pi}'_*\mathbb{Q}$$

and the corresponding isomorphism of cohomology

$$H^1(C, j_*R^2\pi'_*\mathbb{Q}) \xrightarrow{\sim} H^1(C, j_*R^2\,\widetilde{\pi}'_*\mathbb{Q}),$$

which in virtue of (4.3) - (4.4) views as $\sigma^* : H^1(C, R^2\pi_*\mathbb{Q}) \xrightarrow{\sim} H^1(C, R^2\widetilde{\pi}_*\mathbb{Q})$. On the other hand, the map $\nu'^* : R^2\pi'_*\mathbb{Q} \to R^2\,\widetilde{\pi}'_*\mathbb{Q}$ determines the map

$$j_*R^2\pi'_*\mathbb{Q} \to j_*R^2\,\widetilde{\pi}'_*\mathbb{Q}$$

and the corresponding map of cohomology $H^1(C, j_*R^2\pi'_*\mathbb{Q}) \to H^1(C, j_*R^2\,\widetilde{\pi}'_*\mathbb{Q})$, which in virtue of (4.3) - (4.4) and (4.6) views as

$$\nu^* = p^{2m!}\,\sigma^* : H^1(C, R^2\pi_*\mathbb{Q}) \to H^1(C, R^2\widetilde{\pi}_*\mathbb{Q}).$$

The well known formula $\sigma_*\sigma^* = \mathrm{id}_{H^3(X,\mathbb{Q})}$ shows that the linear operator

$$[p^{m!}_{X/C}]^* = \sigma_*\nu^* : H^3(X,\mathbb{Q}) \to H^3(X,\mathbb{Q})$$

induces the multiplication by the number $p^{2m!}$ in the subspace $H^1(C, R^2\pi_*\mathbb{Q}) \subset H^3(X,\mathbb{Q})$.

It is known that, for any smooth complex projective variety W, the Picard variety

$$\mathrm{Pic}^0(W) = H^{0,1}(W,\mathbb{C})/\operatorname{Im}[H^1(W,\mathbb{Z}) \to H^{0,1}(W,\mathbb{C})]$$

and the Albanese variety dual to it ([40, Ch. II, § 3]; [41, Ch. 2, the end of § 6]; [42, P. 171 - 172])

$$\mathrm{Alb}(W) = H^0(W,\Omega^1_W)^\vee/(H_1(W,\mathbb{Z})/\operatorname{tors})$$

are stable under a transition to a smooth projective variety birationally equivalent to the variety W, because a monoidal transformation $f : W' \to W$ with a centre at a Zariski closed smooth subvariety $D \hookrightarrow W$ determines the canonical isomorphism of Hodge structures $H^1(W,\mathbb{Z}) \underset{\sim}{\xrightarrow{f^*}} H^1(W',\mathbb{Z})$ [42, Proposition 13.1] and a birational map of projective non-singular complex varieties is the composite of projective blow-ups and projective contractions with smooth centres [43, Theorem 0.1.1].

Let $\delta \in \Delta \setminus \Delta^{\text{multiplicative}}$. Then the toric rank r_δ belongs to the set $\{1, \dots, d-1\}$. The fibre X_δ is a union of smooth $(d-1)$-dimensional varieties $X_{\delta i}$ of multiplicity 1 with normal crossings, and the variety $X_{\delta i}$ is the closure of the irreducible component $\mathcal{M}_{\delta i}$ of the algebraic group $\mathcal{M}_\delta$ in the Zariski topology of the fibre X_δ. On the other hand,

$$\mathcal{M}_{\delta i} = a_{\delta i} \mathcal{M}_\delta^0$$

for some element $a_{\delta i} \in \mathcal{M}_{\delta i}$, therefore the variety $\mathcal{M}_{\delta i}$ is isomorphic to the connected component $\mathcal{M}_\delta^0$ of the neutral element of the group $\mathcal{M}_\delta$, which is included into an exact sequence of algebraic groups

$$1 \to \mathrm{Gm}^{r_\delta} \to \mathcal{M}_\delta^0 \xrightarrow{f_\delta} A_\delta \to 0, \tag{4.7}$$

where A_δ is an Abelian variety of a strictly positive dimension.

From now on we denote by $\mathrm{alb}_{\delta i} : X_{\delta i} \to \mathrm{Alb}(X_{\delta i})$ the Albanese map, which is determined uniquely up to a translation on the Abelian variety $\mathrm{Alb}(X_{\delta i})$ [40, Ch. II, § 3, Theorem 11].

One may assume that $X_{\delta 1} = \overline{\mathcal{M}_\delta^0}$. It is known that the canonical rational map $F_\delta : X_{\delta 1} - \to A_\delta$, determined by the extension (4.7), in reality is regular [40, Ch. II, § 1, Theorem 2]. It follows from (4.7) that fibres of the morphism f_δ are isomorphic to the torus Gm^{r_δ}. Therefore, for any morphism $\Phi : X_{\delta 1} \to A$ into an arbitrary Abelian variety A, the restriction of the morphism Φ to any fibre of the morphism f_δ is a constant map [40, Ch. II, § 1, P. 25, Corollary]. Thus the morphism $\Phi|_{\mathcal{M}_\delta^0}$ is decomposed as $\mathcal{M}_\delta^0 \xrightarrow{f_\delta} A_\delta \to A$ and, consequently, the morphism Φ is the composite $\overline{\mathcal{M}_\delta^0} \xrightarrow{F_\delta} A_\delta \to A$. Therefore $\mathrm{Alb}(X_{\delta 1}) = A_\delta$. Since the Albanese variety is stable under a transition to a birationally equivalent variety $X_{\delta i}$ of the variety $X_{\delta 1}$, then

$$\forall i \quad \mathrm{Alb}(X_{\delta i}) = A_\delta. \tag{4.8}$$

Let X^{sm} be the set of all points $x \in X$ such that the structure morphism π is smooth at x. It is evident that the special fibre X_δ^{sm} is a disjoint union of semi-Abelian schemes $\mathcal{M}_{\delta i}$ which are isomorphic to the variety $\mathcal{M}_\delta^0$ [17, Section 4.4]. Since reductions are stable in Grothendieck's sense, then, for any ramified covering $\widetilde{C} \to C$, the connected component of the neutral element of the special fibre $\mathcal{M}_\delta$ of the Néron model $\mathcal{M} \to C$ is isomorphic to the connected component of the neutral element of the special fibre $\widetilde{\mathcal{M}}_\delta$ of the Néron model $\widetilde{\mathcal{M}} \to \widetilde{C}$ [15, Corollaries 3.3, 3.9]; in particular, all irreducible components $\widetilde{X}_{\delta j}$ of the special fibre of the Künnemann compactification of the Néron model $\widetilde{\mathcal{M}} \to \widetilde{C}$ are birationally equivalent to the variety $X_{\delta i}$. Consequently, the equalities (4.8) are preserved under the base change $\widetilde{C} \to C$.

The canonical surjection $f_\delta : \mathcal{M}_\delta^0 \to A_\delta$ admits a prolongation to a *surjective morphism* $F_\delta : X_{\delta 1} \to A_\delta$, which in turn yields an injection $F_\delta^* : H^1(A_\delta, \mathbb{Q}) \hookrightarrow H^1(X_{\delta 1}, \mathbb{Q})$ [2, Proposition 1.2.4], so that in reality it follows from (4.8) that the injection F_δ^* is an *isomorphism*

$$H^1(A_\delta, \mathbb{Q}) \xrightarrow{\sim} H^1(X_{\delta 1}, \mathbb{Q}). \tag{4.9}$$

Since the space $H^1(X_{\delta i},\mathbb{Q})$ is stable under a transition to a variety $X_{\delta 1}$ birationally equivalent to the variety $X_{\delta i}$, then (4.8) - (4.9) and the exact sequence

$$0\to H^1(\mathrm{Alb}(X_{\delta i}),\mathbb{Q})\to H^1(X_{\delta i},\mathbb{Q})\to H^0(X_{\delta i},R^1\,\mathrm{alb}_{\delta i*}\,\mathbb{Q}),$$

determined by the Leray spectral sequence

$$E_2^{p,q}(\mathrm{alb}_{\delta i})=H^p(\mathrm{Alb}(X_{\delta i}),R^q\,\mathrm{alb}_{\delta i*}\,\mathbb{Q})$$

of the Albanese map $\mathrm{alb}_{\delta i}$, yield the canonical isomorphism of rational Hodge structures

$$\mathrm{alb}_{\delta i}^*:H^1(\mathrm{Alb}(X_{\delta i}),\mathbb{Q})\,\tilde{\to}\,H^1(X_{\delta i},\mathbb{Q}). \tag{4.10}$$

Finitely, the map $\mathrm{alb}_{\delta i}$ is included into the commutative diagram of rational maps

$$\begin{array}{ccc} X_{\delta 1} & \xrightarrow{\mathrm{alb}_{\delta 1}} & \mathrm{Alb}(X_{\delta 1}) \\ \uparrow^{\times a_{\delta i}^{-1}} & & \simeq\uparrow^{(\mathrm{alb}_{\delta 1}[\times a_{\delta i}^{-1}])_*} \\ | & & \\ X_{\delta i} & \xrightarrow{\mathrm{alb}_{\delta i}} & \mathrm{Alb}(X_{\delta i}), \end{array} \tag{4.11}$$

where for all $x_{\delta i}\in\mathcal{M}_{\delta i}$ we have:

$$\mathrm{alb}_{\delta 1}[\times a_{\delta i}^{-1}](x_{\delta i})=(\mathrm{alb}_{\delta 1}[\times a_{\delta i}^{-1}])_*\,\mathrm{alb}_{\delta i}(x_{\delta i})+c_{\delta 1},$$

$c_{\delta 1}\in\mathrm{Alb}(X_{\delta 1})$, $(\mathrm{alb}_{\delta 1}[\times a_{\delta i}^{-1}])_*$ is the canonical *isomorphism* of Abelian varieties [40, Ch. II, § 3, P. 41], determined in virtue of the universal property of the Albanese map by a *regular* [40, Ch. II, § 1, Theorem 2] morphism

$$\mathrm{alb}_{\delta 1}[\times a_{\delta i}^{-1}]:X_{\delta i}\to\mathrm{Alb}(X_{\delta 1}).$$

From (4.11) we obtain the commutative diagram of isomorphisms

$$\begin{array}{ccc} H^1(X_{\delta 1},\mathbb{Q}) & \xleftarrow[\sim]{\mathrm{alb}_{\delta 1}^*} & H^1(\mathrm{Alb}(X_{\delta 1}),\mathbb{Q}) \\ \simeq\downarrow^{[\times a_{\delta i}^{-1}]^*} & & \simeq\downarrow^{((\mathrm{alb}_{\delta 1}[\times a_{\delta i}^{-1}])_*)^*} \\ H^1(X_{\delta i},\mathbb{Q}) & \xleftarrow[\sim]{\mathrm{alb}_{\delta i}^*} & H^1(\mathrm{Alb}(X_{\delta i}),\mathbb{Q}), \end{array} \tag{4.12}$$

which determines the isomorphism $[\times a_{\delta i}^{-1}]^*$ and it is independent of a choice of a point $c_{\delta 1}$ (because the translation $z\mapsto z+c_{\delta 1}$ on the Albanese variety $\mathrm{Alb}(X_{\delta 1})$ acts identically on the space $H^1(\mathrm{Alb}(X_{\delta 1}),\mathbb{C})$, generated by differentials $dz_i,\overline{dz_i}$ $(i=1,\dots,\dim_{\mathbb{C}}\mathrm{Alb}(X_{\delta 1}))$ in terms of some lattice $L_{\delta 1}\hookrightarrow\mathbb{C}^{\dim_{\mathbb{C}}\mathrm{Alb}(X_{\delta 1})}$ and canonical global coordinates z_i $(i=1,\dots,\dim_{\mathbb{C}}\mathrm{Alb}(X_{\delta 1}))$ on the variety $\mathbb{C}^{\dim_{\mathbb{C}}\mathrm{Alb}(X_{\delta 1})}/L_{\delta 1}=\mathrm{Alb}(X_{\delta 1})$).

Consider the commutative diagrams with exact rows

$$\begin{array}{ccccccccc} 1\to & \mathrm{Gm}^{r_\delta} & \to & \mathcal{M}_\delta^0 & \to & A_\delta & \to 0 \\ & \downarrow^{p_{X/C}^{m!}|_{\mathrm{Gm}^{r_\delta}}} & & \downarrow^{p_{X/C}^{m!}|_{\mathcal{M}_\delta^0}} & & \downarrow^{\times p^{m!}} & \\ 1\to & \mathrm{Gm}^{r_\delta} & \to & \mathcal{M}_\delta^0 & \to & A_\delta & \to 0; \end{array} \tag{4.13}$$

$$\begin{array}{ccccccc} 1\to & \mathcal{M}_\delta^0 & \to & \mathcal{M}_\delta & \to & \mathcal{M}_\delta/\mathcal{M}_\delta^0 & \to 0 \\ & \downarrow^{p_{X/C}^{m!}|_{\mathcal{M}_\delta^0}} & & \downarrow^{p_{X/C}^{m!}|_{\mathcal{M}_\delta}} & & \downarrow^{\times p^{m!}} & \\ 1\to & \mathcal{M}_\delta^0 & \to & \mathcal{M}_\delta & \to & \mathcal{M}_\delta/\mathcal{M}_\delta^0 & \to 0, \end{array} \tag{4.14}$$

where $\mathcal{M}_\delta/\mathcal{M}_\delta^0$ is a finite group (of order m_δ) of *connected components* of the algebraic group $\mathcal{M}_\delta$ [15, Section (1.1.5)]. The evident surjectivity of canonical maps $p_{X/C}^{m!}|_{\mathrm{Gm}^{r_\delta}}$, $A_\delta\xrightarrow{\times p^{m!}}A_\delta$ and the corresponding to the diagram (4.13) exact sequence of the snake-like diagram [34, § 1, Proposition 2] show that the canonical map $p^{m!}_{X/C}|_{\mathcal{M}^0_\delta}$ is surjective. On the other hand, the multiplication by the invertible in the ring $\mathbb{Z}/m_\delta\mathbb{Z}$ element $p \mod m_\delta$ yields a permutation of elements of the finite group $\mathcal{M}_\delta/\mathcal{M}^0_\delta$. Consequently, by Lagrange's theorem, the multiplication by the element $p^{m!} \mod m_\delta$ is the *identity bijection* of the set $\mathcal{M}_\delta/\mathcal{M}^0_\delta$. Therefore from the commutativity of the diagram (4.14), from the exactness of the corresponding sequence of the snake-like diagram and from the surjectivity of the morphism $p^{m!}_{X/C}|_{\mathcal{M}^0_\delta}$ it follows that the morphism $p^{m!}_{X/C}|_{\mathcal{M}_\delta}$ is surjective, and

$$(\forall i) \quad p^{m!}_{X/C}(\mathcal{M}_{\delta i}) = \mathcal{M}_{\delta i}. \tag{4.15}$$

Consequently, there exists a commutative diagram

$$\begin{array}{ccccc} & & W_{\delta i} & & \\ & & \downarrow^{\sigma_{\delta i}} & \searrow^{\nu_{\delta i}} & \\ \overline{\mathcal{M}_{\delta i}} & = & X_{\delta i} & -\xrightarrow{p^{m!}_{X/C}|_{X_{\delta i}}} & X_{\delta i}, \end{array} \tag{4.16}$$

of a resolution of indeterminacies of the rational map $p^{m!}_{X/C}|_{X_{\delta i}}$, where the morphism $\sigma_{\delta i}$ is the composite of monoidal transformations with non-singular centres, lying over the variety $X_{\delta i} \setminus \mathcal{M}_\delta$ [43, Theorem 0.1.1].

For any element $x_{\delta 1} \in \mathcal{M}_{\delta 1}$ we have:

$$a_{\delta i}x_{\delta 1} \in \mathcal{M}_{\delta i};$$

$$p^{m!}_{X/C}|_{X_{\delta i}}(a_{\delta i}x_{\delta 1}) = (a_{\delta i}x_{\delta 1})^{p^{m!}} = a_{\delta i}^{p^{m!}} x_{\delta 1}^{p^{m!}} = p^{m!}_{X/C}|_{X_{\delta i}}(a_{\delta i}) p^{m!}_{X/C}|_{X_{\delta 1}}(x_{\delta 1}) \in \mathcal{M}_{\delta i},$$

and it follows from (4.15) that

$$p^{m!}_{X/C}|_{X_{\delta i}}(a_{\delta i}) = a_{\delta i}b_{\delta 1}$$

for some element $b_{\delta 1} \in \mathcal{M}_{\delta 1}$. Consequently, in accordance with (4.8), the diagram (4.16) is extendable to the commutative diagram of rational maps

$$\begin{array}{ccccc} & & W_{\delta i} & & \\ & & \downarrow^{\sigma_{\delta i}} & \searrow^{\nu_{\delta i}} & \\ \overline{\mathcal{M}_{\delta i}} & = & X_{\delta i} & -\xrightarrow{p^{m!}_{X/C}|_{X_{\delta i}}} & X_{\delta i} \\ & & \Big| \downarrow^{\times a_{\delta i}^{-1}} & & \Big| \downarrow^{\times a_{\delta i}^{-1} b_{\delta 1}^{-1}} \\ \overline{\mathcal{M}_{\delta 1}} & = & X_{\delta 1} & -\xrightarrow{p^{m!}_{X/C}|_{X_{\delta 1}}} & X_{\delta 1} \\ & & \downarrow^{\mathrm{alb}_{\delta 1}} & & \downarrow^{\mathrm{alb}_{\delta 1}} \\ & & \mathrm{Alb}(X_{\delta 1}) & & \mathrm{Alb}(X_{\delta 1}) \\ & & || & & || \\ & & A_\delta & \xrightarrow{z \mapsto p^{m!}z} & A_\delta, \end{array}$$

which in virtue of (4.8), (4.10) and (4.12) yields the commutative diagram of *isomorphisms* of rational Hodge structures

$$
\begin{array}{ccccc}
 & & H^1(W_{\delta i},\mathbb{Q}) & & \\
 & & \uparrow^{[\sigma_{\delta i}]^*} & \nwarrow^{[\nu_{\delta i}]^*} & \\
H^1(\overline{\mathcal{M}_{\delta i}},\mathbb{Q}) & = & H^1(X_{\delta i},\mathbb{Q}) & \xleftarrow{[p^{m!}_{X/C}|_{X_{\delta i}}]^*} & H^1(X_{\delta i},\mathbb{Q}) \\
 & & \uparrow^{[\times\, a_{\delta i}^{-1}]^*} & & \uparrow^{[\times\, a_{\delta i}^{-1} b_{\delta 1}^{-1}]^*} \\
H^1(\overline{\mathcal{M}_{\delta 1}},\mathbb{Q}) & = & H^1(X_{\delta 1},\mathbb{Q}) & \xleftarrow{[p^{m!}_{X/C}|_{X_{\delta 1}}]^*} & H^1(X_{\delta 1},\mathbb{Q}) \\
 & & \uparrow^{\mathrm{alb}^*_{\delta 1}} & & \uparrow^{\mathrm{alb}^*_{\delta 1}} \\
 & & H^1(A_\delta,\mathbb{Q}) & \xleftarrow{\times\, p^{m!}} & H^1(A_\delta,\mathbb{Q}).
\end{array}
\tag{4.17}
$$

Consider the *morphism* $\varphi_{\delta i}: X_{\delta i} \to A_\delta$, which is the composite of rational maps

$$X_{\delta i} \dashrightarrow^{\times\, a_{\delta i}^{-1}} X_{\delta 1} \xrightarrow{\mathrm{alb}_{\delta 1}} A_\delta .$$

It is evident that the composite of rational maps

$$X_{\delta i} \dashrightarrow^{\times\, a_{\delta i}^{-1} b_{\delta 1}^{-1}} X_{\delta 1} \xrightarrow{\mathrm{alb}_{\delta 1}} A_\delta$$

is a morphism and it takes the form

$$X_{\delta i} \xrightarrow{\varphi_{\delta i}} A_\delta \xrightarrow{z \mapsto z - f_\delta(b_{\delta 1})} A_\delta ,$$

where the map f_δ is determined by the exact sequence (4.7) of algebraic groups. On the other hand, using the commentary to the formula (4.12), it is easy to see that the translation $A_\delta \xrightarrow{z \mapsto z - f_\delta(b_{\delta 1})} A_\delta$ induces the *identity map* of the space $H^1(A_\delta,\mathbb{Q})$, therefore it follows from the commutativity of the diagram (4.17) and from (4.8) - (4.9) that there exists the commutative diagram

$$
\begin{array}{ccc}
H^1(W_{\delta i},\mathbb{Q}) & & \\
\simeq\uparrow^{[\sigma_{\delta i}]^*} & \nwarrow^{[\nu_{\delta i}]^*} & \\
H^1(X_{\delta i},\mathbb{Q}) & & H^1(X_{\delta i},\mathbb{Q}) \\
\simeq\uparrow^{\varphi^*_{\delta i}} & & \simeq\uparrow^{\varphi^*_{\delta i}} \\
H^1(A_\delta,\mathbb{Q}) & \xleftarrow{\times\, p^{m!}} & H^1(A_\delta,\mathbb{Q}),
\end{array}
$$

so that $[\nu_{\delta i}]^*|_{H^1(X_{\delta i},\mathbb{Q})} = p^{m!}\,[\sigma_{\delta i}]^*|_{H^1(X_{\delta i},\mathbb{Q})}$. Consequently,

$$[\sigma_{\delta i}]_*[\nu_{\delta i}]^*|_{H^1(X_{\delta i},\mathbb{Q})} = p^{m!}\,[\sigma_{\delta i}]_*[\sigma_{\delta i}]^*|_{H^1(X_{\delta i},\mathbb{Q})} = p^{m!}\,|_{H^1(X_{\delta i},\mathbb{Q})} \tag{4.18}$$

in virtue of the well-known equality $[\sigma_{\delta i}]_*[\sigma_{\delta i}]^*|_{H^1(X_{\delta i},\mathbb{Q})} = \mathrm{id}_{H^1(X_{\delta i},\mathbb{Q})}$.

It is well known that there is a long exact sequence of cohomology with compact supports [29, Ch. III, § 1, Remark 1.30]

$$\cdots \to H^{2d-3}_{\mathrm{c}}(X \setminus \pi^{-1}(\Delta),\mathbb{Q}) \to H^{2d-3}_{\mathrm{c}}(X,\mathbb{Q}) \to H^{2d-3}_{\mathrm{c}}(\pi^{-1}(\Delta),\mathbb{Q}) \to \dots ,$$

therefore the Poincaré duality [29, Ch. VI, § 11, Corollary 11.2] yields the exact sequence [24, Corollary (8.2.8)]

$$\oplus_{\delta\in\Delta,\ i=1,\dots,m_\delta} H^1(X_{\delta i},\mathbb{Q}) = H^1(Z,\mathbb{Q}) \xrightarrow{(i_\Delta f)_*} H^3(X,\mathbb{Q}) \to H^3(X',\mathbb{Q}). \tag{4.19}$$

From now on we identify the divisor $X_{\delta i}$ with the zero section of a normal bundle $\mathcal{N}_{X_{\delta i}/X}$. There is a class $t \in H^2(\mathcal{N}_{X_{\delta i}/X}, \mathcal{N}_{X_{\delta i}/X} \setminus X_{\delta i},\mathbb{Q})$, called *the Thom class* of the normal bundle $\mathcal{N}_{X_{\delta i}/X}$, characterized by the property that it restricts to the chosen generator of the 1-dimensional space $H^2([\mathcal{N}_{X_{\delta i}/X}]_x, [\mathcal{N}_{X_{\delta i}/X}]_x \setminus x, \mathbb{Q})$ for all $x \in X_{\delta i}$. It determines *the Thom isomorphism*

$$H^k(X_{\delta i}, \mathbb{Q}) \widetilde{\rightarrow} H^{k+2}(\mathcal{N}_{X_{\delta i}/X}, \mathcal{N}_{X_{\delta i}/X} \setminus X_{\delta i}, \mathbb{Q}),$$

which is given by $\alpha \mapsto \alpha \smile t$ ([44], P. 3) via the identification of cohomology $H^*(X_{\delta i}, \mathbb{Q}) \widetilde{\rightarrow} H^*(\mathcal{N}_{X_{\delta i}/X}, \mathbb{Q})$ determined by the canonical map $\mathcal{N}_{X_{\delta i}/X} \to X_{\delta i}$.

It is known that, for the embedding $\iota_{X_{\delta i}/X} : X_{\delta i} \hookrightarrow X$ of a divisor $X_{\delta i}$, the *Gysin map* of cohomology $\iota_{X_{\delta i}/X*} : H^k(X_{\delta i}, \mathbb{Q}) \to H^{k+2}(X, \mathbb{Q})$ is the composite ([44], P. 11, Section (iv))

$$H^k(X_{\delta i}, \mathbb{Q}) \xrightarrow[\sim]{\smile t} H^{k+2}(\mathcal{N}_{X_{\delta i}/X}, \mathcal{N}_{X_{\delta i}/X} \setminus X_{\delta i}, \mathbb{Q})$$
$$= H^{k+2}(X, X \setminus X_{\delta i}, \mathbb{Q}) \to H^{k+2}(X, \mathbb{Q}).$$

On the other hand, the image of the element $1 \in H^0(X_{\delta i}, \mathbb{Q})$ with respect to the Gysin map $H^0(X_{\delta i}, \mathbb{Q}) \to H^2(X, \mathbb{Q})$ coincides with the cohomology class $\mathrm{cl}_X(X_{\delta i}) \in H^2(X, \mathbb{Q})$, so one may identify the Thom class of the normal bundle $\mathcal{N}_{X_{\delta i}/X}$ with the class $\mathrm{cl}_X(X_{\delta i})$ [29, Ch. VI, § 6]. Therefore the Gysin map $\iota_{X_{\delta i}/X*}$ is defined by the formula [45, Definition 14]

$$\alpha \mapsto \alpha \smile \mathrm{cl}_X(X_{\delta i}). \tag{4.20}$$

In accordance with (4.18) we have the equalities of operators

$$[p^{m!}_{X/C}|_{X_{\delta i}}]^*|_{H^1(X_{\delta i}, \mathbb{Q})} = [\sigma_{\delta i}]_*[\nu_{\delta i}]^*|_{H^1(X_{\delta i}, \mathbb{Q})} = p^{m!}|_{H^1(X_{\delta i}, \mathbb{Q})}. \tag{4.21}$$

On the other hand, returning to the isomorphism (4.1), we claim that *the morphism* $p^{m!}_{X/C_{|\mathcal{M}}} : \mathcal{M} \to \mathcal{M}$ is étale ([29], Ch. I, § 3, Theorem 3.20), because it is non-ramified ([29], Ch. I, § 3, Proposition 3.2) in virtue of *the smoothness* of its *finite* fibres, which are group schemes over the field of characteristic zero ([46], Lecture 25, Theorem 1), and the canonical morphism of rings $\mathcal{O}_{\mathcal{M}, p^{m!}_{X/C_{|\mathcal{M}}}(y)} \to \mathcal{O}_{\mathcal{M},y}$ is injective for all $y \in \mathcal{M}$, because it is included into the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{M}, p^{m!}_{X/C_{|\mathcal{M}}}(y)} & \to & \mathcal{O}_{\mathcal{M},y} \\ \bigcap & & \bigcap \\ \kappa(\mathcal{M}) & \hookrightarrow & \kappa(\mathcal{M}), \end{array}$$

where $\kappa(\mathcal{M})$ is the field of rational functions on the variety $\mathcal{M}$ and the inclusion of fields $\kappa(\mathcal{M}) \hookrightarrow \kappa(\mathcal{M})$ is determined by the dominant morphism

$$p^{m!}_{X/C_{|\mathcal{M}}} : \mathcal{M} \to \mathcal{M}.$$

As a result, we get from (4.15) the equalities of *smooth* divisors

$$[p^{m!}_{X/C_{|\mathcal{M}}}]^{-1}(\mathcal{M}_{\delta i}) = [p^{m!}_{X/C_{|\mathcal{M}}}]^*(\mathcal{M}_{\delta i}) = \mathcal{M}_{\delta i}. \tag{4.22}$$

It is evident that there is the decomposition of the groups of divisors

$$\mathrm{Div}(\widetilde{X}) = \sigma^*(\mathrm{Div}(X)) \oplus \mathrm{Ker}(\sigma_*), \tag{4.23}$$

where the group $\mathrm{Ker}(\sigma_*)$ is generated by divisors, which are contractible by the morphism σ, and the map $\sigma_* \sigma^*$ is the identity [47, Ch. III, § 3, Section 3.5].

Since the codimension of the set of points of indeterminacy of the rational map $p^{m!}_{X/C}$ is greater than 1, then it follows from the definition of the strict preimage

$X_{\delta i}^{\text{strict}}$ of the divisor $X_{\delta i} = \overline{\mathcal{M}_{\delta i}}$ [47, Ch. III, § 3, Section 3.3] and from (4.22), (4.23) that

$$\nu^*(X_{\delta i}) \in X_{\delta i}^{\text{strict}} + \operatorname{Ker}(\sigma_*);$$

$$[p_{X/C}^{m!}]^*(X_{\delta i}) = \sigma_*\nu^*(X_{\delta i}) = \sigma_*(X_{\delta i}^{\text{strict}}) = X_{\delta i}.$$

Hence one has the equality

$$[p_{X/C}^{m!}]^*|_{H^2(X,\mathbb{Q})}(\text{cl}_X(X_{\delta i})) = \text{cl}_X(X_{\delta i}),$$

therefore it follows from (4.20), (4.21) and from the functoriality of constructions under consideration that, for any element $\alpha \in H^1(X_{\delta i}, \mathbb{Q})$, there are the equalities

$$\begin{aligned} [p_{X/C}^{m!}]^*(\iota_{X_{\delta i}/X*}(\alpha)) &= [p_{X/C}^{m!}]^*|_{H^3(X,\mathbb{Q})}(\alpha \smile \text{cl}_X(X_{\delta i})) \\ &= [p_{X/C}^{m!}|_{X_{\delta i}}]^*|_{H^1(X_{\delta i},\mathbb{Q})}(\alpha) \smile [p_{X/C}^{m!}]^*|_{H^2(X,\mathbb{Q})}(\text{cl}_X(X_{\delta i})) \\ &= p^{m!}\,\alpha \smile \text{cl}_X(X_{\delta i}) = p^{m!}\iota_{X_{\delta i}/X*}(\alpha). \end{aligned} \tag{4.24}$$

Consequently, the lemma follows from (3.23), (4.19) and (4.24).

4.3. By definition of the direct image of cohomology we have [38, Formula (1.2)]:

$$\text{pr}_{2*}(H^i(X,\mathbb{Q}) \otimes_{\mathbb{Q}} H^*(X,\mathbb{Q})) = 0 \quad \text{for all} \quad i \neq 2d. \tag{4.25}$$

Let $u_{3,3}, u_{3,3^\perp}, u_{3^\perp,3}, u_{3^\perp,3^\perp}, h$ be the components of the algebraic correspondence $u = (\iota\sigma)_*\left[[\text{cl}_Y(D^{(1)})]^{\smile 2}\right]$ in direct summands

$$K_{3X} \otimes_{\mathbb{Q}} K_{3X};\ \ldots\ ; K_{3X}^\perp \otimes_{\mathbb{Q}} K_{3X}^\perp;$$

$$H \overset{\text{def}}{=} \oplus_{p+q=6,\ p\neq 3} H^p(X,\mathbb{Q}) \otimes_{\mathbb{Q}} H^q(X,\mathbb{Q})$$

of the Künneth decomposition of the space $H^6(X \times X, \mathbb{Q})$.

4.4. Acting on the algebraic class u by the operator

$$[p_{X/C}^{m!}]^* \otimes_{\mathbb{Q}} [p_{X/C}^{m!}]^* = \sigma_*\nu^* \otimes_{\mathbb{Q}} \sigma_*\nu^*,$$

we obtain in virtue of Lemma 4.2 the *algebraic* class

$$([p_{X/C}^{m!}]^* \otimes_{\mathbb{Q}} [p_{X/C}^{m!}]^*)(u) = p^{4\cdot m!}\cdot u_{3,3} + p^{3\cdot m!}\cdot u_{3,3^\perp} + p^{3\cdot m!}\cdot u_{3^\perp,3} + p^{2\cdot m!}\cdot u_{3^\perp,3^\perp} + h_1,$$

where $h_1 = ([p_{X/C}^{m!}]^* \otimes_{\mathbb{Q}} [p_{X/C}^{m!}]^*)(h) \in H$ (because the operator

$$\sigma_*\nu^* \otimes_{\mathbb{Q}} \sigma_*\nu^* = (\sigma \times \sigma)_*(\nu \times \nu)^*$$

transforms the subspace $H \subset H^6(X \times X, \mathbb{Q})$ into H and it transforms algebraic classes into algebraic cohomology classes [2, Proposition 1.3.7].

Subtracting from this class the element $p^{3\cdot m!}\cdot u$, we obtain an algebraic class $(p^{4\cdot m!} - p^{3\cdot m!})\cdot u_{3,3} + (p^{2\cdot m!} - p^{3\cdot m!})\cdot u_{3^\perp,3^\perp} + (h_1 - p^{3\cdot m!}\cdot h)$. Thus, for some element $h_2 \in H$, the class $p^{m!}\cdot u_{3,3} - u_{3^\perp,3^\perp} + h_2$ is algebraic. It is evident that, for some element $h_3 \in H$, the class

$$([p_{X/C}^{m!}]^* \otimes_{\mathbb{Q}} [p_{X/C}^{m!}]^*)(p^{m!}\cdot u_{3,3} - u_{3^\perp,3^\perp} + h_2) = p^{5\cdot m!}\cdot u_{3,3} - p^{2\cdot m!}\cdot u_{3^\perp,3^\perp} + h_3$$

is algebraic. Subtracting from it the algebraic class

$$p^{4\cdot m!}(p^{m!}\cdot u_{3,3} - u_{3^\perp,3^\perp} + h_2) = p^{5\cdot m!}\cdot u_{3,3} - p^{4\cdot m!}\cdot u_{3^\perp,3^\perp} + p^{4\cdot m!}\cdot h_2,$$

we obtain an algebraic class $(p^{4\cdot m!} - p^{2\cdot m!})\cdot u_{3^\perp,3^\perp} + (h_3 - p^{4\cdot m!}\cdot h_2)$. Therefore, for some element $h_4 \in H$, the class $u_{3^\perp,3^\perp} + h_4$ is algebraic. Consequently, for some element $h_5 \in H$, the class $u_{3,3} + h_5$ is algebraic. As a result, for some element

$h_6 \in H$, the class $u_{3,3^\perp} + u_{3^\perp,3} + h_6$ is algebraic. Acting on this class by the operator $[p^{m!}_{X/C}]^* \otimes_{\mathbb{Q}} [1^*_{X/C}]$, we obtain an algebraic class of the form

$$p^{2\cdot m!} \cdot u_{3,3^\perp} + p^{m!} \cdot u_{3^\perp,3} + h_7 \quad (h_7 \in H),$$

therefore, for some elements $h_8, h_9 \in H$, the classes $u_{3,3^\perp} + h_8$ and $u_{3^\perp,3} + h_9$ are algebraic. Thus, for some element $h_{10} \in H$, the class $u_{3,3} + u_{3,3^\perp} + h_{10}$ is *algebraic.*

Note that the correspondence $u_{3^\perp,3} + u_{3^\perp,3^\perp}$ annihilates the space $K_{(2d-3)X}$ in virtue of (3.16); besides, elements $h, h_{10} \in H$ annihilate this space according to the formula (4.25). Therefore in virtue of (3.9) the algebraic correspondence $u_{3,3} + u_{3,3^\perp} + h_{10}$ yields an isomorphism

$$K_{(2d-3)X} \xrightarrow[\sim]{x \mapsto \mathrm{pr}_{2*}(\mathrm{pr}_1^* x \smile (u_{3,3}+u_{3,3^\perp}+h_{10}))} K_{3X}, \tag{4.26}$$

and by (3.14) and (4.25) it is obvious that the correspondence $u_{3,3} + u_{3,3^\perp} + h_{10}$ annihilates the subspace

$$\mathrm{cl}_X(H)^{\smile d-3} \smile K^\perp_{3X} = K^\perp_{(2d-3)X} \subset H^{2d-3}(X, \mathbb{Q}).$$

On the other hand, by (3.16), (3.18), (3.19) and Lemma 3.8 the *algebraic* correspondence $\wp(K^\perp_{3X})$ annihilates the subspace $K_{(2d-3)X} \subset H^{2d-3}(X, \mathbb{Q})$ and it determines an isomorphism

$$K^\perp_{(2d-3)X} \xrightarrow[\sim]{x \mapsto \mathrm{pr}_{2*}(\mathrm{pr}_1^* x \smile \wp(K^\perp_{3X}))} K^\perp_{3X},$$

therefore by (3.15), (3.17), (3.18) and (4.26) an algebraic correspondence

$$u_{3,3} + u_{3,3^\perp} + h_{10} + \wp(K^\perp_{3X})$$

yields an isomorphism

$$H^{2d-3}(X, \mathbb{Q}) \xrightarrow[\sim]{x \mapsto \mathrm{pr}_{2*}(\mathrm{pr}_1^* x \smile (u_{3,3}+u_{3,3^\perp}+h_{10}+\wp(K^\perp_{3X})))} H^3(X, \mathbb{Q}).$$

The theorem is proved.

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