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TEMPORAL LOGIC WITH OVERLAP TEMPORAL RELATIONS GENERATED BY TIME STATES THEMSELVES

V.V. RYBAKOV

Abstract. We study a temporal logic with non–standard temporal accessibility relations. This logic is generated by semantic underground models, and any such a model has a base formed by a frame with temporal relations generated by temporal states themselves; potentially, any state possesses its own temporal accessibility relation, and it is possible that all of them can be different. We consider this to be the most plausible modelling, because any time state has, in principle, its own view on what is past (or future). Time relations may have non–empty overlaps and they can be totally intransitive. Thus, this approach may be suitable for analysis of the most general cases of reasoning about computation, information flows, reliability, and other areas of AI and CS. The main mathematical question under consideration here is the existence of algorithms for solving satisfiability problems. Here we solve this problem and find the required algorithms. In the final part of our paper we formulate some interesting open problems.

Keywords: temporal logic, non–classical logics, information, knowledge representation, deciding algorithms, computability, information, satisfiability, decidability.

1. Introduction

This paper combines the following two issues: (1) is the pure mathematical one; it consists of construction of mathematical models for time flow and transition of

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information based on Kripke–Hintikka–like relational models and logic syntaxes — formulas and other syntactical instruments (to analyze the laws and properties of such models), and (2) which consists of possible applications to information transfer, correctness proof for reasonings, plausibility, extracting hidden infirmation, consistency, and reliability of knowledge as well as to other areas of AI and CS. The concept of knowledge, which at first glance may look like a kind of a stable, correct, profoundly verified (and supported?) information, is in the centre of studies in CS and philosophy. This and the related areas use instruments of temporal logic for representation and development of computational tools.

Probably, the concept of knowledge in terms of symbolic logic, appeared at about the end of 1950. In 1962 Hintikka has written a book *Knowledge and Belief*, the first book–length work to suggest the use of modalities to capture the semantics of knowledge. This book sets much of the groundwork for the subject, but since that time a great deal of research has taken place. Temporal logic since then became a popular area in mathematical symbolic logic and CS; a lot of impressive results where obtained (for historical outlook see Gabbay, Hodkinson, Reynolds [5, 6], Goldblat [7], Goranko [8], van Benthem [28], Yde Venema [31]).

Since the invention of the linear temporal logic $\mathcal{LTL}$ with operation U — until — by Amir Pnueli that system was studied from many viewpoints due to interesting mathematical representations and useful applications to analysis of protocols for computations and verification of consistency. Automata technique to solve satisfiability in this logic was developed by Vardi [29, 30]). Among reasonably modern results concerning this logic I would mention the solution to the admissibility problem for $\mathcal{LTL}$ in Rybakov [14, 15]; the basis for admissible rules of $\mathcal{LTL}$ was obtained by Babenyshev and Rybakov in [3]. The unification problem for $\mathcal{LTL}$ was solved in [19]. As for applications of logical methods in AI and CS, the tools developed around the temporal logic work well for analysis in multi–agent environment (see, for example [16, 17]).

Up to now, the temporal logic was investigated from many viewpoints. In particular, the extensions of $\mathcal{LTL}$ for the case of non–transitive models, were studied in Rybakov [20, 21, 25] for the case of the interval versions of the logic. Also the modelling of multi–agent reasoning via temporal models was applied in Rybakov [18, 22, 24] for some versions of liner logic.

This paper is devoted to the study of an important modification of $\mathcal{LTL}$, a logic based on non–transitive time with possible time overlaps on temporal accessibility relations; so it is intransitive by its nature. But the most innovative part here is that the temporal relations on the generating models are individual for any time state. This looks as a quite new approach which was not touched upon yet in literature and as the most plausible one for real simulations of time runs. As it was already mentioned above, the non–transitive temporal logics generated over linear time were actively studied. I was trying to resolve the most general case, when the temporal relations may be unbounded, and in addition, not all of them are placed into infinite sequences of fixed intervals of time, since this restriction looked to be a little bit artificial, but my attempt was unsuccessful. Here we found the solution via a new approach which successfully packs all the restrictions into a single new one and enables us to resolve the problem. The main mathematical problem we study is the existence of algorithms for solving the satisfiability problems. We solve this problem and offer algorithms based on a reduction of the problem to special models of computable sizes. At the end of the paper we formulate some interesting open problems.

2. Logical Language and Models with Overlap Relations

As we have noted above, the most innovative point of the paper is the use of temporal accessibility relations separately, so to say individually, for any temporal state. This looks like here we have infinite number of accessibility relations, and it seems that we will need an infinite set of temporal operations in the logical language. But, in fact, it is not the case, and we can model this approach in the usual temporal language.

Our logical language consists of potentially infinite set of propositional letters P, Boolean logical operations, operation $\mathcal{N}$ (next), and the operation U (until). The formation rules for compound formulas are standard: any letter from P is a formula; the set of all formulas is closed with respect to the applications of Boolean logical operations, the unary operation $\mathcal{N}$ (next), and the binary operation U (until); here $\varphi U \psi$ means that φ holds until ψ is true, $\mathcal{N}\varphi$ means that φ is true in the next temporal state.

To model the temporal flow we will use new modified Kripke–Hintikka–like models based on linear order on the natural numbers.

Definition 1. *A linear temporal non-transitive frame is a tuple*

$$\mathcal{F} := \langle N, \{R_x \mid x \in N\}, Nxt\rangle, \ \textit{such that}$$

for all $x \in N$, R_x is a linear order on the interval $[x, a_x]$ for some $a_x \geq x, a_x \in N$ (R_a can also be a linear order on the whole interval $[a, \infty]$), and for all $x, y \in N$ holds $x \ Nxt \ y \Leftrightarrow y = x + 1$.

It may happen that xR_xa_x, $y \in (x, a_x)$, and $\neg(yR_ya_x)$. So to say, y is a state situated earlier than x but y remembers even less than x. Besides, it is clear that all the relations form together a non–transitive relation: it may happen that xR_xa_x, $x < y < a_x$, so (xR_xy), (yR_ya_y) but $\neg(xR_ya_y)$.

A model $\mathcal{M}$ on any $\mathcal{F}$ is defined by introduction a valuation V on $\mathcal{F}$: for a set of propositional letters p we let $V(p) \subseteq N$, and V is extended to all the formulas as follows:

Definition 2. *For any $a \in N$ we put*

$$(\mathcal{F}, a) \Vdash_V p \quad \Leftrightarrow \quad p \in V(p);$$

$$(\mathcal{F}, a) \Vdash_V \neg\varphi \quad \Leftrightarrow \quad (\mathcal{F}, a) \nVdash_V \varphi;$$

$$(\mathcal{F}, a) \Vdash_V (\varphi \ \wedge \ \psi) \quad \Leftrightarrow \quad ((\mathcal{F}, a) \Vdash_V \varphi) \wedge ((\mathcal{F}, a), \Vdash_V \psi);$$

$$(\mathcal{F}, a) \Vdash_V (\varphi \ \vee \ \psi) \quad \Leftrightarrow \quad ((\mathcal{F}, a) \Vdash_V \varphi) \vee ((\mathcal{F}, a) \Vdash_V \psi);$$

$$(\mathcal{F}, a) \Vdash_V (\varphi \ \rightarrow \ \psi) \quad \Leftrightarrow \quad ((\mathcal{F}, a) \Vdash_V \psi) \vee ((\mathcal{F}, a) \nVdash_V \varphi);$$

for formulas of kind $\varphi U \psi$, we define the truth values as follows:

$$(\mathcal{F}, c) \Vdash_V (\varphi \ U \ \psi) \quad \Leftrightarrow$$

$$\exists b \in N[(cR_cb) \wedge ((\mathcal{F}, b) \Vdash_V \psi) \wedge$$

$$\forall y[(c \leq y, \& y < b) \Rightarrow (\mathcal{F}, y) \Vdash_V \varphi]];$$

$$(\mathcal{F}, a) \Vdash_V \mathcal{N}\varphi \quad \Leftrightarrow \quad [(a \ Nxt \ b) \Rightarrow (\mathcal{F}, b) \Vdash_V \varphi].$$

The notation $(\mathcal{F}, a) \Vdash_V \varphi$ means that *the formula* φ *is true (valid)* at the state a with respect to the valuation V. We see that the truth of any formula with main temporal operation U at a state a refers to the unique accessibility relation R_a, for a only. Sometimes, we will use notation $Nxt(a) = b$ or $Next(a) = b$ to say that $a\ Nxt\ b$.

Definition 3. *The logic* T_L^{Ov} *is the set of all formulas which are valid at any state of any model based on any linear temporal non–transitive frame* $\mathcal{F}$.

General illustrations of the ideas why time flow may be considered to be non–transitive and how such an approach might be used are given in Rybakov [20, 21, 22, 24, 25].

3. A Technique via Reduced Forms

Our aim is to show that the satisfiability problem for the introduced logic is decidable. The usual technique based on filtration, the usage of temporal degrees of formulas and dropping points fail to work for this semantics since the relations are generally are non–transitive and the rules for computation of truth values of formulas with U differ from the standard ones. We will use a modification of our old technique for reduction of formulas to rules (which we have already used earlier many times for different purposes (cf. e. g. [17, 15]) and a transformation of the latter ones to the so–called reduced forms. We now briefly recall this technique.

A *rule* is an expression $\mathbf{r} := \varphi_1(x_1, \ldots, x_n), \ldots, \varphi_s(x_1, \ldots, x_n) \ / \ \psi(x_1, \ldots, x_n)$, where all $\varphi_k(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ are formulas constructed from letters (variables) $x_1, \ldots, x_n$.

Formulas $\varphi_k(x_1, \ldots, x_n)$ are called *premises* and $\psi(x_1, \ldots, x_n)$ is called the *conclusion*. The rule $\mathbf{r}$ means that $\psi(x_1, \ldots, x_n)$ (conclusion) follows (logically follows) from the assumptions $\varphi_1(x_1, \ldots, x_n), \ldots, \varphi_s(x_1, \ldots, x_n)$. The definition of the validness of a rule is the same for any relational model. To recall it, assume that a model $\mathcal{M}$ and a rule $\mathbf{r}$ are given.

Definition 4. *A rule* $\mathbf{r} := \varphi_1(x_1, \ldots, x_n), \ldots, \varphi_s(x_1, \ldots, x_n) \ / \ \psi(x_1, \ldots, x_n)$, *is valid in the model* $\mathcal{M}$ *based at a frame* $\mathcal{F}$ *iff*

$$\left[\forall a \left((\mathcal{F}, a) \Vdash_V \textstyle\bigwedge_{1 \le i \le s} \varphi_i\right)\right] \ \Rightarrow \ \left[\forall a \left((\mathcal{F}, a) \Vdash_V \psi\right)\right].$$

If $\forall a \left((\mathcal{F}, a) \Vdash_V \bigwedge_{1 \le i \le s} \varphi_i\right)$ *but* $\exists a \left((\mathcal{F}, a) \nVdash_V \psi\right)$, *then we say that* $\mathbf{r}$ *is refuted in* $\mathcal{F}$ *by* V *and we denote this fact as* $\mathcal{F} \nVdash_V \mathbf{r}$.

Definition 5. *A rule* $\mathbf{r}$ *is valid (or true) on a frame* $\mathcal{F}$ *iff* $\mathbf{r}$ *is true (valid) in any model based on* $\mathcal{F}$.

Definition 6. *A formula* φ *is satisfiable iff there is a frame* $\mathcal{F}$ *and a valuation* V *on* $\mathcal{F}$ *such that* φ *is true w.r.t.* V *at some state from* $\mathcal{F}$.

Lemma 1. *For a formula* φ, φ *is satisfiable iff the rule* $x \to x/\neg\varphi$ *may be refuted in some model* $\mathcal{M}$.

The follows immediately from the definitions. Thus we have

Lemma 2. *If there is an algorithm verifying for any given rule* r *if this rule is valid in all models* $\mathcal{M}$ *then there exists an algorithm verifying if any given formula is satisfiable.*

Now we need the rules in some uniform simple form, in particular, without nested temporal operations.

Definition 7. *A rule* $\mathbf{r}$ *is said to be in reduced normal form if* $\mathbf{r} = \varepsilon/x_1$ *where*

$$\varepsilon = \bigvee_{1\leq j\leq m} \Big[\bigwedge_{1\leq i\leq n} x_i^{t(j,i,0)} \wedge \bigwedge_{1\leq i\leq n} (Nx_i)^{t(j,i,1)} \wedge \wedge \bigwedge_{1\leq i,k\leq n} (x_i U x_k)^{t(j,i,k,2)} \Big],$$

$t(j,i,0)$, $t(j,i,1)$, $t(j,i,k,2) \in \{0,1\}$ *and, for any formula* α, $\alpha^0 := \alpha$, $\alpha^1 := \neg\alpha$.

Definition 8. *For any given rule* $\mathbf{r}$*, a rule* $\mathbf{r_{nf}}$ *in the reduced normal form is said to be a reduced normal form of* $\mathbf{r}$ *iff*

for any frame $\mathcal{F}$*, the rule* $\mathbf{r}$ *is valid in* $\mathcal{F}$ *if and only if the rule* $\mathbf{r_{nf}}$ *is valid in* $\mathcal{F}$*.*

Theorem 1. *There exists an algorithm running in (single) exponential time which given any rule* $\mathbf{r}$ *constructs some its reduced form* $\mathbf{r_{nf}}$*.*

Proof. The proofs of similar statements for various relative relational models and rules was suggested by us quite a while ago since 1984 (for instance, see Lemma 5 in [3] or the proofs of similar statements in [14]).

The reduced normal forms of rules constructed by the algorithm from the proof of this theorem are uniquely defined.

Thus, if we are interested in the study of the problem of refutation for rules, we may restrict ourselves with consideration of rules in reduced form only.

4. Main Proofs, Results

Firstly we will need some special auxiliary models. Recall that a *linear temporal non–transitive frame* $\mathcal{F}$ is a tuple $\mathcal{F} := \langle N, \{R_x \mid x \in N\}, Nxt\rangle$ such that for any $x \in N$, R_x is a linear order on the interval $[x, a_x]$, for some a_x chosen for each x. It might be also that R_x is a linear order on the whole interval $[x, \infty)$. A model $\mathcal{M}$ based on $\mathcal{F}$ is obtained by introduction of some valuation V in $\mathcal{F}$ of a set of letters.

Definition 9. *Any* $\mathcal{M}_{+Lp}$ *model has the following structure. For* $m, m > 1, n > m$, $\mathcal{M}_{+Lp} = \langle [0,n], \leq, \mathrm{Next}, V\rangle$*, where* $\mathrm{Next}(n) := m+1$.

Relations R_x *in such models are defined as follows: any* R_x *is a linear order on* $[x, a_x]$ *where (1)* $x \leq m$ *and* $a_x \leq n$*, or (2)* $x \geq m$ *and* $a_x \leq n$ *or (3) as in (2) but in addition* R_x *is extended by the linear order on* $[m+1, b]$*,* $b \leq n$*, and all elements from the second interval* $[m+1, b]$ *considered to be strictly bigger than the states of the first one (so we do a loop). Here* V *is just a valuation as above.*

The rules for computation of the truth values of formulas in such models with respect to any given valuation V are defined exactly as it was described earlier for the usual models; simply for states x bigger than m, the orders R_x within $\leq$, in a sense, are replaced by possible sequences by $Next$ and they use these new R_x for the existence of a solution for 'until'.

Theorem 2. *If a rule* r *in normal reduced form is refuted in a model* $\mathcal{M}$ *by a valuation* V*, then there exists a finite model of kind* $\mathcal{M}_{+Lp}$ *disproving* r *by its own valuation* V *(the size of such model is yet not specified).*

Proof. Let $\mathcal{M} := \langle N, \{R_x \mid x \in N\}, Nxt, V\rangle$, and the rule in reduced normal form is $r = \varepsilon/x_1$, where $\varepsilon = \bigvee_{1\le j\le v} \theta_j$ and

$$\theta_j = [\bigwedge_{1\le i\le n} x_i^{t(j,i,0)} \wedge \bigwedge_{1\le i\le n} (N\ x_i)^{t(j,i,1)} \wedge \bigwedge_{1\le i,k\le n} (x_i U x_k)^{t(j,i,k,2)}];$$

let r be refuted in a $\mathcal{M}$ by a valuation V: $\neg(\mathcal{M} \Vdash_V r)$. That is, all the formulas from the premise of r are true at all states, but the conclusion fails to be true at some s. Clearly, we may assume that $s = 0$.

Thus, for any $a \in \mathcal{F}$ there is exactly one unique θ_j which is true at a with respect to V. Denote this θ_j by $\theta(a)$. Now we need to define some special sets. For any $b \in \mathcal{F}$ and any formula $\varphi := x_i U x_j$ from the premise of the rule such that $(\mathcal{M}, b) \Vdash_V x_i U x_j$, we let

$$Ev(\varphi, b) := min\{k \mid b \le k, bR_b k, (\mathcal{M}, k) \Vdash_V x_j, \forall c(b \le c < k)(\mathcal{M}, c) \Vdash_V x_i\}.$$

So, $Ev(\varphi, b)$ is the minimal evidence state saying that $x_i U x_j$ is true at b. And vice versa, for any $b \in \mathcal{M}$ such that $(\mathcal{M}, b) \nVdash_V x_i U x_j$, we let

$$Disp(\varphi, b) := min\{k \mid b \le k, bR_b k, [(\mathcal{M}, k) \Vdash_V x_j \Rightarrow \exists c(b \le c < k)(\mathcal{M}, c) \nVdash_V x_i]\}.$$

That is, $Disp(\varphi)$ is the minimal element disproving the formula φ.

Let Dm be the set of all disjunctive members of the premise of the rule r. Due to the infinity of N, there is a number m and a subset Dm_1 of Dm such that for any number $m_1 \ge m$ there is exactly one $\theta \in Dm_1$ which is true with respect to V at m_1 and for any θ from Dm_1 there are infinitely many numbers bigger than m at which θ is true with respect to V. In other worlds, the following conditions hold

$$\begin{aligned}
(1)\quad & \forall m_1 \ge m \exists \theta \in Dm_1\big[(\mathcal{M}, m_1) \Vdash_V \theta \& \\
& [\forall \theta_1 \in Dm_1 (\mathcal{M}, m_1) \Vdash_V \theta_1 \Rightarrow \theta = \theta_1]\big],
\end{aligned}$$

$$\begin{aligned}
(2)\quad & \forall m_1 \ge m \forall \theta \in Dm_1\big[(\mathcal{M}, m_1) \Vdash_V \theta \Rightarrow \\
& \exists m_2 > (m_1 + m + ||Dm||)\ (\mathcal{M}, m_2) \Vdash_V \theta\big].
\end{aligned}$$

Now consider a smallest a where $a > m$ and $a > b$, where

$$(3)\qquad b = max\Big\{n + 1 \mid n \in \bigcup_{\varphi}\{Disp(\varphi, m)\} \cup \bigcup_{\varphi}\{Ev(\varphi, m)\}\Big\}$$

and $\theta(m+1) = \theta(a)$.

Now we modify our model. Let $\mathcal{M}_{+Lp}$ be a model obtained form $\mathcal{M}$ as follows:

$$\mathcal{M}_{+Lp} = \langle\, [0, m] \cup [m, a]\,\rangle\,,$$

where $\text{Next}(a) := m + 1$ and the model is defined as earlier for models of kind $\mathcal{M}_{+Lp}$ and, in addition, it has the following structure concerning the accessibility relations $R_x, x \in N$:

For all $x \ge m, x \in N$, if $[x, a_x]$ is located inside $[0, a]$ we do not change R_x, otherwise

$$(4)\qquad a_x := b.$$

We show now that the truth values for formulas from Dm in the modified model are the same as earlier.

Lemma 3. *For any $x \in [0, a]$ and $\theta(x)$ defined in the model $\mathcal{M}$ holds*

$$(\mathcal{M}, x) \Vdash_V \theta(x) \Leftrightarrow (\mathcal{M}_{+Lp}, x) \Vdash \theta(x).$$

Proof goes by induction on the structure of formula $\theta(x)$. For components of such formulas not containing operations U, the similar statement are shown by straightforward simple induction of the length of formulas. For formulas $\varphi := x_i U x_j$, the equivalence

$$(\mathcal{M}, x) \Vdash_V \ \Leftrightarrow (\mathcal{M}_{+Lp}, x) \Vdash x_i U x_j$$

follows from our definition (3) above:

$$b = max\Big\{n+1 \mid n \in \bigcup_{\varphi}\{Disp(\varphi, m)\} \cup \bigcup_{\varphi}\{Ev(\varphi, m)\}\Big\},$$

because due to the presence of all evidence states and of all disproving states for operation U, they are all included into the modified model and this is sufficient to keep truth values of formulas of kind $x_i U x_j$ the same. Lemma is complete.

This concludes the proof of our theorem.

Now we need to find (compute) upper bounds for the sizes of finite models refuting the rules.

Theorem 3. *If a rule r in a normal reduced form is refuted in a model $\mathcal{M}_{+Lp}$ then it is refuted in some such model of a polynomial size computable from the length of r.*

Proof. Let $\mathcal{M}_{+Lp} := \langle\, [0, m] \cup [m, a], \leq, \mathrm{Next}, V\,\rangle$, where $\mathrm{Next}(a) := m + 1$, $\mathbf{r} = \varepsilon / x_1$, in which

$$\varepsilon \quad = \quad \bigvee_{1 \leq j \leq m} \left[\bigwedge_{1 \leq i \leq n} x_i^{t(j,i,0)} \wedge \bigwedge_{1 \leq i \leq n} (N x_i)^{t(j,i,1)} \wedge \bigwedge_{1 \leq i,k \leq n} (x_i U x_k)^{t(j,i,k,2)} \right],$$

and $Dm(r)$ is the set of all disjunctive members of the premise of the rule r, and for any $x \in [0, a]$, $\theta(x)$ is the member of $Dm(r)$ which is true on x.

Now, as in the previous lemma, consider the following definitions in this new model. Consider the chosen branching state $m \in \mathcal{M}_{+Lp}$; for any formula $\varphi := x_i U x_j$ from the premise of the rule if $(\mathcal{M}_{+Lp}, m) \Vdash_V x_i U x_j$, we set

$$\begin{aligned} Ev(\varphi, m) \quad := \quad & min\{k \mid m R_m k,\ k \leq a,\ (\mathcal{M}_{+Lp}, k) \Vdash_V x_j, \\ & \forall c (m \leq c < k)(\mathcal{M}, c) \Vdash_V x_i\}. \end{aligned}$$

So, $Ev(\varphi, m)$ is the minimal evidence state saying that $x_i U x_j$ is true at m. Vice versa, if $(\mathcal{M}_{+Lp}, m) \nVdash_V x_i U x_j$ then we define

$$\begin{aligned} Disp(\varphi, m) \quad := \quad & min\{k \mid m \leq k \leq a, m R_m k, [(\mathcal{M}_{+Lp}, k) \Vdash_V x_j \Rightarrow \\ & \exists c (m \leq c < k)(\mathcal{M}_{+Lp}, c) \nVdash_V x_i]\}. \end{aligned}$$

That is, $Disp(\varphi)$ is the minimal element disproving the formula φ.
Let $\{a_1, \ldots, a_n\}$ be the increasing sequence of all elements from all sets $Disp(\varphi, m)$ and all $Ev(\varphi, m)$. Now we are ready to start the rarefication procedure in order to reduce the size of the model $\mathcal{M}_{+Lp}$ to a computable one (from the size of r).

STEP 1. If $a_n = a - 1$ then we do nothing. Otherwise consider $\theta(a-1)$ and any minimal $b \in [a_n + 1, a - 1]$, where $\theta(a-1) = \theta(b)$, if exists. And now we delete all the elements situated strictly between $a-1$ and $b-1$ and redefine relations R_x as follows: if a_x does not exceed $b-1$ or if $a_x \geq a_1$ then we let R_x intact. Otherwise

$$R_x := [x, b-1] \cup [a-1, a_{a-1}].$$

Let $\mathcal{M}_1$ be the model modified as above.

Lemma 4. *For all $x \in \mathcal{M}_1$ and $\theta(x)$ defined for $\mathcal{M}_{+Lp}$ holds*

$$(\mathcal{M}_{+Lp}, x) \Vdash_V \theta(x) \Leftrightarrow (\mathcal{M}_1, x) \Vdash_V \theta(x).$$

Proof follows from a straightforward computation using $\theta(a-1) = \theta(b)$ valid in $\mathcal{M}_{+Lp}$. Lemma is complete.

Now we consider some c with the property $Next(c) = a-1$ instead of b as above and execute a similar transformation for it doing proper rarefication. After this, we continue doing such transformations until we delete all the states x with the same $\theta(x)$ moving to a_n. So, such a transformation will be completed in at most $\|Dm(r)\|$ steps and the resulting model $\mathcal{M}_2$ by Lemma 4 will disprove r.

Now we will reduce the size of $\mathcal{M}_2$ by executing a rarefication procedure within $[m, a_n]$. To do this, we separately consider all the intervals $[a_i, a_i+1]$ while moving down from $[a_n-1, a_n]$ to $[m, a_1]$.

We do this for $[a_n-1, a_n]$ as we did for $[b, a-1]$ above and so on. After we complete this procedure, we will have a computable upper bound for the number of states situated between a_n and m it does not exceed $n \times k \times \|Dm(r)\| + \|Dm(r)\|$, where k is the number of all formulas of kind $x_i U x_j$ in the rule r. Denote the so obtained model by $\mathcal{M}_3$. Again, it will disprove r.

STEP 2. Now we will apply the same rarefication technique to the model $\mathcal{M}_3$ while moving downwards from m to 0; that is, we rarefy the interval $[0, m]$ exactly by the same procedure as we used for the interval $[b, a-1]$ above. Inasmuch as we do not need disproving (and proving) states since already we do not have a loop by $Next$, we need to consider only this interval itself in only one run. So, after the completion of this procedure, we will have a model $\mathcal{M}_4$ which will still disprove r and will have the size at most $n \times k \times ||Dm(r)|| + ||Dm(r)|| + k \times ||Dm(r)||$. Theorem is complete.

Theorem 4. If a rule r in normal form is refuted in a model $\mathcal{M}_{+Lp}$ then it can be refuted in some usual model $\mathcal{M}$.

Proof. We need to apply a simple modification of the standard unraveling technique only. Let $\mathcal{M}_{+Lp}$ be based at the set $[0, m] \cup [m, a]$, where $\text{Next}(a) := m+1$, $\mathbf{r} = \varepsilon / x_1$. In fact, now it is sufficient only to roll towards the future the cyclic part $[m, a]$ starting from the first occurrence of m in the model.

Using Lemmas 1, 2 and Theorems 1, 2, 3, and 4, we immediately obtain:

Theorem 5. *The satisfiability problem for T_L^{Ov} is decidable. The logic T_L^{Ov} itself is decidable.*

Notice that we may consider the reduced version of this logic T_L^{Ov}, namely, the logic $T_L^{Ov-Next}$ without the logical operation $\mathcal{N}$-next. Since we did never use this operation in our proofs, the following theorem holds.

Theorem 6. *The satisfiability problem for $T_L^{Ov-Next}$ is decidable. The logic $T_L^{Ov-Next}$ itself is decidable.*

Now we would like to present several open problems we think to be of interest. (1) To extend the obtained results to branching time logic whose linear parts by operation Next look as frames of this paper. A similar question is answered by Rybakov in [25] for frames which are still within the old paradigm of a kind of interval logic. (2) To study a problem of unification for logics studied in our paper. The logical unification problem is important for applications in AI and CS and it may be considered as algebraic problem of finding solutions for equations in free algebras. This problem was actively studied earlier (see Baader [1, 2], Ghilardi [9, 10], Rybakov [19]), and finding its solution for the logic we have introduced here seems to be interesting. (3) To study the admissibly problem for it. After the paper of H. Fridman [4], which contained a list of open logical problems, the problem of admissibility was investigated for many logics (see [26, 27, 14, 11, 12]). As for the nontransitive temporal linear logic, the most progress was achieved only for a logic with uniform limitations on time intervals with transitivity in a paper by Rybakov [23]. (4) To consider a question of axiomatization for our logic. (5) To embed the agents' logic components into this temporal logic.

References

[1] F. Baader, P. Marantidis, A. Mottet, A. Okhotin. *Extensions of unificationmodulo ACUI,* Mathematical Structures in Computer Science, (2019), 1–30.

[2] F. Baader, P. Narendran, *Unification of concept terms,* Proceedings of the 11th International Workshop on Unification, UNIF-97, LIFO Technical Report, (2001), 97–98.

[3] S.Babenyshev, V. Rybakov, *Linear Temporal Logic LTL: Basis for Admissible Rules,* J. Log. Comput., **21**:2 (2011), 157–177. Zbl 1233.03026

[4] H. Friedman, *One Hundred and Two Problems in Mathematical Logic,* J. Symb. Log., **40** (1975), 113–129. Zbl 0318.02002

[5] D.M. Gabbay, I.M. Hodkinson, *An axiomatization of the temporal logic with until and since over the real numbers*, J. Log. Comput.,**1**:2 (1990), 229–259. Zbl 0744.03018

[6] D.M. Gabbay, I.M. Hodkinson, M.A. Reynolds, *Temporal Logic. Vol 1. Mathematical foundations and computational aspects*, Clarendon Press, Oxford, (1994). Zbl 0921.03023

[7] R. Goldblatt, *Logics of Time and Computations*, CSLI Lecture Notes, **7** (1992), Chapter 6. MR1191162

[8] V. Goranko, *Hierarchies of Modal and Temporal Logics with Reference Pointers*, J. Log. Lang. Inf., **5**:1 (1996), 1–24. Zbl 0851.03003

[9] S. Ghilardi, *Unification in Intuitionistic Logic*, J. Symb. Log., **64**:2 (1999), 859–880. Zbl 0930.03009

[10] S. Ghilardi, *Unification Through Projectivity*, J. Log. Comput., **7**:6 (1997), 733–752. Zbl 0894.08004

[11] S. Odintsov, V. Rybakov, *Inference Rules in Nelson's Logics, Admissibility and Weak Admissibility*, Log. Univers., **9**:1 (2015), 93-120. Zbl 1336.03036

[12] S. Odintsov, V. Rybakov, *Unification and admissible rules for paraconsistent minimal Johanssons' logic J and positive intuitionistic logic* $\mathbf{IPC}^+$, Ann. Pure Appl. Logic, **164**:7-8 (2013), 771-784. Zbl 1323.03029

[13] V.Rybakov, *Refined common knowledge logics or logics of common information*, Arch. Math. Log., **42**:2 (2003), 179–200. Zbl 1030.03015

[14] V. Rybakov, *Logical Consecutions in Discrete Linear Temporal Logic*, J. Symb. Log., **70**:4 (2005), 1137–1149. Zbl 1110.03010

[15] V. Rybakov, *Linear temporal logic with until and next, logical consecutions*, Ann. Pure Appl. Logic, **155**:1 (2008), 32–45. Zbl 1147.03008

[16] V. Rybakov, *Logic of knowledge and discovery via interacting agents–Decision algorithm for true and satisfiable statements*, Inf. Sci., **179**:11 (2009), 1608–1614. Zbl 1179.68151

[17] V. Rybakov, *Linear Temporal Logic* LTL_K *extended by Multi-Agent Logic* K_n *with Interacting Agents*, J. Log. Comput., **19**:6 (2009), 989–1017. Zbl 1208.03023

[18] V. Rybakov, *Logical Analysis for Chance Discovery in Multi-Agents' Environment*, KES 2012, Conference Proceedings, Springer, (2012), 1593–1601.

[19] V. Rybakov, *Writing out unifiers in linear temporal logic*, J. Log. Comput., **22**:5 (2012), 1199–1206. Zbl 1259.03029

[20] V.Rybakov, *Non-transitive linear temporal logic and logical knowledge operations*, J. Log. Comput., **26**:3 (2016), 945–958. Zbl 1403.03028

[21] V.V. Rybakov, *Nontransitive temporal multiagent logic, information and knowledge, deciding algorithms*, Sib. Math. J., **58**:5 (2017), 875–886. Zbl 1420.03060

[22] V. Rybakov, *Multiagent temporal logics with multivaluations*, Sib. Math. J., **59**:4 (2018), 710–720. Zbl 06976648

[23] V. Rybakov, *Linear Temporal Logic with Non-transitive Time, Algorithms for Decidability and Verification of Admissibility*, in *Outstanding Contributions to Logic*, Springer, 2018, 219–243. Zbl 1429.03074

[24] V. Rybakov, *Temporal multi-valued logic with lost worlds in the past*, Sib. Electron. Math. Izv., **15** (2018), 436–449. Zbl 06931667

[25] V. Rybakov, *Branching time agents' logic, satisfiability problem by rules in reduced form*, Sib. Electron. Math. Izv., **16** (2019), 1158–1170. Zbl 07122035

[26] R. Schmidt, D. Tishkovsky, *Solving rule admissibility problem for S4 by a tableau method*, Automated Reasoning Workshop, (2011), 30.

[27] S. Babenyshev, V. Rybakov, R. Schmidt, D. Tishkovsky, *A tableau method for checking rule admissibility in S4*, Electronic Notes in Theoretical Computer Science, **262** (2010), 17–32. Zbl 1345.03033

[28] J. van Benthem, *Tense logic and time*, Notre Dame J. Formal Logic, **25**:1 (1984), 1–16. Zbl 0536.03008

[29] M. Vardi, *An automata-theoretic approach to linear temporal logic,* in: Y. Banff Higher Order Workshop (1995), 238–266. Available at http://citeseer.ist.psu.edu/vardi96automatatheoretic.html.

[30] M. Vardi, Reasoning about the past with two-way automata, in: Larsen K.G.(ed.) et al., ICALP, LNCS, **1443** (1998), 628–641. Zbl 0909.03019

[31] Yde Venema, *Temporal Logic,* chapter in: Lou Goble (ed), *The Blackwell Guide on Philosophical Logic*, Blackwell Publishers, Oxford, 2001, 203–223. Zbl 0997.03019

Vladimir Vladimirovich Rybakov
Institute of Mathematics and Computer Science
Siberian Federal University,
79, Svobodny ave.,
Krasnoyarsk, 660041, Russia
A.P. Ershov Institute of informatics systems SB RAS,
6, Acad. Lavrentjev ave.,
Novosibirsk, 630090, Russia
E-mail address: Vladimir_Rybakov@mail.ru