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**THE TARSKI-LINDENBAUM ALGEBRA OF THE CLASS OF
ALL PRIME STRONGLY CONSTRUCTIVIZABLE MODELS OF
ALGORITHMIC DIMENSION ONE**

M.G. PERETYAT'KIN

ABSTRACT. We study the class of all prime strongly constructivizable models of algorithmic dimension 1 in a fixed finite rich signature. It is proved that the Tarski-Lindenbaum algebra of this class considered together with a Gödel numbering of the sentences is a Boolean Π_3^0 -algebra whose computable ultrafilters form a dense subset in the set of all ultrafilters; moreover, this algebra is universal with respect to the class of Boolean Σ_2^0 -algebras whose computable ultrafilters represent a dense subset in the set of arbitrary ultrafilters in the algebra. This gives a characterization to the Tarski-Lindenbaum algebra of the class of all prime strongly constructivizable models of algorithmic dimension 1 in a fixed finite rich signature.

Keywords: first-order logic, Tarski-Lindenbaum algebra, computable isomorphism, semantic class of models, algorithmic complexity estimate.

Algorithmic complexity estimates of elementary theories of some semantic classes of models have been studied in [1], [2], [3], and [4]. Furthermore, in these works, we initiate investigations on the problem of characterization of the Tarski-Lindenbaum algebras of some most interesting semantic classes of models. In particular, the paper [3] establishes algorithmic complexity of the Tarski-Lindenbaum algebra of the class of all strongly constructive prime models. In this work, we describe the Tarski-Lindenbaum algebra of the class of all strongly constructive prime models of algorithmic dimension 1.

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We consider theories in first-order predicate logic *with equality* and use general concepts of model theory, algorithm theory, constructive models, and Boolean algebras found in [5], [6], and [7]. Generally, *incomplete theories* are considered. In the work, the signatures are considered only, which admit Gödel's numberings of the formulas. Such a signature is called *enumerable*.

A finite signature is called *rich*, if it contains at least one n -ary predicate or function symbol for $n > 1$, or two unary function symbols. The following notations are used: $FL(\sigma)$ is the set of all formulas of signature σ , $SL(\sigma)$ is the set of all sentences (i.e., closed formulas) of signature σ . In the work, we use a finite rich signature σ , and consider a fixed Gödel numbering Φ_i , $i \in \mathbb{N}$, of the set $SL(\sigma)$. A theory F is called *finitely axiomatizable* if it is defined by a finite set of axioms, and its signature is finite. Generally, *incomplete theories* are considered.

Let T be a theory of signature σ . On the set of sentences $SL(\sigma)$, an equivalence relation \sim_T is defined by the rule $\Phi \sim_T \Psi \Leftrightarrow T \vdash (\Phi \leftrightarrow \Psi)$. The logical connectives \vee , $\&$, and \neg generate Boolean operations \cup , \cap , and $-$ on the quotient set $SL(\sigma)/\sim_T$; One can easily check that, these operations are well-defined on the \sim_T -classes. Thereby, we obtain an algebra of the form

$$L(T) = (SL(\sigma)/\sim_T; \cup, \cap, -, \mathbf{0}, \mathbf{1}),$$

that, in fact, is a Boolean algebra. It is called the *Tarski-Lindenbaum algebra* of the theory T . By $\mathcal{L}(T)$, we denote the Tarski-Lindenbaum algebra $L(T)$ considered together with a Gödel numbering γ ; thereby, the concept of a computable isomorphism is applicable to such objects. For a class of models M , we write briefly $\mathcal{L}(M)$ instead of $\mathcal{L}(\text{Th}(M))$.

The set of all finite tuples α of the form $\alpha = \langle \alpha_0, \alpha_1, \dots, \alpha_n \rangle$, $\alpha_i \in \{0, 1\}$, is denoted by $2^{<\omega}$. The empty tuple is denoted by \emptyset . The *canonical (Gödel) index* of a finite tuple of zeros and ones of the form $\varepsilon = \langle \varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1} \rangle$, $\varepsilon_i \in \{0, 1\}$, is the number $\text{Nom}(\varepsilon) = 2^n + \varepsilon_0 2^{n-1} + \varepsilon_1 2^{n-2} + \dots + \varepsilon_{n-1} - 1$. Obviously, the following relation is satisfied:

$$(0.1) \quad \text{Len}(\varepsilon) < \text{Len}(\varepsilon') \Rightarrow \text{Nom}(\varepsilon) < \text{Nom}(\varepsilon'), \text{ for all } \varepsilon, \varepsilon' \in 2^{<\omega}.$$

We often write shortly $\langle \varepsilon \rangle$ instead of $\text{Nom}(\varepsilon)$.

We consider Boolean algebras in the signature $\sigma_{BA} = \{\cup, \cap, -, \mathbf{0}, \mathbf{1}\}$. Besides, we consider two following binary relations, which are first-order definable in the theory of Boolean algebras by formulas $a \subseteq b \Leftrightarrow (a \cap b = a)$, $a \supseteq b \Leftrightarrow (a \cap b = b)$. Let \mathcal{B} be a Boolean algebra, and $a \in \mathcal{B}$. By $\mathcal{B}[a]$, we denote the restriction of the Boolean algebra \mathcal{B} on the set of all subelements of the element $a \in \mathcal{B}$ counting that $\mathbf{1} = a$ and $-x$ is defined as $a \setminus x$ in $\mathcal{B}[a]$. If b is an element of a Boolean algebra and $\alpha \in \{0, 1\}$, then b^α means b for $\alpha = 1$ and $-b$ for $\alpha = 0$. Similarly, if Φ is a formula and $\alpha \in \{0, 1\}$, then Φ^α means Φ for $\alpha = 1$ and $\neg \Phi$ for $\alpha = 0$. We use the notation $\mathcal{P}(A)$ for the power-set of A , and $\text{Card}(A)$ for cardinality of the set A .

A model \mathfrak{N} is said to be strongly constructivizable if it has at least one strong constructivization. The model \mathfrak{N} has algorithmic dimension 1, symbolically

$$\dim_{s.c.}(\mathfrak{N}) = 1,$$

if \mathfrak{N} is strongly constructivizable and any two strong constructivizations ν_1 and ν_2 of \mathfrak{N} are equivalent; i.e, the numerated models (\mathfrak{N}, ν_1) and (\mathfrak{N}, ν_2) are constructively isomorphic, written: $(\mathfrak{N}, \nu_1) \cong (\mathfrak{N}, \nu_2)$. The model \mathfrak{N} has algorithmic dimension ω ,

symbolically $\dim_{s.c.}(\mathfrak{N}) = \omega$, if there is an infinite sequence of strong constructivizations of this model $\nu_i, i < \omega$, such that any two models in the sequence $(\mathfrak{N}, \nu_i), i < \omega$, are not constructively isomorphic. In [8], it is shown that the case of any finite dimension $\dim_{s.c.}(\mathfrak{N}) = n, 1 < n < \omega$, is impossible.

Definition of the concept of a compact binary tree can be found in the work [1, Sec. 2.1]. In the work, we use a more specialized term *compact binary trees* for them. An element n of a compact binary tree \mathcal{D} such that $L(n) \notin \mathcal{D}$ is called a *dead end* of the tree \mathcal{D} . The set of all dead-end elements of a tree \mathcal{D} is denoted by $\text{Dend}(\mathcal{D})$. A tree is called *atomic* if above each of its elements, there is at least one dead-end element. If \mathcal{D} is a compact binary tree, we denote by $\Pi(\mathcal{D})$ the set of all maximal chains, while $\Pi^{fin}(\mathcal{D})$ denotes the set of all maximal finite chains in the tree \mathcal{D} . A notation 'rank' stands for ranks of separate chains in a tree, while 'Rank' means rank of the tree itself, i.e., supremum of ranks of its maximal chains.

Following Rogers, [6], we use the notation W_n for n th computably enumerable set in Post's numbering of the family of all c.e. sets. Moreover, we denote by W_n^t a finite part of the set W_n that can be computed in t steps. By \mathcal{D}_n , we denote the closure of the set W_n up to a compact binary tree; and \mathcal{D}_n^X is the closure of the set W_n^X up to a compact binary tree, where W_n denotes the computably enumerable set with c.e. index n , while W_n^X denotes the computably enumerable set with c.e. index n relative to computability with an oracle $X \subseteq \mathbb{N}$, cf. [6]. It can be easily checked that the tree \mathcal{D}_n is computably enumerable, while the tree \mathcal{D}_n^X is computably enumerable with oracle X ; moreover, each c.e. tree is presented in the sequence $\mathcal{D}_n, n \in \mathbb{N}$, and each c.e. tree in computation with oracle X is presented in the sequence $\mathcal{D}_n^X, n \in \mathbb{N}$. The number n is considered as a c.e. index for the tree \mathcal{D}_n , and n is considered as a c.e. index for the tree \mathcal{D}_n^X considered in computability with an oracle X .

We denote by \mathfrak{P}_i a table condition with the Gödel number $n, n \in \mathbb{N}; A \models \mathfrak{P}_i$, means that the table condition is satisfied in the set $A, A \subseteq \mathbb{N}$. The set

$$\Omega(m) = \{A \subseteq \mathbb{N} \mid (\forall i \in W_m) A \models \mathfrak{P}_i\},$$

is called the *parametric Stone space* with an index m ;

Main statement on the *canonical construction* of finitely axiomatizable theories can be found in [1, Th. 3.1.1]. Its part involved in construction of this work states the following:

Statement 0.1. *Effectively in a pair of integers (m, s) and a finite rich signature σ , it is possible to construct a finitely axiomatizable theory $F = \mathbb{F}\mathbb{C}(m, s, \sigma)$ of signature σ together with an effective sequence $\theta_n, n \in \mathbb{N}$, of sentences of the signature σ such that the family of extensions of F defined by*

$$(0.2) \quad F[A] = F \cup \{\theta_i \mid i \in A\} \cup \{\neg \theta_j \mid j \in \mathbb{N} \setminus A\}, \quad A \subseteq \mathbb{N},$$

satisfies the following properties:

- (A) *for any $A \subseteq \mathbb{N}$, the theory $F[A]$ is either complete or contradictory;*
- (B) *the theory $F[A], A \subseteq \mathbb{N}$, is consistent if and only if $A \in \Omega(m)$;*
- (C) *for an arbitrary $A \in \Omega(m)$, the following statements are satisfied:*
 - (a) *theory $F[A]$ has a prime model if and only if the tree \mathcal{D}_s^A is atomic,*
 - (b) *a prime model of the theory $F[A]$, if it exists, is strongly constructivizable if and only if the set A is computable and the family of chains $\Pi^{fin}(\mathcal{D}_s^A)$ is computable,*

- (c) a prime model of the theory $F[A]$, if it exists and is strongly constructivizable, has algorithmic dimension 1 if and only if the tree \mathcal{D}_s^A is computable; otherwise, the model has algorithmic dimension ω .

Finally, we formulate a technical statement.

Lemma 0.2. *Given a set $X \subseteq \mathbb{N}$ that is considered as an oracle. For an arbitrary numerated Boolean Σ_1^X -algebra (\mathcal{B}, ν) , there is a numeration v of \mathcal{B} such that (\mathcal{B}, v) is a Boolean Σ_1^X -algebra whose computable ultrafilters form a dense set in the set of all ultrafilters of the algebra (\mathcal{B}, v) .*

Proof. See Lemma 2 in [4]. □

1. MAIN CLAIM

Hereafter, we fix a finite rich signature σ . We denote by $P(\sigma)$ the class of all prime models of signature σ , and by $P_{s.c}^1(\sigma)$, the class of all prime strongly constructivizable models of algorithmic dimension 1 signature σ .

Theorem 1.1. *The following assertions hold :*

- (a) $\mathcal{L}(P_{s.c}^1(\sigma))$ is a Boolean Π_3^0 -algebra with respect to its Gödel numbering,
- (b) computable ultrafilters of $\mathcal{L}(P_{s.c}^1(\sigma))$ represent a dense set among arbitrary ultrafilters in the algebra,
- (c) there is a numbering v such that $(\mathcal{L}(P_{s.c}^1(\sigma)), v)$ is a Boolean Σ_2^0 -algebra,
- (d) for an arbitrary Boolean Σ_2^0 -algebra (\mathcal{B}, ν) whose computable ultrafilters represent a dense set in the set of all ultrafilters, there is a sentence Φ of signature σ , such that $(\mathcal{B}, \nu) \cong (\mathcal{L}(\text{Th}(\text{Mod}(\Phi) \cap P_{s.c}^1(\sigma))), \gamma)$, where γ is a Gödel numbering of the sentences of signature σ ,
- (e) for an arbitrary Boolean Σ_2^0 -algebra (\mathcal{B}, ν) , there is a sentence Φ of signature σ , such that $(\mathcal{B}, \nu) \cong (\mathcal{L}(\text{Th}(\text{Mod}(\Phi) \cap P_{s.c}^1(\sigma))), \gamma)$, where γ is a Gödel numbering of the sentences of signature σ .

Proof. Due to Goncharov and Harrington, [9], [10], we have that a prime model \mathfrak{M} of a complete decidable theory T is strongly constructivizable if and only if the family of principal types realized in \mathfrak{M} is computable. Moreover, the prime model \mathfrak{M} has algorithmic dimension 1 if and only if there is a computably enumerable set A presenting the family \mathcal{A} of formulas generating principal types in \mathfrak{M} (by construction, complement of \mathcal{A} is a c.e. set, thus, the set \mathcal{A} itself turns out to be computable).

From this, we obtain that, a sentence Ψ of signature σ has a strongly constructivizable prime model of algorithmic dimension 1 if and only if there exist integers m, n, p , satisfying the following properties:

1. $W_m \cap W_n = \emptyset$ & $W_m \cup W_n = \mathbb{N}$, \forall & $\forall \exists$
2. $T = \{\Phi_i \mid i \in W_m\}$ is a complete theory, $\forall \exists$
3. $\Psi \in T$, \exists
4. $(\forall i \in W_p) \varphi_i(\bar{x})$ is a consistent in T formula, $\forall \exists$
5. $(\forall i \in W_p) \varphi_i(\bar{x})$ is an atomic in T formula, $\forall \exists$
6. $(\forall j) (T \vdash \neg(\exists \bar{x})\varphi_j(\bar{x}) \text{ or } (\exists i \in W_p) [T \vdash \varphi_i(\bar{x}) \rightarrow \varphi_j(\bar{x})])$. $\forall \exists$

Thus, we obtain a prefix $\exists\forall\exists$ for this condition. Finally, sentences Φ and Ψ are equivalent on the class $P_{s.c}^1$ if and only if $(\Phi \& \neg\Psi) \vee (\Psi \& \neg\Phi)$ does not have a model in this class. This gives the required prefix $\forall\exists\forall$ for (a).

Let T be an arbitrary complete theory extending $\text{Th}(P_{s.c}^1)$, and Ψ be a sentence provable in T . Obviously, Ψ has a model $\mathfrak{N} \in P_{s.c}^1$. From this, we have that complete decidable theory $T' = \text{Th}(\mathfrak{N})$ presenting a computable ultrafilter in $\text{St}(\text{Th}(P_{s.c}^1))$ is found in the neighborhood Ψ in the given arbitrary ultrafilter T of the Stone space. This gives the required density property posed in (b).

Part (c) is a corollary of (a) due to [11, Sec.2, Th.1, Th.2].

Consider a numerated Boolean Σ_2^0 -algebra (\mathcal{B}, ν) . By Lemma 0.2, there is a numeration ν' of \mathcal{B} such that (\mathcal{B}, ν') is a Boolean Σ_2^0 -algebra whose computable ultrafilters represent a dense set within the set of all ultrafilters in the algebra. This establishes implication (d) \Rightarrow (e).

Proof of Part (d) is considered in Sections 2-3.

2. A TECHNICAL STATEMENT

By K , we denote a fixed c.e. set, which is not computable. In further consideration, we use the following set

$$(2.1) \quad A_2 = \{n \mid W_n \text{ is infinite}\},$$

which is Π_2^0 -complete, cf. [6, Ch.13, Th.VIII].

In a parallel way, we also use its complement $E_2 = \mathbb{N} \setminus A_2$, which is a Σ_2^0 -complete set.

Lemma 2.1. *There is a total computable operator $\Theta : \mathbb{N} \rightarrow \mathbb{N}$ satisfying the following properties:*

- (a) $\mathcal{D} = \Theta(X)$ is an atomic tree, for all $X \subseteq \mathbb{N}$.
- (b) $\mathcal{D} = \Theta(X)$ is a computably enumerable tree, if the set $X \subseteq \mathbb{N}$ is computable.
- (c) For any computable $X \subseteq \mathbb{N}$, the family $\Pi^{fin}(\mathcal{D})$, $\mathcal{D} = \Theta(X)$, is computable.
- (d) For any computable $X \subseteq \mathbb{N}$, the tree $\mathcal{D} = \Theta(X)$ is computable if and only if $(\forall k)[k \in X \Rightarrow W_n \text{ is infinite}]$.

Proof. Consider a tree \mathcal{D}^* of Rank 3 depicted in Fig. 1. The tree includes the most-right chain $\{R^i(0) \mid i < \omega\}$ having rank 2 together with a sequence of infinite chains $\pi_m^* = \{R^i(0) \mid 0 \leq i < m\} \cup \{R^m L^j(0) \mid j < \omega\}$ each having rank 1. The other chains in the tree \mathcal{D}^* are finite; i.e., they have rank 0.

The tree \mathcal{D}^* has a series of families of dead-end elements

$$(2.2) \quad B_n = \{RL^{i+1}R^n(0) \mid 1 \leq i < \omega\}, \quad n = 0, 1, 2, \dots,$$

playing an important role in our construction.

We define a tree $\Theta(X)$ depending on an input parameter $X \subseteq \mathbb{N}$ as follows:

$$(2.3) \quad \Theta(X) = \mathcal{D}^* \cup \bigcup_{n < \omega} \{L(a), R(a) \mid a = RL^{i+1}R^n(0), i \in H_n\},$$

where

$$(2.4) \quad H_n = \begin{cases} \emptyset, & \text{if } n \notin X, \\ K \cup \{k \mid (\exists t < \omega)[k < \text{Card}(W_n^t)]\}, & \text{if } n \in X. \end{cases}$$

Obviously, the set H_n is computably enumerable uniformly in n , whenever X is a fixed computable set. The following properties are satisfied:

- (2.5) (a) if $n \notin X$, then $H_n = \emptyset$,
- (b) if $n \in X$ & $n \in A_2$, then $H_n = \mathbb{N}$,
- (c) if $n \in X$ & $n \notin A_2$, then $H_n = K \cup \{0, 1, \dots, t\}$, for some $t < \omega$,

Indeed, in the case $n \notin X$, we have $H_n = \emptyset$ by definition. In the case $n \in X$ and $n \in A_2$, the set W_n is infinite ensuring that $\text{Card}(W_n^t)$ gets arbitrarily large values whenever $t \rightarrow \infty$. Thus, we have $H_n = \mathbb{N}$ in this case. In the case $n \in X$ and $n \notin A_2$, the set W_n is finite ensuring that $\text{Card}(W_n^t)$ is limited whenever $t \rightarrow \infty$. Thus, we have in this case $H_n = K \cup Z$, where Z is a finite set of the form $\{0, 1, \dots, t\}$ for some $t < \omega$.

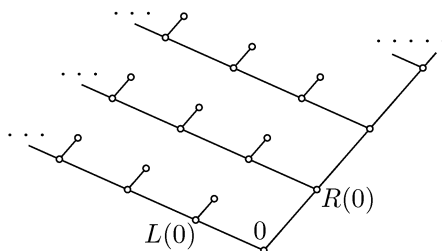


Fig. 1. The tree \mathcal{D}^* , a standard tree of Rank 3

Main aim of the construction is to map each set H_n , cf. (2.4), onto successive elements of family B_n in the tree \mathcal{D}^* . In the case when an element $a \in H_n$ is mapped on a dead-end element $\mathfrak{a} \in B_n$, we add to the tree a pair of new elements $L(\mathfrak{a})$ and $R(\mathfrak{a})$, cf. Fig. 2.

We are going to check requirements of Lemma 2.1.

(a) The tree \mathcal{D} plays the role of a framework in our construction. For some dead-end elements \mathfrak{a} in \mathcal{D}^* , we add their successors $L(\mathfrak{a})$ and $R(\mathfrak{a})$ to the tree. This ensures that the target tree $\Theta(X)$ is atomic.

(b) Immediately, from construction.

(c) The set of elements under the mapping represents a computably enumerable family of dead-end elements \mathfrak{a} of the framework tree \mathcal{D}^* . By construction, we add successors $L(\mathfrak{a})$ and $R(\mathfrak{a})$, to these elements \mathfrak{a} , cf. Fig. 2. From this, we obtain that the family $\Pi^{fin}(\mathcal{D})$ is computable.

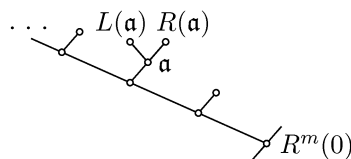


Fig. 2. Addition of a branching to a dead-end element in H_n

(d) Suppose that the set X is computable. In the case when all elements in X are c.e. indices of infinite sets, properties (2.5)(a,b) provide that the set of dead-ends in the tree $\Theta(X)$ is computable, thus, the tree $\mathcal{D} = \Theta(X)$ itself is computable. Now we consider the other case when X contains at least one index n_0 of a finite set. By virtue of (2.5)(c), in this case, set H_{n_0} differs from K just on a finite interval of \mathbb{N} , thus, we obtain reducibility

$$i \in K \setminus \{0, 1, \dots, t\} \Leftrightarrow R^2 L^{i+1} R^{n_0}(0) \in \text{Dend}(\Theta(X))$$

for some finite t . Since K is not computable, we obtain that the set of dead-end elements $\text{Dend}(\Theta(X))$ cannot be computable. Therefore, the tree $\mathcal{D} = \Theta(X)$ itself is not computable. \square

3. PROOF TO PART (D) OF THEOREM 1.1

Given a numerated Boolean Σ_2^0 -algebra (\mathcal{B}, ν) satisfying

$$(3.1) \quad \text{computable ultrafilters of } (\mathcal{B}, \nu) \text{ form a dense set in } \text{St}(\mathcal{B}).$$

We assume, that \mathcal{B} is a nontrivial algebra. By definition, signature operations \cup , \cap and $-$ in \mathcal{B} are presentable by computable functions on ν -numbers, and the equality relation is a Σ_2^0 -relation in numeration ν :

$$\nu(x) = \nu(y) \Leftrightarrow H(x, y), \quad H \in \Sigma_2^0.$$

Consequently, there exists a unary Π_2^0 -relation H^* such that for any finite tuple of zeros and ones $\alpha = \langle \alpha_0, \alpha_1, \dots, \alpha_n \rangle$, we have

$$\nu(0)^{\alpha_0} \cap \nu(1)^{\alpha_1} \cap \dots \cap \nu(n)^{\alpha_n} = \mathbf{0} \Leftrightarrow \langle \alpha_0, \alpha_1, \dots, \alpha_n \rangle \notin H^*,$$

or, equivalently,

$$(3.2) \quad \nu(0)^{\alpha_0} \cap \nu(1)^{\alpha_1} \cap \dots \cap \nu(n)^{\alpha_n} \neq \mathbf{0} \Leftrightarrow \langle \alpha_0, \alpha_1, \dots, \alpha_n \rangle \in H^*, \quad H^* \in \Pi_2^0.$$

Since any Π_2^0 -set is m -reducible to A_2 , there is a general computable function $f(x)$ such that for an arbitrary tuple $\alpha \in 2^{<\omega}$, $\alpha = \langle \alpha_0, \dots, \alpha_n \rangle$, we have

$$(3.3) \quad \nu(0)^{\alpha_0} \cap \nu(1)^{\alpha_1} \cap \dots \cap \nu(n)^{\alpha_n} \neq \mathbf{0} \Leftrightarrow W_{f(\alpha)} \text{ is infinite.}$$

Now, our goal is to choose a pair (m, s) of integer parameters.

Choice of m . We choose m such that $\Omega(m) = \mathcal{P}(\mathbb{N})$ (cf. preliminaries). For this, it is enough to fix any m satisfying $W_m = \emptyset$.

Choice of s . For this purpose, we first describe a computable functional Ψ from $\mathcal{P}(\mathbb{N})$ to $\mathcal{P}(\mathbb{N})$ yielding compact binary trees.

Given a set $A \subseteq \mathbb{N}$. Let $\alpha = \langle \alpha_0, \alpha_1, \dots, \alpha_k, \dots \rangle$ be the characteristic sequence for A ; that is, a sequence of zeros and ones such that the following is satisfied:

$$(3.4) \quad \alpha_k = \begin{cases} 1, & \text{if } k \in A, \\ 0, & \text{if } k \notin A. \end{cases}$$

Taking A as an input parameter, let us construct the following set

$$(3.5) \quad Q_A = \{f(\emptyset), f(\langle \alpha_0 \rangle), \dots, f(\langle \alpha_0, \dots, \alpha_n \rangle), \dots\}.$$

By virtue of (0.1), the set Q_A is computable whenever A is computable. Use the set Q_A as an input parameter X for the construction $X \mapsto \Theta(X)$ described in Lemma 2.1. As a result, we obtain a subset \mathcal{D} of \mathbb{N} , which actually is a tree by Lemma 2.1(a). Thereby, the transformation $A \mapsto \Psi(A)$ is presented by the following rule:

$$(3.6) \quad A \mapsto Q_A \mapsto \Theta(Q_A) = \mathcal{D} = \Psi(A).$$

End of description of the operator Ψ .

Transformation $A \mapsto \Psi(A)$ can be considered as a computation by an algorithm \mathcal{M} working with an oracle A . Let s be a Gödel number of the algorithm \mathcal{M} . In accordance with the basic definitions of algorithm theory (cf. preliminaries), we obtain the following form of the operator Ψ :

$$(3.7) \quad \Psi(A) = \mathcal{D}_s^A, \quad A \subseteq \mathbb{N}.$$

Choice of the pair of parameters (m, s) is finished.

Now, we study main properties of the transformation $\Psi : A \mapsto \mathcal{D}_s^A$.

Lemma 3.1. *The following assertions hold:*

- (a) *For any $A \subseteq \mathbb{N}$, \mathcal{D}_s^A is an atomic tree,*
- (b) *For any computable $A \subseteq \mathbb{N}$, \mathcal{D}_s^A is a computably enumerable tree.*
- (c) *For any computable $A \subseteq \mathbb{N}$, $\Pi^{fin}(\mathcal{D}_s^A)$ is computable.*

Proof. Statement of Part (a) is provided by Lemma 2.1(a), Part (b) is provided by Lemma 2.1(b), while Part (c) is followed from Lemma 2.1(c). \square

Now we immediately pass to the proof of Part (d) of Theorem 1.1.

First of all, we have to point out a sentence Φ of the given finite rich signature σ in accordance with requirements in Part (d) of Theorem 1.1. For this, we use the canonical construction, cf. Statement 0.1. Apply this construction to the pair (m, s) specifying also the signature σ . As a result, we obtain a finitely axiomatizable theory $F = \mathbb{F}c(m, s, \sigma)$ of signature σ . As Φ , we get a conjunction of axioms of the theory F . After that, our principal aim is to show that sentence Φ satisfies all requirements listed in Theorem 1.1(d).

Consider an arbitrary finite tuple of zeros and ones $\alpha = \langle \alpha_0, \dots, \alpha_k \rangle$. Construct an elementary intersection of elements in \mathcal{B} by the rule

$$(3.8) \quad b_\alpha = \nu(0)^{\alpha_0} \cap \nu(1)^{\alpha_1} \cap \dots \cap \nu(k)^{\alpha_k},$$

and, concurrently, an elementary conjunction of corresponding sentences θ_i involved in Statement 0.1 by the rule

$$(3.9) \quad \beta_\alpha = \theta_0^{\alpha_0} \& \theta_1^{\alpha_1} \& \dots \& \theta_k^{\alpha_k}.$$

The key idea of the construction is to establish the following relation:

Lemma 3.2. *For any tuple $\alpha \in 2^{<\omega}$, we have $b_\alpha \neq \mathbf{0}$ if and only if $\Phi \& \beta_\alpha$ has a strongly constructivizable prime model of algorithmic dimension 1.*

Proof. First, we assume that $b_\alpha \neq \mathbf{0}$. By (3.1), computable ultrafilters form a dense set among arbitrary ultrafilters in the Boolean algebra (\mathcal{B}, ν) . From this we obtain that there is an infinite sequence $\alpha^* = \langle \alpha_i \mid i < \omega \rangle$ extending α such that the set A linked with α^* by rule (3.4) is computable, and

$$(3.10) \quad \nu(0)^{\alpha_0} \cap \dots \cap \nu(i)^{\alpha_i} \neq \mathbf{0}, \quad \text{for all } i \in \mathbb{N}.$$

By virtue of (3.3), we obtain that $W_{f(\langle \alpha_0, \dots, \alpha_i \rangle)}$ is infinite for each $i \in \mathbb{N}$; thereby, the set (3.5) consists of indices of infinite sets. By Lemma 2.1(a,b,c), the tree $\Psi(A) = \mathcal{D}_s^A$ defined in (3.6) and (3.7) is an atomic computable tree with a computable family $\Pi^{fin}(\mathcal{D}_s^A)$. By Statement 0.1(A,B,C), theory $F[A]$ is consistent, complete, and has a prime model \mathfrak{N} , which is strongly constructivizable and has algorithmic dimension 1. This ensures that formula $\Phi \& \beta_\alpha$ is satisfied in the model $\mathfrak{N} \in P_{s,c}^1(\sigma)$ since the formula is provable in theory $F[A]$.

Now, we assume that sentence $\Phi \& \beta_\alpha$ has a strongly constructivizable prime model \mathfrak{N} of algorithmic dimension 1. Consider the set

$$(3.11) \quad A = \{\theta_i \mid \mathfrak{N} \models \theta_i\},$$

which is obviously computable. Build an infinite sequence $\alpha^* = \langle \alpha_i \mid i < \omega \rangle$ that is linked with A via the rule (3.4). Since $A \in \Omega(m)$, by Statement 0.1(A,B), theory $F[A]$ is consistent and complete. Moreover, this theory is decidable by Janiczak Theorem since it is computably axiomatizable. By (3.11), all axioms of $F[A]$ are

satisfied in the model \mathfrak{N} . Thereby, we have that A is computable and $F[A]$ has a strongly constructivizable prime model of algorithmic dimension 1. By applying Statement 0.1(C), we conclude that tree \mathcal{D}_s^A is atomic and computable, and its family $\Pi^{fin}(\mathcal{D}_s^A)$ is computable. In accordance with rules (3.6) and (3.7), we have $\mathcal{D}_s^A = \Theta(Q_A)$, where Q_A is defined by rule (3.5). By Lemma 2.1(d), Q_A consists of indices of infinite sets. By applying (3.3), we finally obtain $b_\alpha \neq \mathbf{0}$.

Lemma 3.2 is proved. \square

Let us map elements $\nu(i)$, $i \in \mathbb{N}$, of Boolean algebra \mathcal{B} onto sentences θ_i , $i \in \mathbb{N}$, of signature σ by the rule:

$$(3.12) \quad \lambda^*(\nu(k)) = \theta_k, \quad k \in \mathbb{N}.$$

Now, we will extend the partial mapping (3.12) up to a computable isomorphism of the algebras under consideration. Namely, we define a mapping

$$\lambda : \mathcal{B} \rightarrow \mathcal{L}(\text{Th}(\text{Mod}(\Phi) \cap P_{s.c}^1(\sigma)))$$

by the rule: for an arbitrary finite sequence of tuples $\alpha_0, \alpha_1, \dots, \alpha_n \in 2^{<\omega}$, we put

$$(3.13) \quad \lambda(b_{\alpha_0} \cup b_{\alpha_1} \cup \dots \cup b_{\alpha_n}) = \beta_{\alpha_0} \cup \beta_{\alpha_1} \cup \dots \cup \beta_{\alpha_n}.$$

The mapping λ is total on the algebra \mathcal{B} since the set of expressions involved in (3.13) concerns all elements of this algebra. Moreover, the relation stated in Lemma 3.2 allows us to omit zero terms. Therefore, μ is a one-to-one correspondence. Boolean operations are produced by the same rules with respect to the unions of elements b_α of the form (3.8) and disjunctions of elementary conjunctions β_α of the form (3.9). Therefore, λ is an isomorphism between the Boolean algebras. Obviously, it is a computable isomorphism with respect to numerations ν and γ ensuring finally the required computable isomorphism

$$\lambda : (\mathcal{B}, \nu) \rightarrow (\mathcal{L}(\text{Th}(\text{Mod}(\Phi) \cap P_{s.c}^1(\sigma))), \gamma).$$

Thereby, Part (d) of Theorem 1.1 is completely proved. \square

4. FINAL REMARKS

Theorem 1.1 gives an answer to Question 10 in [2, p.237]. The Tarski-Lindenbaum algebra $\mathcal{L}(P_{s.c}^1)$ is characterized in terms of second and third levels of arithmetic hierarchy. It has less algorithmic complexity in comparison with other results in this direction; for instance, cf. [3] and [4]. A common scheme of the proof in this work may be useful in solving similar problems with respect to other semantic classes of sentences.

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MIKHAIL G. PERETYATKIN
INSTITUTE OF MATHEMATICS AND MATHEMATICAL MODELING,
125, PUSHKIN STR.,
ALMATY, 050010, KAZAKHSTAN
E-mail address: peretyatkin@math.kz