S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 17, стр. 954–963 (2020) УДК 519.233
DOI 10.33048/semi.2020.17.070 MSC 62F03

ASYMPTOTICS OF AN EMPIRICAL BRIDGE OF REGRESSION ON INDUCED ORDER STATISTICS

A.P. KOVALEVSKII

Abstract. We develop a class of statistical tests for analysis of multivariate data. These statistical tests verify the hypothesis of a linear regression model. To solve the question of the applicability of the regression model, one needs a statistical test to determine whether the actual multivariate data corresponds to this model. If the data does not correspond to the model, then the latter should be corrected. The developed statistical tests are based on an ordering of data array by some null variable. With this ordering, all observed variables become concomitants (induced order statistics). Statistical tests are based on functionals of the process of sequential (under the introduced ordering) sums of regression residuals. We prove a theorem on weak convergence of this process to a centered Gaussian process with continuous trajectories. This theorem is the basis of an algorithm for analysis of multivariate data for matching a linear regression model. The proposed statistical tests have several advantages compared to the commonly used statistical tests based on recursive regression residuals. So, unlike the latter, the statistics of the new tests are invariant to a change in ordering from direct to reverse. The proof of the theorem is based on the Central Limit Theorem for induced order statistics by Davydov and Egorov (2000).

Keywords: concomitants, weak convergence, regression residuals, empirical bridge.

Kovalevskii, A.P., Asymptotics of an empirical bridge of regression on induced order statistics.

The research is supported by Mathematical Center in Akademgorodok under agreement No. 075-15-2019-1675 with the Ministry of Science and Higher Education of the Russian Federation.

Received April, 27, 2020, published July, 17, 2020.

1. Introduction

Galton (1889) proposed the term regression to describe the dependence of offsprings' height on parents' height. To convert women's height to men's height, the author used a coefficient 1.08 since he had determined that men are on average 1.08 times higher than women (p. 78). After that, Galton calculated the average height of parents and estimated the coefficient of dependence of the height of an offspring (separately for sons and for daughters) on the average height of parents. For sons, he estimated the coefficient as $2/3$ (p. 102). Assuming the equal contribution of a mother and a father to the height of their son, Galton estimated the contribution of each parent by $1/3$. He called this dependence a regression because the coefficient $2/3$ is less than 1. Galton called it 'regression to the mean', meaning that parental height is not inherited completely. The fact that height is not fully inherited also makes sense in terms of genetics: an individual's height is determined by several genes; the offspring's genotype is made up of the genotypes of the parents, but not all the genes appear in the phenotype. For describing dependence of height of the descendant Y_i (separately for a son and a daughter) on height of the father X_{i1} and that of the mother X_{i2}, it seems natural to use the following model:

$$Y_i = \theta_1 X_{i1} + \theta_2 X_{i2} + \theta_3 + \varepsilon_i, \quad i = 1, \dots, n. \tag{1}$$

Regression models have been significantly developed over the past 130 years and are used in many fields of knowledge. The unknown coefficients θ_1, θ_2, θ_3 are estimated by the least squares method. Random errors ε_i are assumed to be independent and identically distributed with zero mean and finite nonzero variance. Standard procedures are used for testing hypotheses on zero regression coefficients (see, for example, Draper and Smith (1998), Chapters 5 and 6), as well as on the absence of correlations of regression errors (Ch. 7 of the same book).

We will discuss the methods for testing the regression hypothesis in general. These methods are not sufficiently developed yet. The null hypothesis states the correspondence to the model (1) while the alternative hypothesis proposes nonlinearity of dependence of the response on one or both regressors. The standard methods of R^2 use the ratio of the explained variance to the total variance. Unfortunately, these methods do not allow us to test the correspondence: for an one-parameter model, the Example 1 in Kovalevskii and Shatalin (2015) shows that the standard statistics R^2 can be made arbitrarily close to 1, but at the same time the model can actually be nonlinear. Moreover, it is remarkable that this nonlinearity is consistently tested by the empirical bridge method, i.e. the analysis of the process for self-centered and self-normalized sequential sums of regression residuals.

The most common method for analyzing data on the correspondence to regression in general is a method of recursive regression residuals. Brown, Durbin, Ewans (1975) proposed the process of recursive regression residuals. In order to apply this method, the data must be ordered (for example, nondecreasing in one of the regressors). After that, recursive regression residuals are constructed sequentially: the estimates of the regression parameters are constructed from the previous observations, and based on those, the residuals are calculated for the following observations. After the appropriate normalization, recursive residuals become independent and identically distributed normal random variables with zero expectation. However, this is only true for normal distribution of regression errors and nonrandom regressors. Aside from these limitations, another disadvantage of this method is a lack of symmetry: for the same regressor, if one orders the data in descending order instead of the ascending one, he will get a different set of recursive regression residuals and another p-value. Zeileis et al. (2002) implemented the process in R package, calling it empirical fluctuation process.

On the other hand, there are methods of time series analysis. MacNeill (1978) studied the linear regression for time series. He obtained limit processes for sequences of partial sums of regression residuals. Later, Bischoff (1998) showed that MacNeill's theorem also holds in a more general setting.

Aue et al. (2008) introduced a new test for polynomial regression functions which is analogous to the classical likelihood test. This approach was developed in Aue et al. (2013, 2018).

Stute (1997) proposed a class of tests for one-parametric case.

Our approach is employed specifically for the regression on induced order statistics (concomitants). This model arises in applications, see Kovalevskii (2013). Partial results are proposed by Kovalevskii and Shatalin (2015, 2016). In reference to the example (1), our method is to order the data by one of the regressors (height of a father or a mother) and to study the process of sequential sums of regression residuals. We use a self-standardized version of the method called empirical bridge. One can base a test on statistics of omega-squared type, that is, the integral of the square of the empirical bridge. The distribution of this statistics can be found by using the methods proposed by Martynov (1978), Deheuvels and Martynov (1996). Our proof uses the Central Limit Theorem for induced order statistics by Davydov and Egorov (2000).

The concept of induced order statistics is a development of the idea of order statistics. Gastwirth (1971) proposed a general definition for the Lorentz curve, which is a general inverse empiric function. Goldie (1977) proved fundamental asymptotic results for the general Lorentz curve. David (1973) and Bhattacharya (1974) introduced the induced order statistics (concomitants) simultaneously. David (1973) initially focused on obtaining precise results, while Bhattacharya (1974) proved the first asymptotic theorems. There are numerous papers on the precise theory of concomitants. David, O'Connell and Yang (1977), Yang (1977) proposed the small sample theory of distribution and the expected value for the rank of the induced order statistics. Davydov and Egorov (2000) proved the Central Limit Theorem and the Law of Iterated Logarithm for the induced order statistics.

For specific classes of multivariate distributions, the interest in developing explicit formulas for the distributions of concomitants from generalized order statistics is not decreasing. Among modern works, we should mention EryIlmaz and Bairamov (2003), Shahbaz et al. (2010), Domma and Giordano (2016). The asymptotic theory of induced order statistics has been developed by many authors in a number of different directions. Bhattacharya (1976) proposed a statistical test for a known regression function. Sen (1976) proved fundamental theorems on the invariance principle. Egorov and Nevzorov (1983) studied the distribution of the induced order statistics obtained by ordering the values of a function f and got an approximation by a mixture of normal distributions. Egorov and Nevzorov (1984) studied the distribution of two-sided truncated sums of the induced vector order statistics and established their rate of convergence to the multivariate normal law. Zamanzade and Vock (2015) proposed a new statistical application of concomitants, while Stepanov, Berred and Nevzorov (2016) studied the concomitants of records.

Good reviews of precise and asymptotic results on the induced order statistics with applications can be found in Bhattacharya (1984), Shorack and Wellner (1986), Balakrishnan and Cohen (1991, Chapter 9), David and Nagaraja (1998, 2003 (section 11.7)), Davydov and Zitikis (2004).

Strong convergence to a corresponding Gaussian process can be proved by developing the methods of Shorack and Wellner (1986), Einmah and Mason (1988), Koul (2002), Sakhanenko and Sukhovershina (2015).

2. Main result and corollaries

Let $(\xi_i, \eta_i) = (\xi_{i1}, \ldots, \xi_{im}, \eta_i)$ be independent and identically distributed random vector rows, $i = 1, \ldots, n$. The rows (ξ_i, η_i) form the matrix (ξ, η).

We assume the linear regression hypothesis H_0:

$$\eta_i = \xi_i \theta + e_i = \sum_{j=1}^{m} \xi_{ij} \theta_j + e_i,$$

$\{e_i\}_{i=1}^n$ and $\{\xi_i\}_{i=1}^n$ are independent, $\mathbf{E}\, e_1 = 0$, $\mathbf{Var}\, e_1 = \sigma^2 > 0$.

The vector $\theta = (\theta_1, \ldots, \theta_m)^T$ and the constant σ^2 are unknown. We consider the orderings of the rows of the matrix (ξ, η). We study a class of orderings, such that their results coincide in distribution with the ordering in increasing order of the extended strings $(\delta_i, \xi_i, \eta_i)$ of the random variables δ_i, $i = 1, \ldots, n$. The random variables δ_i are introduced artificially so that $(\delta_i, \xi_i, \eta_i) = (\delta_i, \xi_{i1}, \ldots, \xi_{im}, \eta_i)$ are independent and identically distributed random vector rows, δ_i has uniform distribution on $[0, 1]$, $i = 1, \ldots, n$, δ_i and e_i are independent. The result of the ordering is a matrix (U, X, Y) with rows $(U_i, \mathbf{X}_i, Y_i) = (U_i, X_{i1}, \ldots, X_{im}, Y_i)$. So $U_1 < \ldots < U_n$ a.s., $U_1, \ldots, U_n$ are the order statistics from uniform distribution on $[0, 1]$. Elements of the matrix (X, Y) are concomitants (induced order statistics).

The examples on the introduction of $\{\delta_i\}_{i=1}^n$ corresponding to the required order are presented below.

Example 1. We do not order the rows (ξ_i, η_i). *Suppose that* δ_i *and* (ξ_i, η_i) *are independent. Then we have a random permutation of rows, that is, the rows* $(\mathbf{X}_i, Y_i)$ *are independent and identically distributed with* (ξ_i, η_i). *Thus,* $(X, Y) = (\xi, \eta)$ *in distribution.*

Example 2. We order the rows (ξ_i, η_i) by the first component in ascending order. For instance, we can use a bubble sorting algorithm (Wirth, 1986, pp. 81–82) as follows. We start with ξ_{11} and ξ_{12}. If the first element is greater than the second one, we swap the rows (ξ_1, η_1) and (ξ_2, η_2). We continue doing this for each pair of adjacent elements to the end of the first column. Then we start over from the first two elements of the first column, repeating until no swaps have occurred during the last pass.

For Example 2, let $F_1(x) = \mathbf{P}(\xi_{i1} \le x)$ *be a cdf of the first component, and* $F_1^{-1}(p) = \sup\{x : F_1(x) < p\}$ *be its quantile function. We suppose that*

$$\xi_{i1} = F_1^{-1}(\delta_i). \tag{2}$$

Then the ordering of δ_i*s corresponds to the ordering of* ξ_{i1}*s.*

Example 3. We reason similarly to Example 2, using the descending order this time. *We designate* $\xi_{i1} = F_1^{-1}(1 - \delta_i)$.

Example 4. We do the same as in Example 2 but in ascending order of some measurable function g of the first three components. *We insert a new first column* *with random variables $g(\xi_{i1}, \xi_{i2}, \xi_{i3})$ and act similarly to (2) with the new first column.*

Example 5. We order the rows (ξ_i, η_i) by the second component in ascending order, and then by the first discrete component in ascending order *(note that if we ordered the rows by a component with a continuous distribution, we would forget a.s. all previous orderings)*. The result is the ascending order in the first column, and for the equal values in the first column, variables in the second column are in ascending order.

For Example 5, suppose that ξ_{i1} takes values $a_1 < a_2 < \ldots$ with probabilities $p_1,\ p_2, \ldots$. Let

$$F_{2,a_j}(x) = \mathbf{P}(\xi_{i2} \le x,\ \xi_{i1} = a_j),$$
$$F_{2,a_j}^{-1}(p) = \sup\{x:\ F_{2,a_j}(x) < p\}, \quad 0 < p < p_j.$$

We put ξ_{i1} as in (2), and

$$\xi_{i2} = F_{2,a_j}^{-1}\left(\delta_i - \sum_{k<j} p_k\right) \text{ on the event } \{F_1^{-1}(\delta_i) = a_j\}.$$

We have the required order.

Let $\widehat{\theta}$ be LSE:

$$\widehat{\theta} = (X^T X)^{-1} X^T Y = (\xi^T \xi)^{-1} \xi^T \eta.$$

It does not depend on the order of the rows.

Suppose that $h(x) = \mathbf{E}\{\xi_1 | \delta_1 = x\}$ is a conditional expectation, $L(x) = \int\limits_0^x h(s)\,ds$ is the *induced theoretical generalised Lorentz curve* (see Davydov and Zitikis (2004)),

$$b^2(x) = \mathbf{E}\left((\xi_1 - h(x))^T (\xi_1 - h(x)) \mid \delta_1 = x\right)$$

is a matrix of conditional covariances.

If $g_{ij} = \mathbf{E}\xi_{1i}\xi_{1j},\ \ G = (g_{ij})_{i,j=1}^m$, then $G = \int_0^1 (b^2(x) + h^T(x)h(x))\,dx$.

Suppose that $\widehat{\varepsilon}_i = Y_i - X_i\widehat{\theta}$, $\widehat{\Delta}_k = \sum\limits_{i=1}^k \widehat{\varepsilon}_i$, $\widehat{\Delta}_0 = 0$.

Let $Z_n = \{Z_n(t),\, 0 \le t \le 1\}$ be a piecewise linear random function with the nodes

$$\left(\frac{k}{n},\ \frac{\widehat{\Delta}_k}{\sigma\sqrt{n}}\right).$$

We designate the weak convergence in $C(0,1)$ with uniform metrics by $\Longrightarrow$.

Theorem 1. *If $\mathbf{E}\xi_{1j}^2 < \infty$ for all $1 \le j \le m$ and $\det G \ne 0$, then $Z_n \Longrightarrow Z$. Here Z is a centered Gaussian process with covariance function*

$$K(s,t) = \min(s,t) - L(s)G^{-1}L^T(t), \quad s,t \in [0,1].$$

Suppose that Z_n^0 is an empirical bridge (see [28], [29], [30]):

$$Z_n^0(t) = \frac{\sigma}{\widehat{\sigma}}(Z_n(t) - tZ_n(1)), \quad 0 \le t \le 1,$$

with $\widehat{\sigma}^2 = \sum_{i=1}^n \widehat{\varepsilon}_i^2/n$. Let $L^0(t) = L(t) - tL(1)$.

Let $L_{n,j}$ be an empirical induced generalised Lorentz curve:

$$L_{n,j}(t) = \frac{1}{n}\sum_{i=1}^{[nt]} X_{ij},$$

$L_n = (L_{n,1}, \dots, L_{n,m})$, $L_n^0(t) = L_n(t) - tL_n(1)$.

Corollary 1. *Suppose that the assumptions of Theorem 1 hold.*

1) Then $Z_n^0 \Longrightarrow Z^0$, *a centered Gaussian process with covariation function*

$$K^0(s,t) = \min\{s,t\} - st - L^0(s)G^{-1}(L^0(t))^T, \quad s,t \in [0,1].$$

2) Let $d \geq 1$ *be an integer,*

$$\mathbf{q} = (Z_n^0(1/(d+1)), \dots, Z_n^0(d/(d+1))),$$

$\widehat{g}_{ij} = \overline{X_i X_j} = \frac{1}{n}\sum_{k=1}^n X_{ki}X_{kj}$, $\widehat{G} = (\widehat{g}_{ij})_{i,j=1}^m$,

$$\widehat{K}^0(s,t) = \min(s,t) - st - L_n^0(s))^T \widehat{G}^{-1}(L_n^0(t))^T,$$

$Q = (\widehat{K}^0(i/(d+1), j/(d+1)))_{i,j=1}^d$. *Then* $\mathbf{q}Q^{-1}\mathbf{q}^T$ *converges weakly to a chi-squared distribution with* d *degrees of freedom.*

If we order by ξ_{i1}, $i = 1, \dots, n$, then $h_1(x) = F_{\xi_{11}}^{-1}(x)$ (see Example 2). In this case, $L_1(t) = \int_0^t F_{\xi_{11}}^{-1}(x)\,dx$.

The next corollary was proved by Kovalevskii and Shatalin (2015).

Corollary 2. *Suppose that* $Y_i = \theta_1 X_{i1} + \varepsilon_i$, $i = 1, \dots, n$, $\theta_1 \in \mathbf{R}$, $(X_{11}, \dots, X_{n1})$ *are order statistics of i.i.d.* $(\xi_{11}, \dots, \xi_{n1})$, *random variables* $(\varepsilon_1, \dots, \varepsilon_n)$ *are i.i.d. and independent of them,* $0 < \mathbf{E}\,\xi_{11}^2 < \infty$, $\mathbf{E}\,\varepsilon_1 = 0$, $0 < \mathbf{Var}\,\varepsilon_1 = \sigma^2 < \infty$. *Then* $Z_n \Rightarrow Z$, *a centered Gaussian process with covariance function*

$$\min(s,t) - L_1(s)L_1(t)/\mathbf{E}\,\xi_{11}^2.$$

The next corollary is a special case of Theorem 1 in Kovalevskii and Shatalin (2016).

Corollary 3. *Suppose that* $Y_i = \theta_1 X_{i1} + \theta_2 + \varepsilon_i$, $i = 1, \dots, n$, θ_1, $\theta_2 \in \mathbf{R}$, $(X_{11}, \dots, X_{n1})$ *are order statistics of i.i.d.* $(\xi_{11}, \dots, \xi_{n1})$, *random variables* $(\varepsilon_1, \dots, \varepsilon_n)$ *are i.i.d. and independent of them,* $0 < \mathbf{Var}\,\xi_{11} < \infty$, $\mathbf{E}\,\varepsilon_1 = 0$, $0 < \mathbf{Var}\,\varepsilon_1 = \sigma^2 < \infty$. *Then* $Z_n \Rightarrow Z$, *a centered Gaussian process with covariance function*

$$\min(s,t) - st - L_1^0(s)L_1^0(t)/\mathbf{Var}\,\xi_{11}.$$

3. Proof of Theorem 1

Let $\varepsilon_i = Y_i - \mathbf{X}_i\theta$, $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)^T$. Note that ε_i are i.i.d. and have the same distribution as e_1. Sequences $\{\varepsilon_i\}_{i=1}^n$ and $\{(U_i, X_i)\}_{i=1}^n$ are independent.

Observe that

$$\begin{aligned}
\widehat{\Delta}_k &= \sum_{i=1}^k (Y_i - X_i\widehat{\theta}) = \sum_{i=1}^k (X_i(\theta - \widehat{\theta}) + \varepsilon_i) \\
&= \sum_{i=1}^k (X_i(\theta - (X^TX)^{-1}X^TY) + \varepsilon_i) \\
&= \sum_{i=1}^k (X_i(\theta - (X^TX)^{-1}X^T(X\theta + \vec{\varepsilon})) + \varepsilon_i) \\
&= \sum_{i=1}^k (\varepsilon_i - X_i(X^TX)^{-1}X^T\vec{\varepsilon}).
\end{aligned}$$

Note that $X_{[nt]}/n \to L(t)$ a.s. uniformely on compact sets, and $X^TX/n \to G$ a.s.

So we study the process

$$\left\{\sum_{i=1}^{[nt]}(\varepsilon_i - L(t)G^{-1}X^T\vec{\varepsilon}),\quad t\in[0,1]\right\}.$$

This process is a bounded linear functional of an $(m+1)$-dimensional process

$$\left\{\sum_{i=1}^{[nt]}(X_i\varepsilon_i,\ \varepsilon_i),\quad t\in[0,1]\right\}.$$

We use the functional central limit theorem for induced order statistics by Davydov and Egorov (2000).

We assume that $\eta_i = \xi_i\theta + e_i$, $\{e_i\}_{i=1}^n$ and $\{\xi_i\}_{i=1}^n$ are independent, $\{e_i\}_{i=1}^n$ are i.i.d., $\mathbf{E}\,e_1 = 0$, $\mathbf{Var}\,e_1 = \sigma^2 > 0$.

Consider the rows $(\delta_i, \xi_i e_i, e_i) = (\delta_i, \xi_{i1}e_i, \ldots, \xi_{im}e_i, e_i)$. We have

$$\mathbf{E}(\xi_1 e_1 \mid \delta_1 = x) = 0,\quad \mathbf{E}(e_1 \mid \delta_1 = x) = 0,\quad x\in[0,1].$$

The conditional covariance matrix of the vector $(\xi_1 e_1, e_1)$ is

$$\tilde{b}^2(x) = \mathbf{E}\left((\xi_1 e_1, e_1)^T(\xi_1 e_1, e_1) \mid \delta_1 = x\right) = \sigma^2\begin{pmatrix} b^2(x) + h^T(x)h(x) & h^T(x) \\ h(x) & 1 \end{pmatrix}.$$

Let $\tilde{b}(x)$ be an upper triangular matrix such that $\tilde{b}(x)\tilde{b}(x)^T = \tilde{b}^2(x)$. Then

$$\tilde{b}(x) = \sigma\begin{pmatrix} b(x) & h^T(x) \\ \mathbf{0} & 1 \end{pmatrix}.$$

Here $b(x)$ is an upper triangular matrix such that $b(x)b(x)^T = b^2(x)$. By Theorem 1 of Davydov and Egorov (2000) the process

$$\left\{\frac{1}{\sqrt{n}}\left(\sum_{i=1}^{[nt]}X_i\varepsilon_i,\ \sum_{i=1}^{[nt]}\varepsilon_i\right)^T,\ t\in[0,1]\right\}$$

converges weakly in the uniform metrics to the Gaussian process

$$\left\{\int_0^t \tilde{b}(x)\,d\mathbf{W}_{m+1}(x),\ t\in[0,1]\right\}.$$

Here $\mathbf{W}_{m+1} = (W_1,\ldots,W_{m+1})^T$ is an $(m+1)$-dimensional standard Wiener process.

So the process

$$\left\{\frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{[nt]}(\varepsilon_i - L(t)G^{-1}X^T\vec{\varepsilon})^T,\quad t\in[0,1]\right\}$$

converges weakly in the uniform metrics to the Gaussian process $Z = \{Z(t),\ t\in[0,1]\}$,

$$Z(t) = W_{m+1}(t) - L(t)G^{-1}\left(\int_0^1 b(x)\,d\mathbf{W}_m(x) + \int_0^1 h^T(x)\,dW_{m+1}(x)\right)^T,$$

$\mathbf{W}_m = (W_1,\ldots,W_m)^T$.

By the noted convergencies $X_{[nt]}/n \to L(t)$ a.s. uniformely on compact sets, $X^TX/n \to G$ a.s., the process Z_n has the same weak limit Z.

The covariance function of the limiting Gaussian process Z is

$$K(s,t) = \mathbf{E}Z(s)Z(t)$$

$$= \min(s,t) - L(s)G^{-1}\int_0^t h^T(x)\,dx - L(t)G^{-1}\int_0^s h^T(x)\,dx$$

$$+L(s)G^{-1}\int_0^1 (b^2(x) + h^T(x)h(x))\,dx\,G^{-1}L^T(t)$$

$$= \min(s,t) - L(s)G^{-1}L^T(t).$$

The proof is complete.

Acknowledgements

The work is supported by Mathematical Center in Akademgorodok under agreement No. 075-15-2019-1675 with the Ministry of Science and Higher Education of the Russian Federation. The author would like to thank an anonimous referee for helpful and constructive comments and suggestions.

References

[1] A. Aue, L. Horvath, M. Huskova, P. Kokoszka, *Testing for changes in polynomial regression*, Bernoulli, **14** (2008), 637–660. Zbl 1155.62027

[2] A. Aue, L. Horvath, *Structural breaks in time series*, Journal of Time Series Analysis, **34**:1 (2013), 1–16. Zbl 1274.62553

[3] A. Aue, G. Rice, O. Sonmez, *Detecting and dating structural breaks in functional data without dimension reduction*, Journal of the Royal Statistical Society: Series B (Statistical Methodology), **80**:3 (2018), 509–529. Zbl 1398.62152

[4] N. Balakrishnan, A.C. Cohen, *Order Statistics and Inference: Estimation Methods*, Academic Press, Boston, MA, 1991. Zbl 0732.62044

[5] P.K. Bhattacharya, *Convergence of sample paths of normalized sums of induced order statistics*, The Annals of Statist., **2** (1974), 1034–1039. Zbl 0307.62036

[6] P.K. Bhattacharya, *An invariance principle in regression analysis*, The Annals of Statist., **4** (1976), 621–624. Zbl 0331.62016

[7] P.K. Bhattacharya, *Induced order statistics : Theory and applications*, in : P.R. Krishnaiah and P.K. Sen, eds., Handbook of Statistics, **4** (North-Holland, Amsterdam), 383–403 (1984). Zbl 0596.62056

[8] W. Bischoff, *A functional central limit theorem for regression models*, Ann. Stat., **26** (1998), 1398–1410. Zbl 0936.62072

[9] R.L. Brown, J. Durbin, J.M. Evans, *Techniques for testing the constancy of regression relationships over time*, J. R. Statist. Soc., **37** (1975), 149–192. Zbl 0321.62063

[10] H.A. David, *Concomitants of order statistics*, Bull. Internat. Statist. Inst., **45** (1973), 295–300. Zbl 0827.62043

[11] H.A. David , J. Galambos, *The asymptotic theory of concomitants of order statistics*, J. Appl. Probab., **11** (1974), 762–770. Zbl 0299.62026

[12] H.A. David, H N. Nagaraja, *Concomitants of order statistics*, in: *Order Statistics: Theory and Methods* (1998), 487–513. Zbl 0905.62055

[13] H.A. David, H N. Nagaraja, *Order Statistics*, New Jersey: John Wiley & Sons, 2003. Zbl 1053.62060

[14] H.A. David, M.J. O'Connell, S.S. Yang, *Distribution and expected value of the rank of a concomitant of an order statistics*, The Annals of Statist., **5** (1977), 216–223. Zbl 0356.62040

[15] Y. Davydov, V. Egorov, *Functional limit theorems for induced order statistics*, Mathematical Methods of Statistics, **9**:3 (2000), 297–313. Zbl 1010.60031

[16] Y. Davydov, R. Zitikis, *Convex rearrangements of random elements*, Fields Institute Communications, **44** (2004), 141–171. Zbl 1080.60055

[17] P. Deheuvels, G.V. Martynov, *Cramer-von mises-type tests with applications to tests of independence for multivariate extreme-value distributions*, Communications in Statistics — Theory and Methods,**25**:4 (1996), 871–908. Zbl 0875.62190

[18] F. Domma, S. Giordano, *Concomitants of m-generalized order statistics from generalized Farlie-Gumbel-Morgenstern distribution family*, Journal of Computational and Applied Mathematics, **294**, 413–435 (2016). Zbl 1330.62216

[19] N.R. Draper, H. Smith, *Applied Regression Analysis*, Wiley Series in Probability and Statistics, 1998. Zbl 0895.62073

[20] V.A. Egorov, V.B. Nevzorov, *Some theorems on induced order statistics*, Theory Prob. Appl., **27**:3 (1983), 633–639. Zbl 0516.62048

[21] V.A. Egorov, V.B. Nevzorov, *Rate of convergence to the normal law of sums of induced order statistics*, J. of Soviet Math., **25**:3 (1984), 1139–1146. Zbl 0549.62019

[22] J. Einmah, D. Mason, *Strong limit theorems for weighted quantile processes*, Annals. Probab., **16**:4 (1988), 1623–1643. Zbl 0659.60052

[23] S. EryIlmaz, I.G. Bairamov, *On a new sample rank of an order statistics and its concomitant, Statistics & Probability Letters*, **63**:2 (2003), 123–131. Zbl 1116.62348

[24] F. Galton, *Natural Inheritance*, London: Macmillan, 1889. http://galton.org/books/natural-inheritance/index.html

[25] J.L. Gastwirth, *A general definition of the Lorenz curve*, Econometrica, **39** (1971), 1037–1039. Zbl 0245.62082

[26] C.M. Goldie, *Convergence theorems for empirical Lorenz curves and their inverses*, Adv. Appl. Prob., **9** (1977), 765–791. Zbl 0383.60003

[27] H.L. Koul, *Weighted Empirical Processes in Dynamic Nonlinear Models*, SpringerVerlag, New York, (2002). Zbl 1007.62047

[28] A. Kovalevskii, *A regression model for prices of second-hand cars. Applied methods of statistical analysis*, Applications in survival analysis, reliability and quality control, (2013) 124–128. ISBN 978-5-7782-2311-0

[29] A.P. Kovalevskii, E.V. Shatalin, *Asymptotics of Sums of Residuals of One-Parameter Linear Regression on Order Statistics*, Theory of probability and its applications, **59**: 3 (2015) 375–387. Zbl 1342.62077

[30] A. Kovalevskii, E. Shatalin, *A limit process for a sequence of partial sums of residuals of a simple regression on order statistics*, Probability and Mathematical Statistics, **36**:1 (2016), 113–120. Zbl 1343.60032

[31] I.B. MacNeill, *Limit processes for sequences of partial sums of regression residuals*, Ann. Prob., **6** (1978), 695–698. Zbl 0377.60028

[32] G.V. Martynov, *Omega-square tests*, Nauka, Moscow (in Russian), 1978. MR 0527912

[33] A.I. Sakhanenko, O.A. Sukhovershina, *On accuracy of approximation in Koul's theorem for weighted empirical processes*, Sib. Elektron. Mat. Izv., **12** (2015), 784–794. Zbl 1342.62079

[34] P.K. Sen, *A note on invariance principles for induced order statistics*, The Annals of Probab., **4** (1976), 474–479. Zbl 0336.60022

[35] M.Q. Shahbaz, S. Shahbaz, M. Mohsin, A. Rafiq, *On distribution of bivariate concomitants of records*, Applied Mathematics Letters, **23**: 5 (2010), 567–570. Zbl 1186.62072

[36] G. Shorack, J. Wellner, *Empirical processes with applications to statistics*, Wiley N. Y., 1986. Zbl 1171.62057

[37] A. Stepanov, A. Berred, V. B. Nevzorov, *Concomitants of records: Limit results, generation techniques, correlation*, Statistics & Probability Letters, **109** (2016), 184–188. Zbl 1383.62150

[38] W. Stute, *Nonparametric model checks for regression*, Ann. Statist., **25** (1997), 613–641. Zbl 0926.62035

[39] N. Wirth, *Algorithms and Data Structures*, NJ: Prentice-Hall, 1986 Zbl 0634.68002

[40] S.S. Yang, *General distribution theory of the concomitants of order statistics*, The Annals of Statist., **5** (1977), 996–1002. Zbl 0367.62017

[41] E. Zamanzade, M. Vock, *Variance estimation in ranked set sampling using a concomitant variable*, Statistics & Probability Letters, Volume **105** (2015), 1–5. Zbl 1328.62065

[42] A. Zeileis, F. Leisch, K. Hornik, Ch. Kleiber, *Strucchange: An R Package for Testing for Structural Change in Linear Regression Models*, Journal of Statistical Software, **7**:2 (2002), 1–38. DOI 10.18637/jss.v007.i02

Artyom Pavlovich Kovalevskii
Novosibirsk State Technical University,
20, K. Marks ave.,
Novosibirsk, 630073, Russia
Novosibirsk State University,
1, Pirogova str.,
Novosibirsk, 630090, Russia
Email address: artyom.kovalevskii@gmail.com