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ON FUNCTION SPACES

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Abstract. For certain properties $\mathfrak{P}$ of topological T_0-spaces, we prove that an arbitrary T_0-space $\mathbb{Y}$ has property $\mathfrak{P}$ if and only if the function space $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ endowed with the pointwise convergence topology possesses $\mathfrak{P}$ for some (and therefore, for each) $[\alpha^*$-$]$space $\mathbb{X}$.

Keywords: d-space, essentially complete space, function space, injective space, sober space, T_0-space.

1. Introduction

The question if certain properties of topological spaces are preserved when passing to function spaces is connected with the definition of Cartesian closed category of topological spaces and is therefore of natural interest. We mention some results connected with this question. In [3, 4, 5] (see also Theorems 7.3.2, 7.3.4 and Corollary 7.3.5 in [12]), the first author proved that for arbitrary sober A-spaces [f-spaces] $\mathbb{X}$ and $\mathbb{Y}$ with least element, the space $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ of continuous functions from $\mathbb{X}$ to $\mathbb{Y}$ endowed with the pointwise convergence topology is also a sober A-space [f-space] having a least element. The first author established in [6] that the property of being a d-space is also preserved when passing to function spaces. Moreover, the first author showed in [8] that for arbitrary Δ-spaces $\mathbb{X}$ and $\mathbb{Y}$, $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ is also a Δ-space. These results allow, in particular, to construct natural models of the λ-calculus. Other results in this direction can be found in G. Gierz *et al.* [14] as well as in the monograph of the first author [12].

The interconnection of a series of properties of topological T_0-spaces and those of function spaces is studied in this paper. Specifically, we consider the following properties of topological T_0-spaces:

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(1) to possess a least element;
(2) to be a d-space;
(3) to be a topological join-semilattice;
(4) to be a sober space;
(5) to be an essentially complete space;
(6) to be a [densely] injective space;
(7) to be a [sober] Δ-space.

Let $\mathfrak{P}$ denote one of the properties (1)–(4). We show that an arbitrary topological T_0-space $\mathbb{Y}$ possesses $\mathfrak{P}$ if and only if the function space $\mathbb{C}(\mathbb{X},\mathbb{Y})$, endowed with the pointwise convergence topology, has the property $\mathfrak{P}$ for some (equivalently, for each) T_0-space $\mathbb{X}$, see Lemma 1, Theorems 3, 6, 10 and Proposition 9. Let $\mathfrak{Q}$ denote one of the properties (5)–(6). Then an arbitrary topological T_0-space $\mathbb{Y}$ has the property $\mathfrak{Q}$ if and only if the function space $\mathbb{C}(\mathbb{X},\mathbb{Y})$ has $\mathfrak{Q}$ for some (equivalently, for each) α^*-space $\mathbb{X}$, see Theorems 19 and 20. Finally, a T_0-space $\mathbb{Y}$ is a [sober] Δ-space if and only if $\mathbb{C}(\mathbb{X},\mathbb{Y})$ is a [sober] Δ-space for some (equivalently, for each) Δ-space $\mathbb{X}$, see Theorem 23 and Corollary 24.

2. d-SPACES $\mathbb{C}(\mathbb{X},\mathbb{Y})$

Given topological spaces $\mathbb{X}$ and $\mathbb{Y}$, let $C(\mathbb{X},\mathbb{Y})$ denote the set of all continuous functions from $\mathbb{X}$ to $\mathbb{Y}$. An arbitrary function $f \in C(\mathbb{X},\mathbb{Y})$ can be considered as an element of the Cartesian power Y^X. *The pointwise convergence topology* on $C(\mathbb{X},\mathbb{Y})$ is induced by the Tychonoff topology on Y^X. Therefore, the set

$$V_{x,U} = \{f \in C(\mathbb{X},\mathbb{Y}) \mid f(x) \in U\}, \text{ where } x \in X \text{ and } \varnothing \neq U \in \mathcal{T}(\mathbb{Y}),$$

forms a subbasis of the pointwise convergence topology; we denote this topology by $\mathcal{P}$. By $\mathbb{C}(\mathbb{X},\mathbb{Y})$, we denote the space $\langle C(\mathbb{X},\mathbb{Y}),\mathcal{P}\rangle$. Thus for $f,\ g \in C(\mathbb{X},\mathbb{Y})$, the relation $f \leq_{\mathcal{P}} g$ holds if and only if $f(x) \leq_{\mathbb{Y}} g(x)$ for all $x \in X$. We write $f \leq g$ instead of $f \leq_{\mathcal{P}} g$. Consider the mapping

$$\xi\colon \mathbb{Y} \to \mathbb{C}(\mathbb{X},\mathbb{Y}); \quad \xi\colon y \mapsto \xi_y, \quad \text{where } \xi_y(x) = y \text{ for all } x \in X.$$

It is straightforward to verify that $\xi(Y) \cap V_{x,U} = \xi(U)$ for all $x \in X$ and $U \in \mathcal{T}(\mathbb{Y})$. Hence ξ is a homeomorphic embedding of $\mathbb{Y}$ into $\mathbb{C}(\mathbb{X},\mathbb{Y})$. The following statement follows in an obvious way from the definition of the pointwise convergence topology.

Lemma 1. *A T_0-space $\mathbb{Y}$ contains a least element* (*with respect to the specialization order*) *if and only if $\mathbb{C}(\mathbb{X},\mathbb{Y})$ contains a least element for some* (*equivalently, for every*) *T_0-space $\mathbb{X}$.*

Lemma 2. [12, Lemma 8.4.1] *Let $\mathbb{X}$ and $\mathbb{Y}$ be T_0-spaces and let $F = \{f_i \in C(\mathbb{X},\mathbb{Y}) \mid i \in I\}$ be an up-directed set with respect to the specialization order. If a function $f\colon X \to Y$ is such that $f(x) = \sup_{\mathbb{Y}}\{f_i(x) \mid i \in I\}$ and $f(x) \in \mathrm{cl}_{\mathbb{Y}}\{f_i(x) \mid i \in I\}$ for all $x \in X$, then $f = \sup F$ in $\mathbb{C}(\mathbb{X},\mathbb{Y})$.*

Proof. It suffices to prove that f continuous. Indeed, let $x \in f^{-1}(U)$ for some $U \in \mathcal{T}(\mathbb{Y})$. Then $f(x) \in U$, whence $f_i(x) \in U$ for some $i \in I$ as $f(x)$ is a limit point of up-directed set $\{f_i(x) \mid i \in I\}$. If $u \in f_i^{-1}(U) \in \mathcal{T}(\mathbb{X})$ then $f(u) \geq f_i(u) \in U$, which implies that $f(u) \in U$. We demonstrated therefore that $x \in f_i^{-1}(U) \subseteq f^{-1}(U)$, whence $f^{-1}(U) \in \mathcal{T}(\mathbb{X})$. □

Definition 1. A T_0-space $\mathbb{X}$ is a *d-space*, if for each nonempty up-directed set $D \subseteq X$, $\sup_{\mathbb{X}} D$ exists and $\sup D \in \operatorname{cl}_{\mathbb{X}} D$.

A space $\mathbb{X}$ is an *A_d-space*, if $\mathbb{X}$ is an A-space and a d-space simultaneously.

Theorem 3. *For T_0-spaces $\mathbb{X}$ and $\mathbb{Y}$, $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ is a d-space if and only if $\mathbb{Y}$ is a d-space.*

Proof. Suppose that $\mathbb{Y}$ is a d-space. According to Lemma 2, each nonempty set $F = \{f_i \in C(\mathbb{X}, \mathbb{Y}) \mid i \in I\}$ which is up-directed with respect to the specialization order has a least upper bound f. Let $x_0, \ldots, x_n \in X$, let $U_0, \ldots, U_n \in \mathcal{T}(\mathbb{Y})$, and let $f \in V_{x_0,U_0} \cap \ldots \cap V_{x_n,U_n}$. This means that $f(x_k) \in U_k$ for all $k \leqslant n$. Since $f(x_k) \in \operatorname{cl}\{f_i(x_k) \mid i \in I\}$ by Lemma 2, there are $i_0, \ldots, i_n \in I$ such that $f_{i_k}(x_k) \in U_k$ for all $k \leqslant n$. If $i \geqslant i_0, \ldots, i_n$ then $f_i(x_k) \geq f_{i_k}(x_k) \in U_k$; that is, $f_i(x_k) \in U_k$ for all $k \leqslant n$, whence $f_i \in V_{x_0,U_0} \cap \ldots \cap V_{x_n,U_n}$. This yields that f is a limit point for F and $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ is a d-space.

Suppose now that $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ is a d-space. Let a set $D \subseteq Y$ be up-directed with respect to the specialization order. Then the set $\xi(D) = \{\xi_d \mid d \in D\}$ is also up-directed with respect to the specialization order in $\mathbb{C}(\mathbb{X}, \mathbb{Y})$. According to our assumption, there exists $f = \sup \xi(D) \in C(\mathbb{X}, \mathbb{Y})$ and $f \in \operatorname{cl} \xi(D)$. We show that $f \in \xi(Y)$. Indeed, let $f(x_0) \in U \in \mathcal{T}(\mathbb{Y})$; we choose an arbitrary element $x_1 \in X$. Then $f \in V_{x_0,U}$. Thus, there exists $d \in D$ such that $\xi_d \in V_{x_0,U}$, whence $f(x_1) \geq \xi_d(x_1) = d = \xi_d(x_0) \in U$. It follows that $f(x_1) \in U$ whence $f(x_0) \leq f(x_1)$. A similar argument shows that $f(x_1) \leq f(x_0)$. Therefore $f = \xi_y$ for some $y \in Y$. Furthermore, the argument above also shows that $y = \sup_{\mathbb{Y}} D$ and $y \in \operatorname{cl}_{\mathbb{Y}} D$. □

The fact that $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ is a d-space whenever $\mathbb{Y}$ is a d-space was established in [6].

3. Sober spaces $\mathbb{C}(\mathbb{X}, \mathbb{Y})$

Definition 2. [11] Consider a topological space $\mathbb{X}$ and a set $Y \subseteq X$. An element $x \in X$ is a *solimit point* for Y in $\mathbb{X}$, if for each $U \in \mathcal{T}(\mathbb{X})$ such that $x \in U$, there exists an element $y \in U \cap Y$ such that $y \leq_{\mathcal{T}(\mathbb{X})} x$.

Let $\operatorname{sob}_{\mathbb{X}} Y$ denote the set of all solimit points for Y in $\mathbb{X}$.

Definition 3. [9] An extension $\mathbb{X} \leq \mathbb{Y}$ of topological T_0-spaces is a *u-extension*, if for each space $\mathbb{Z}$ and arbitrary continuous functions $f, g \colon \mathbb{Y} \to \mathbb{Z}$ such that $f|_{\mathbb{X}} = g|_{\mathbb{X}}$ the equality $f = g$ holds.

For the next theorem, see [11, Theorem 3.2] and [12, Theorem 5.2.2].

Theorem 4. [12, Theorem 5.2.2] *For an arbitrary extension $\mathbb{X} \leq \mathbb{Y}$ of topological T_0-spaces, the following conditions are equivalent.*

(1) *$\mathbb{X} \leq \mathbb{Y}$ is a u-extension.*
(2) *For each $U \in \mathcal{T}(\mathbb{X})$, there is a unique set $V \in \mathcal{T}(\mathbb{Y})$ such that $V \cap X = U$.*
(3) *Each element $y \in Y$ is a solimit point for X in $\mathbb{Y}$; that is, $Y = \operatorname{sob}_{\mathbb{Y}} X$.*

Definition 4. [11] A subset $Y \subseteq X$ is *sober in* $\mathbb{X}$ if $Y = \operatorname{sob}_{\mathbb{X}} Y$. A topological T_0-space $\mathbb{X}$ is *sober* if its ground set X is sober in each T_0-extension $\mathbb{Y} \geq \mathbb{X}$.

For the next theorem, see [11, Theorem 3.6] and [12, Theorem 5.3.2].

Theorem 5. [12, Theorem 5.3.2] *For an arbitrary T_0-space $\mathbb{X}$, the following conditions are equivalent.*

(1) *$\mathbb{X}$ is sober.*

(2) *If $S \subseteq X$ is a nonempty irreducible set in $\mathbb{X}$, then $S = {\downarrow_{\mathbb{X}}} x$ for some $x \in S$.*
(3) *For an arbitrary u-extension $\mathbb{Y}_0 \leq \mathbb{Y}$, each continuous function $f_0 \colon \mathbb{Y}_0 \to \mathbb{X}$ has a* [*unique*] *continuous extension $f \colon \mathbb{Y} \to \mathbb{X}$.*
(4) *$\mathbb{X}$ has no proper u-extension.*

We establish our first main result.

Theorem 6. *A T_0-space $\mathbb{Y}$ is sober if and only if $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ is a sober space for some* (*equivalently, for each*) *T_0-space $\mathbb{X}$.*

Proof. Let $\mathbb{X}$ be a T_0-space, let $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ be a sober space, and let a subspace $\mathbb{Z}_0 \leq \mathbb{Z}$ be such that $Z = \mathrm{sob}_{\mathbb{Z}} Z_0$. Consider an arbitrary continuous function $f_0 \colon \mathbb{Z}_0 \to \mathbb{Y}$. Recall that $\xi \colon \mathbb{Y} \to \mathbb{C}(\mathbb{X}, \mathbb{Y})$ is also continuous. Thus, the function $\xi f_0 \colon \mathbb{Z}_0 \to \mathbb{C}(\mathbb{X}, \mathbb{Y})$ is continuous. As $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ is a sober space, there is a continuous function $f \colon \mathbb{Z} \to \mathbb{C}(\mathbb{X}, \mathbb{Y})$ such that $f|_{\mathbb{Z}_0} = \xi f_0$. We prove that $f(Z) \subseteq \xi(Y)$. Indeed, let $z \in Z$; then $z = \sup({\downarrow} z \cap Z_0)$ and $z \in \mathrm{cl}({\downarrow} z \cap Z_0)$. It is straightforward to verify that $f(z) \in \mathrm{cl}\, f({\downarrow} z \cap Z_0)$ and $f(z) = \sup f({\downarrow} z \cap Z_0)$. Let $x_0,\, x_1 \in X$ and let $[f(z)](x_0) \in U \in \mathcal{T}(\mathbb{Y})$. Then $f(z) \in V_{x_0,U}$, whence $z \in f^{-1}(V_{x_0,U}) \in \mathcal{T}(\mathbb{Z})$. Therefore there is $z_0 \in {\downarrow} z \cap Z_0$ such that $f(z_0) \in V_{x_0,U}$. Since the function $f(z_0)$ is constant, $f(z_0) \in V_{x_1,U}$. But then $f(z) \in V_{x_1,U}$, as $f(z_0) \leq f(z)$. Therefore $[f(z)](x_1) \in U$. Similarly, $[f(z)](x_1) \in U$ implies that $[f(z)](x_0) \in U$; that is, $[f(z)](x_0) = [f(z)](x_1)$, which is our desired conclusion. Inclusion $f(Z) \subseteq \xi(Y)$ implies that $\xi^{-1} f|_{\mathbb{Z}_0} = f_0$. In view of Theorem 5, $\mathbb{Y}$ is a sober space.

Conversely, let $\mathbb{Y}$ be a sober space and let $\mathbb{X}$ be an arbitrary T_0-space. Let also $F \subseteq C(\mathbb{X}, \mathbb{Y})$ be an irreducible set. We prove that for each $x \in X$, the set $F(x) = \{f(x) \mid f \in F\}$ is irreducible in $\mathbb{Y}$. Indeed, we consider arbitrary open sets $U_0,\, U_1 \in \mathcal{T}(\mathbb{Y})$ and suppose that $F(x) \subseteq F_0 \cup F_1$, where $F_i = Y \backslash U_i$ for each $i < 2$. Then $F \cap V_{x,U_0} \cap V_{x,U_1} = F \cap V_{x,U_0 \cap U_1} = \varnothing$. According to our assumption, $F \cap V_{x,U_i} = \varnothing$ for some $i < 2$. This means that $F(x) \subseteq F_i$, which proves that the set $F(x)$ is irreducible. Furthermore, the sobriety of $\mathbb{Y}$ and Theorem 5 imply that for each $x \in X$, there is $g(x) \in Y$ such that ${\downarrow} g(x) = \mathrm{cl}_{\mathbb{Y}} F(x)$. We show that g is continuous. Indeed, let $U \in \mathcal{T}(\mathbb{Y})$ and let $x \in g^{-1}(U)$. Then $g(x) \in U$, whence $f(x) \in U$ for some $f \in F$. Suppose that $x' \in f^{-1}(U)$. This means that $f(x') \in U$. Since $f(x') \leq g(x')$, we conclude that $g(x') \in U$. Therefore $x \in f^{-1}(U) \subseteq g^{-1}(U)$ and $g^{-1}(U) \in \mathcal{T}(\mathbb{X})$. Hence the function g is continuous. Let $h \in \mathrm{cl}\, F$; then for $x \in X$, we have $h(x) \in \mathrm{cl}_{\mathbb{Y}} F(x)$. Indeed, let $h(x) \in U \in \mathcal{T}(\mathbb{Y})$. Then $h \in V_{x,U}$ and $f \in V_{x,U}$ for some $f \in F$, whence $f(x) \in U$, which was to be proved. Thus, $h(x) \leq g(x)$ for all $x \in X$, whence $h \leq_{\mathcal{P}} g$. This means that g is an upper bound of $\mathrm{cl}\, F$.

In order to prove the sobriety of $\mathbb{C}(\mathbb{X}, \mathbb{Y})$, in view of Theorem 5, it suffices to establish that g is a limit point for F. Indeed, let $g \in U$, where the set $U \subseteq C(\mathbb{X}, \mathbb{Y})$ is open in the pointwise convergence topology. This means by definition that $g \in V_{x_0,U_0} \cap \ldots \cap V_{x_n,U_n} \subseteq U$ for some $x_0,\, \ldots,\, x_n \in X$ and some $U_0,\, \ldots,\, U_n \in \mathcal{T}(\mathbb{Y})$. Thus, $g(x_i) \in U_i$ for all $i \leqslant n$. Since $g(x_i)$ is a limit point of the set $F(x_i)$, we conclude that $F \cap V_{x_i,U_i} \neq \varnothing$ for all $i \leqslant n$. Therefore $F \cap V_{x_0,U_0} \cap \ldots \cap V_{x_n,U_n} \neq \varnothing$, whence $F \cap U \neq \varnothing$, which is our desired conclusion. □

4. Essentially complete spaces $\mathbb{C}(\mathbb{X}, \mathbb{Y})$

Following [7], we consider several properties of a T_0-space $\mathbb{X}$.

(H_0) $\mathbb{X}$ has a least element 0 with respect to the specialization order $\leq$.

(H_1) $\mathbb{X}$ is a join semilattice with respect to $\leq$; $\vee$ denotes the join operation in X.

(H_2) The join operation $\vee\colon \mathbb{X}^2 \to \mathbb{X}$ is continuous; that is, $\langle X, \vee, \mathcal{T}(\mathbb{X})\rangle$ is a topological join-semilattice.

The proof of the following statement is straightforward.

Lemma 7. *If $\mathbb{X}$ is an A_d-space and $x_0 \vee x_1 \in U \in \mathcal{T}(\mathbb{X})$ for some $x_0, x_1 \in X$, then there are $U_0, U_1 \in \mathcal{T}(\mathbb{X})$ such that $x_0 \in U_0$, $x_1 \in U_1$ and $U_0 \cap U_1 \subseteq U$.*

In [7], the following statement was proved, see also [12, Corollary 10.4.2].

Theorem 8. [12, Corollary 10.4.2] *A T_0-space $\mathbb{X}$ is essentially complete if and only if $\mathbb{X}$ is a d-space with properties* (H_0)–(H_2).

Proposition 9. *A T_0-space $\mathbb{Y}$ possesses the properties* (H_1)–(H_2) *if and only if the function space $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ possesses* (H_1)–(H_2) *for some* (*equivalently, for each*) *space $\mathbb{X}$. Moreover, $\mathbb{Y}$ possesses the property* (H_0) *if and only if $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ possesses* (H_0) *for some* (*equivalently, for each*) *space $\mathbb{X}$.*

Proof. Suppose that $\mathbb{Y}$ has the properties (H_1)–(H_2). Let f_0, $f_1 \in C(\mathbb{X}, \mathbb{Y})$. We put

$$g(x) = f_0(x) \vee f_1(x) \in Y, \quad x \in X.$$

We show that g is continuous. Indeed, let $g(x) \in U \in \mathcal{T}(\mathbb{Y})$. Since the function $\vee$ is continuous on $\mathbb{Y}$ according to our assumption, there are sets U_0, $U_1 \in \mathcal{T}(\mathbb{Y})$ such that $f_i(x) \in U_i$, $i < 2$, and $U_0 \cap U_1 \subseteq U$. This means that $x \in W = f_0^{-1}(U_0) \cap f_1^{-1}(U_1) \in \mathcal{T}(\mathbb{X})$. For all $x' \in W$ and all $i < 2$, we have $g(x') \geq f_i(x') \in U_i$, whence $g(x') \in U_0 \cap U_1 \subseteq U$. Therefore $x \in W \subseteq g^{-1}(U)$, and the set $g^{-1}(U)$ is open in $\mathbb{X}$. Besides that, it is not hard to verify that $g = \sup\{f_0, f_1\}$. Thus, the space $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ possesses the property (H_1).

We show that the function $\vee$ is continuous on $\mathbb{C}(\mathbb{X}, \mathbb{Y})$. Indeed, if $f_0 \vee f_1 \in V_{x_0,U_0} \cap \ldots \cap V_{x_n,U_n}$ for some elements $x_0, \ldots, x_n \in X$ and some sets $U_0, \ldots, U_n \in \mathcal{T}(\mathbb{Y})$, then $f_0(x_i) \vee f_1(x_i) \in U_i$ for all $i \leqslant n$. Since $\vee$ is continuous on $\mathbb{Y}$ according to our assumption, for each $i \leqslant n$, there are sets U_{0i}, $U_{1i} \in \mathcal{T}(\mathbb{Y})$ such that $f_0(x_i) \in U_{0i}$, $f_1(x_i) \in U_{1i}$, and $U_{0i} \cap U_{1i} \subseteq U_i$. We put

$$V_0 = V_{x_0,U_{00}} \cap \ldots \cap V_{x_n,U_{0n}}, \quad V_1 = V_{x_0,U_{10}} \cap \ldots \cap V_{x_n,U_{1n}}.$$

Then $f_0 \in V_0$ and $f_1 \in V_1$. If $h \in V_0 \cap V_1$ then $h(x_i) \in U_{0i} \cap U_{1i} \subseteq U_i$ for all $i \leqslant n$. It follows that $h \in V_{x_0,U_0} \cap \ldots \cap V_{x_n,U_n}$. Thus, $V_0 \cap V_1 \subseteq V_{x_0,U_0} \cap \ldots \cap V_{x_n,U_n}$ and the space $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ possesses the property (H_2).

Conversely, assume that $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ has the properties (H_1)–(H_2). According to our assumption, for arbitrary y_0, $y_1 \in Y$, there is $f = \xi_{y_0} \vee \xi_{y_1} \in C(\mathbb{X}, \mathbb{Y})$. We prove that f is a constant function. Indeed, let $f(x_0) \in U \in \mathcal{T}(\mathbb{Y})$; choose an arbitrary element $x_1 \in X$. Then $f \in V_{x_0,U}$. Since $\vee$ is a continuous function, there are elements $u_0, \ldots, u_n, w_0, \ldots, w_m \in X$ and sets $U_0, \ldots, U_n, W_0, \ldots, W_m \in \mathcal{T}(\mathbb{Y})$ such that $\xi_{y_0} \in V_0 = V_{u_0,U_0} \cap \ldots \cap V_{u_n,U_n}$, $\xi_{y_1} \in V_1 = V_{w_0,W_0} \cap \ldots \cap V_{w_m,W_m}$, and $V_0 \cap V_1 \subseteq V_{x_0,U}$. We put $A_0 = U_0 \cap \ldots \cap U_n \in \mathcal{T}(\mathbb{Y})$, $A_1 = W_0 \cap \ldots \cap W_m \in \mathcal{T}(\mathbb{Y})$. Since ξ_{y_0}, ξ_{y_1} are constant functions, $\xi_{y_0} \in V_{x_1,A_0}$ and $\xi_{y_1} \in V_{x_1,A_1}$. Therefore $f \in V_{x_1,A_0} \cap V_{x_1,A_1} = V_{x_1,A_0 \cap A_1}$, whence $f(x_1) \in A_0 \cap A_1$. Moreover, if $y' \in A_0 \cap A_1$ then $\xi_{y'} \in V_0 \cap V_1 \subseteq V_{x_0,U}$, whence $y' \in U$. We have therefore proved that $f(x_1) \in A_0 \cap A_1 \subseteq U$. This means that $f(x_0) \leq f(x_1)$. A similar argument shows that $f(x_1) \leq f(x_0)$. Thus, the space $\mathbb{Y}$ has the property (H_1).

Finally, we prove that $\mathbb{Y}$ has the property (H_2). Suppose that $y = y_0 \vee y_1 \in U \in \mathcal{T}(\mathbb{Y})$. Then $\xi_{y_0} \vee \xi_{y_1} = \xi_y \in V_{x,U}$ for each $x \in X$. Since $\vee$ is a continuous function, there are elements $u_0, \dots, u_n$, $w_0, \dots, w_m \in X$ and sets $U_0, \dots, U_n$, $W_0, \dots, W_m \in \mathcal{T}(\mathbb{Y})$ such that $\xi_{y_0} \in V_0 = V_{u_0,U_0} \cap \dots \cap V_{u_n,U_n}$, $\xi_{y_1} \in V_1 = V_{w_0,W_0} \cap \dots \cap V_{w_m,W_m}$, and $V_0 \cap V_1 \subseteq V_{x_0,U}$. We put $A_0 = U_0 \cap \dots \cap U_n \in \mathcal{T}(\mathbb{Y})$, $A_1 = W_0 \cap \dots \cap W_m \in \mathcal{T}(\mathbb{Y})$. Then we establish as above that $y_0 \in A_0$, $y_1 \in A_1$ and $A_0 \cap A_1 \subseteq U$.

The last statement is obvious. □

Corollary 3, Theorem 8, and Proposition 9 yield

Theorem 10. *A T_0-space $\mathbb{Y}$ is essentially complete if and only if the function space $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ is essentially complete for some* (*equivalently, for each*) *space $\mathbb{X}$.*

5. Injective spaces $\mathbb{C}(\mathbb{X}, \mathbb{Y})$

The proof of the following statement is straightforward.

Corollary 11. *If $\{\mathbb{X}_i \mid i \in I\}$ is a family of* [*densely*] *injective spaces, then $\prod_{i \in I} \mathbb{X}_i$ is a* [*densely*] *injective space.*

Theorem 12. [12, Theorem 4.2.3] *A topological T_0-space $\mathbb{X}$ is injective if and only if the following conditions hold:*

(1) *$\mathbb{X}$ is a d-space;*
(2) *$\mathbb{X}$ is an α-space;*
(3) *$\langle X; \leq_{\mathbb{X}} \rangle$ is a complete lattice.*

Theorem 13. [12, Theorem 4.2.4] *A topological T_0-space $\mathbb{X}$ is densely injective if and only if the following conditions hold:*

(1) *$\mathbb{X}$ is a d-space;*
(2) *$\mathbb{X}$ is an α-space;*
(3) *$\mathbb{X}$ is a* bc-*domain.*

In particular, the following statement follows, see [12, Chapter 7].

Corollary 14. *A topological T_0-space $\mathbb{X}$ is* [*densely*] *injective if and only if $\mathbb{X}$ is a A_d-space with a least and a greatest element* [*with a least element*].

Definition 5. A space $\mathbb{X}$ is an *α^*-space*, if for each set $U \in \mathcal{T}(\mathbb{X})$ and each element $x \in U$, there are elements $x_0, \dots, x_n \in U$ such that $x \in \operatorname{int}(\uparrow x_0 \cup \dots \cup \uparrow x_n)$.

Given topological spaces $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, let $M(\mathbb{X}, \mathbb{Y})$ denote the set of all functions from X to Y. We put

$$\lambda \colon M(\mathbb{Z} \times \mathbb{X}, \mathbb{Y}) \to M\big(\mathbb{Z}, M(\mathbb{X}, \mathbb{Y})\big), \quad \lambda(f) \colon z \mapsto f(z, x);$$
$$\lambda^* \colon M\big(\mathbb{Z}, M(\mathbb{X}, \mathbb{Y})\big) \to M(\mathbb{Z} \times \mathbb{X}, \mathbb{Y}), \quad \lambda^*(g) \colon (z, x) \mapsto [g(z)](x).$$

It is not hard to verify that λ and λ^* are mutually inverse mappings. Hence they establish a one-to-one correspondence between sets

$$M(\mathbb{Z} \times \mathbb{X}, \mathbb{Y})$$

and

$$M\big(\mathbb{Z}, M(\mathbb{X}, \mathbb{Y})\big).$$

For the following definition, we refer to [1]. A topology $\mathcal{T}$ on $C(\mathbb{X}, \mathbb{Y})$ is *proper*, if $\lambda\big(C(\mathbb{Z} \times \mathbb{X}, \mathbb{Y})\big) \subseteq C\big(\mathbb{Z}, \mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})\big)$ for an arbitrary space $\mathbb{Z}$. A topology $\mathcal{T}$ on $C(\mathbb{X}, \mathbb{Y})$ is *admissible*, if $\lambda^*\big(C\big(\mathbb{Z}, \mathbb{C}_{\mathcal{T}}(\mathbb{X}, \mathbb{Y})\big)\big) \subseteq C(\mathbb{Z} \times \mathbb{X}, \mathbb{Y})$ for an arbitrary space $\mathbb{Z}$. A topology $\mathcal{T}$ on $C(\mathbb{X}, \mathbb{Y})$ is *exponential*, if $\mathcal{T}$ is both proper and admissible.

Theorem 15. [2] *The following conditions are equivalent for a T_0-space $\mathbb{X}$.*

(1) *$\mathbb{X}$ is an α^*-space.*
(2) *For all spaces $\mathbb{Y}$, $\mathbb{Z}$, a function $f\colon \mathbb{X}\times\mathbb{Y}\to\mathbb{Z}$ is continuous if and only if f is separately continuous.*
(3) *For an arbitrary space $\mathbb{Y}$, the pointwise convergence topology $\mathcal{P}$ on $C(\mathbb{X},\mathbb{Y})$ is exponential.*

The reader can find a proof of Theorem 15 in [12], see [12, Theorem 6.3.3].

Proposition 16. *Let $\mathcal{T}$ be an exponential topology on $C(\mathbb{X},\mathbb{Y})$.*

(1) *If $\mathbb{Y}$ is [densely] injective, then $\mathbb{C}_{\mathcal{T}}(\mathbb{X},\mathbb{Y})$ is [densely] injective.*
(2) *If $\mathbb{Y}$ is sober, then $\mathbb{C}_{\mathcal{T}}(\mathbb{X},\mathbb{Y})$ is sober.*

Proof. Suppose that a subspace $\mathbb{Z}_0\le\mathbb{Z}$ is such that $Z=\operatorname{sob}_{\mathbb{Z}} Z_0$ and that $f\colon \mathbb{Z}_0\to\mathbb{C}_{\mathcal{T}}(\mathbb{X},\mathbb{Y})$ is a continuous function. Since $\mathcal{T}$ is an admissible topology, $\lambda^*(f)\colon \mathbb{Z}_0\times\mathbb{X}\to\mathbb{Y}$ is a continuous function. It is straightforward to see that $Z\times X=\operatorname{sob}_{\mathbb{Z}\times\mathbb{X}}(Z_0\times X)$. By Theorem 5, there is a continuous function $g\colon \mathbb{Z}\times\mathbb{X}\to\mathbb{Y}$ such that $g|_{\mathbb{Z}_0\times\mathbb{X}}=\lambda^*(f)$. Since $\mathcal{T}$ is a proper topology, $\lambda(g)\colon\mathbb{Z}\to\mathbb{C}_{\mathcal{T}}(\mathbb{X},\mathbb{Y})$ is a continuous function. It is obvious that $\lambda(g)|_{\mathbb{Z}_0}=f$. According to Theorems 4 and 5, $\mathbb{C}_{\mathcal{T}}(\mathbb{X},\mathbb{Y})$ is a sober space. Therefore, statement (ii) is proved. Statement (i) has a similar proof. □

Statement (i) of Proposition 16 was proved in [13].

Proposition 17. *If $\mathbb{C}(\mathbb{X},\mathbb{Y})$ is an α^*-space [an α-space] for some space $\mathbb{X}$, then $\mathbb{Y}$ is also an α^*-space [an α-space].*

Proof. Let $\mathbb{C}(\mathbb{X},\mathbb{Y})$ be an α^*-space. Suppose that $y\in U\in\mathcal{T}(\mathbb{Y})$ and fix an element $x\in X$; then $\xi_y\in V_{x,U}$. According to our assumption, there are continuous functions $f_0,\ldots,f_m\in V_{x,U}$, elements $x_0,\ldots,x_n\in X$, and open sets $W_0,\ldots,W_n\in\mathcal{T}(\mathbb{Y})$ such that $\xi_y\in V\subseteq{\uparrow}f_0\cup\ldots\cup{\uparrow}f_m$, where $V=V_{x_0,W_0}\cap\ldots\cap V_{x_n,W_n}$. Then $y\in W_0\cap\ldots\cap W_n\subseteq{\uparrow}f_0(x)\cup\ldots\cup{\uparrow}f_m(x)$ and $f_0(x),\ldots,f_m(x)\in U$. If $\mathbb{C}(\mathbb{X},\mathbb{Y})$ is an α-space, we assume in the argument above that $m=0$. □

Corollary 18. *If $\mathbb{X}$ is an α^*-space and $\mathbb{Y}$ is a densely injective [injective] space, then $\mathbb{C}(\mathbb{X},\mathbb{Y})$ is a densely injective [injective] space.*

Proof. According to Theorem 15, the pointwise convergence topology on $C(\mathbb{X},\mathbb{Y})$ is exponential. The desired conclusion follows from Proposition 16. □

The following two statements generalize Proposition 16(i) for the pointwise convergence topology.

Theorem 19. *A space $\mathbb{Y}$ is densely injective if and only if $\mathbb{C}(\mathbb{X},\mathbb{Y})$ is a densely injective space for some (equivalently, for each) α^*-space $\mathbb{X}$.*

Proof. Let $\mathbb{Y}$ be a densely injective space and let $\mathbb{X}$ be an α^*-space. According to Corollary 18, $\mathbb{C}(\mathbb{X},\mathbb{Y})$ is densely injective.

Conversely, let the space $\mathbb{C}(\mathbb{X},\mathbb{Y})$ be densely injective for some T_0-space $\mathbb{X}$. According to Theorem 13, $\mathbb{C}(\mathbb{X},\mathbb{Y})$ is an α-space, a d-space, and a bc-domain. According to Proposition 17, $\mathbb{Y}$ is an α-space. According to Corollary 3, $\mathbb{Y}$ is a d-space. In view of Theorem 13, in order to prove the dense injectivity of $\mathbb{Y}$, it suffices to show that $\mathbb{Y}$ is a partial join-semilattice with respect to the specialization order. Indeed, let $y_0,\ y_1\le y$ in $\mathbb{Y}$. This means that $\xi_{y_0},\ \xi_{y_1}\le\xi_y$ in $\mathbb{C}(\mathbb{X},\mathbb{Y})$.

As $\mathbb{C}(\mathbb{X},\mathbb{Y})$ is a bc-domain, there is a continuous function $f = \xi_{y_0} \vee \xi_{y_1}$. We prove that f is constant. Indeed, let $f(x_0) \in U \in \mathcal{T}(\mathbb{Y})$ and let $x_1 \in X$ be an arbitrary element. Then $f \in V_{x_0,U}$. According to Lemma 7 and the definition of the pointwise convergence topology, there are elements $u_0,\dots,u_n$, $w_0,\dots,w_m \in X$ and sets $U_0,\dots,U_n$, $W_0,\dots,W_m \in \mathcal{T}(\mathbb{Y})$ such that $\xi_{y_0} \in V_0 = V_{u_0,U_0} \cap \dots \cap V_{u_n,U_n}$, $\xi_{y_1} \in V_1 = V_{w_0,W_0} \cap \dots \cap V_{w_m,W_m}$, and $V_0 \cap V_1 \subseteq V_{x_0,U}$. We put $A_0 = U_0 \cap \dots \cap U_n \in \mathcal{T}(\mathbb{Y})$ and $A_1 = W_0 \cap \dots \cap W_m \in \mathcal{T}(\mathbb{Y})$. Since the functions ξ_{y_0} and ξ_{y_1} are constant, $\xi_{y_0} \in V_{x_1,A_0}$ and $\xi_{y_1} \in V_{x_1,A_1}$. Thus, $f \in V_{x_1,A_0} \cap V_{x_1,A_1} = V_{x_1,A_0\cap A_1}$, whence $f(x_1) \in A_0 \cap A_1$. Moreover, if $y' \in A_0 \cap A_1$ then $\xi_{y'} \in V_0 \cap V_1 \subseteq V_{x_0,U}$, whence $y' \in U$. We have therefore proved that $f(x_1) \in A_0 \cap A_1 \subseteq U$. This implies that $f(x_0) \leq f(x_1)$. A symmetric argument shows that $f(x_1) \leq f(x_0)$. It follows that $f = \xi_y$ for some $y \in Y$, whence $y = y_0 \vee y_1$ in $\mathbb{Y}$. □

Theorem 20. *A space $\mathbb{Y}$ is injective if and only if $\mathbb{C}(\mathbb{X},\mathbb{Y})$ is an injective space for some (equivalently, for each) α^*-space $\mathbb{X}$.*

Proof. Let $\mathbb{Y}$ be an injective space and let $\mathbb{X}$ be an α^*-space. Then according to Corollary 18, $\mathbb{C}(\mathbb{X},\mathbb{Y})$ is an injective space.

Conversely, let the space $\mathbb{C}(\mathbb{X},\mathbb{Y})$ be injective for some T_0-space $\mathbb{X}$. According to Theorem 12 $\mathbb{C}(\mathbb{X},\mathbb{Y})$ is an α-space, a d-space, and a complete lattice with respect to the specialization order. By Proposition 17, $\mathbb{Y}$ is an α-space. By Theorem 3, $\mathbb{Y}$ is a d-space. Since the space $\mathbb{C}(\mathbb{X},\mathbb{Y})$ possesses the properties (H_0)–(H_2) by Lemma 7, the space $\mathbb{Y}$ also possesses the properties (H_0)–(H_2) by Proposition 9. This means that $\mathbb{Y}$ is a complete lattice with respect to the specialization order. Applying Theorem 12 again, we conclude that $\mathbb{Y}$ is an injective space. □

6. Δ-SPACES $\mathbb{C}(\mathbb{X},\mathbb{Y})$

Definition 6. [8] A continuous function $\delta\colon \mathbb{X} \to \mathbb{X}$ from a topological space $\mathbb{X}$ into itself is a *deflation*, if the set $\delta(X)$ is finite and $\delta(x) \leq_{\mathbb{X}} x$ for all $x \in X$.

A T_0-space $\mathbb{X}$ is a Δ-*space*, if there is an up-directed family $\{\delta_i\colon \mathbb{X} \to \mathbb{X} \mid i \in I\}$ of deflations of $\mathbb{X}$ with the property that for every $U \in \mathcal{T}(\mathbb{X})$ and every $x \in U$, there is $i \in I$ such that $\delta_i(x) \in U$.

A space $\mathbb{X}$ is a Δ_d-*space*, if $\mathbb{X}$ is a Δ-space and a d-space simultaneously.

From Definition 6, we obtain

Corollary 21. [12, Corollary 9.1.4] *Every Δ-space is a α-space.*

In view of Corollary 21 and Corollary 8.2.11 from [12], we obtain

Corollary 22. *A Δ-space $\mathbb{X}$ is sober if and only if $\mathbb{X}$ is a Δ_d-space.*

The following generalization of Theorem 1 from [8] holds.

Theorem 23. *A space $\mathbb{Y}$ is a Δ-space if and only if the function space $\mathbb{C}(\mathbb{X},\mathbb{Y})$ is a Δ-space for some (equivalently, for each) Δ-space $\mathbb{X}$.*

Proof. Let $\mathbb{X}$ and $\mathbb{Y}$ be Δ-spaces. According to [8, Theorem 1], $\mathbb{C}(\mathbb{X},\mathbb{Y})$ is a Δ-space.

Conversely, let $\mathbb{C}(\mathbb{X},\mathbb{Y})$ be a Δ-space for some T_0-space $\mathbb{X}$. Let D denote an up-directed family of deflations of the space $\mathbb{C}(\mathbb{X},\mathbb{Y})$ which satisfies all the requirements of Definition 6. We fix an element $a \in X$. For each $\delta \in D$, consider the mapping

$$\kappa_\delta\colon \mathbb{Y} \to \mathbb{Y}, \quad \kappa_\delta\colon y \mapsto \delta(\xi_y)(a).$$

The set $\kappa_\delta(Y)$ is finite, as the set $\{\delta(\xi_y) \mid y \in Y\}$ is finite. Moreover, for every $y \in Y$, we have $\kappa_\delta(y) = \delta(\xi_y)(a) \leq \xi_y(a) = y$, since δ is a deflation of $\mathbb{C}(\mathbb{X}, \mathbb{Y})$. Let $U \in \mathcal{T}(\mathbb{X})$; then

$$\kappa_\delta^{-1}(U) = \{y \in Y \mid \delta(\xi_y)(a) \in U\} = \{y \in Y \mid \delta(\xi_y) \in V_{a,U}\} = \\ = \{y \in Y \mid \xi_y \in \delta^{-1}(V_{a,U})\} = \xi^{-1}\delta^{-1}(V_{a,U}) \in \mathcal{T}(\mathbb{Y}),$$

as ξ and δ are continuous. Thus, κ_δ is continuous, whence it is a deflation of $\mathbb{Y}$. Moreover, if $\delta, \delta' \in D$ are such that $\delta \leq \delta'$, then $\kappa_\delta(y) = \delta(\xi_y)(a) \leq \delta'(\xi_y)(a) = \kappa_{\delta'}(y)$ for all $y \in Y$. Therefore $\{\kappa_\delta \mid \delta \in D\}$ is an up-directed family of deflations of $\mathbb{C}(\mathbb{Y}, \mathbb{Y})$. Finally, if $y \in U \in \mathcal{T}(\mathbb{X})$ then $\xi_y \in V_{a,U}$. In view of the choice of D, there is a deflation $\delta \in D$ such that $\delta(\xi_y) \in V_{a,U}$. Then $\kappa_\delta(y) = \delta(\xi_y)(a) \in U$, and the proof is complete. □

From Theorems 3 and 23, we get

Corollary 24. *A space $\mathbb{Y}$ is a Δ_d-space if and only if $\mathbb{C}(\mathbb{X}, \mathbb{Y})$ is a Δ_d-space for some (equivalently, for every) Δ-space $\mathbb{X}$.*

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