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KOLMOGOROV DIFFERENTIAL SYSTEMS WITH PRESCRIBED ALGEBRAIC LIMIT CYCLES

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Abstract. For a given algebraic curve, we exhibit Kolmogorov differential systems, and we show that these systems admit precisely the bounded components of the curve as hyperbolic limit cycles if certain conditions on the parameters of the system are satisfied.

Keywords: Sixteenth problem of Hilbert, planar differential system, invariant curve, periodic solution, hyperbolic limit cycle.

1. Introduction

The aim of the second part of sixteenth problem of Hilbert is to find the maximum number of limit cycles of the differential system

$$\begin{aligned} \dot{x} &= \frac{dx}{dt} = P(x,y), \\ \dot{y} &= \frac{dy}{dt} = Q(x,y), \end{aligned} \tag{1.1}$$

where P and Q are two polynomials of any degree. Several articles and books have been published on the analysis of the existence, number and stability of limit cycles of system(1.1), (see for instance [7, 9, 11, 13, 18]).

Many mathematical models in biology science and population dynamics, frequently involve the systems of ordinary differential equations having the form

$$\begin{aligned} \dot{x} &= \frac{dx}{dt} = xF\left(x,y\right), \\ \dot{y} &= \frac{dy}{dt} = yG\left(x,y\right), \end{aligned} \tag{1.2}$$

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where $x(t)$ and $y(t)$ represent the population density of two species at time t, and $F(x,y)$, $G(x,y)$ are the capita growth rate of each specie, usually, such systems are called Kolmogorov systems.

Kolmogorov models are widely used in ecology to describe the interaction between two populations, and a limit cycle corresponds to an equilibrium state of the system.

In mathematical modeling of ecological systems and population dynamics, more mathematicians and scientists were attracted to the subject and several articles have been published, (see for instance [5, 10, 16]).

When $F(x,y)$ and $G(x,y)$ are polynomials of degrees ≥ 2, limit cycles can occur and there is an extensive literature dealing with their existence, number and stability, (see for instance [12, 14, 15]).

The exact analytic expressions of the limit cycles for a given kolmogorov system is still unknown except for specific cases.

This paper is a contribution in the direction of analyzing the existence, determining the number of limit cycles and giving their explicit form for classes of Kolmogorov systems.

Motivated by some research papers exhibiting planar polynomial systems with one or more algebraic limit cycles analytically given (see for instance[1, 2, 4]), and mainly based on the papers of C.Christopher[6], and S. Benyoucef[3], we will extend the same concept to Kolmogorov systems, where just by choosing the components of Kolmogorov system satisfying certain conditions, we can conclude directly the number and the explicit form of limit cycles.

2. Introductory concepts

Let us recall some useful notions.

For $U \in \mathbb{R}[x,y]$, the algebraic curve $U=0$ is called an invariant curve of the polynomial system(1.2), if for some polynomial $K \in \mathbb{R}[x,y]$, called the cofactor of the algebraic curve, we have

$$xF(x,y)\frac{\partial U}{\partial x} + yG(x,y)\frac{\partial U}{\partial y} = KU. \tag{2.1}$$

The curve $\Gamma = \left\{(x,y) \in \mathbb{R}^2, U(x,y)=0\right\}$ is a non-singular of system(1.2), if the equilibrium points of the system that satisfy

$$\begin{aligned} xF(x,y) &= 0, \\ yG(x,y) &= 0, \end{aligned} \tag{2.2}$$

are not contained on the curve Γ.

If the curve Γ is non-singular of system(1.2), the equilibrium points of the system are contained either in its unbounded components or are located on the curve $K(x,y)=0$.

A limit cycle $\gamma = \{(x(t), y(t)), t \in [0,T]\}$ is a $T-$periodic solution isolated with respect to all other possible periodic solutions of the system.

A $T-$periodic solution γ is a hyperbolic limit cycle if $\int_0^T div(\gamma(t))dt$ is different from zero.

3. The main results

Let $\Gamma = \left\{(x,y) \in \mathbb{R}^2, U(x,y)=0\right\}$, a curve of degree n.

We consider a polynomial differential system of degree greater than n

$$\begin{aligned}\dot{x} &= x\left(R(x,y)U - y\Phi(x,y)U_y\right),\\ \dot{y} &= y\left(S(x,y)U + x\Phi(x,y)U_x\right),\end{aligned} \tag{3.1}$$

where $R(x,y)$, $S(x,y)$, $\Phi(x,y)$ are polynomial functions.

Our contribution consist to show that the system(3.1) admits all the bounded components of Γ as hyperbolic limit cycles if certain conditions on the parameters are satisfied.

Theorem 1. *Let U, C^1 function in open subset $V = \left\{(x,y) \in \mathbb{R}^2,\ x>0, y>0\right\}$.*
If $U = 0$ is non-singular of polynomial differential system

$$\begin{aligned}\dot{x} &= x\left(\left(P(y) + axy + b\right)U - \alpha y U_y\right),\\ \dot{y} &= y\left(\left(Q(x) + cxy + d\right)U + \alpha x U_x\right),\end{aligned} \tag{3.2}$$

where $P(y)$ and $Q(x)$ are polynomial functions of any degree, α, a, b, c, d are real numbers satisfying $\alpha \neq 0, a + c \neq 0$, then the system (3.2) admits all the bounded components of $U = 0$ as hyperbolic limit cycles.

Proof. Let Γ the curve of $U = 0$.

Note that Γ is non-singular of system (3.2).

To show that all the bounded components of Γ are hyperbolic limit cycles of system (3.2), we will prove that Γ is an invariant curve of the system(3.2), and $\int_0^T div(\Gamma)dt \neq 0$. (See for instance Perko[17, Pages 216-217]).

i) Γ is an invariant curve of the system(3.2):

$\frac{dU}{dt} = U_x\left(\dot{x}\right) + U_y\left(\dot{y}\right)$
$= U_x\left(x\left(\left(P(y) + axy + b\right)U - \alpha y U_y\right)\right) + U_y\left(y\left(\left(Q(x) + cxy + d\right)U + \alpha x U_x\right)\right)$
$= U\left(xU_x\left(P(y) + axy + b\right) + yU_y\left(Q(x) + cxy + d\right)\right),$

where the cofactor is $K(x,y) = xU_x\left(P(y) + axy + b\right) + yU_y\left(Q(x) + cxy + d\right)$.

ii) $\int_0^T div(\Gamma)dt$ is nonzero.

To see this, note that

$$\int_0^T div(\Gamma)dt = \int_0^T K(x(t), y(t))dt, \tag{3.3}$$

see for instance Giacomini & Grau [8, theo 2].

$\int_0^T K(x(t),y(t))dt = \oint_\Gamma \frac{xU_x(P(y)+axy+b)}{\alpha xyU_x}dy + \oint_\Gamma \frac{yU_y(Q(x)+cxy+d)}{-\alpha xyU_y}dx$
$= \oint_\Gamma \frac{(P(y)+axy+b)}{\alpha y}dy - \oint_\Gamma \frac{(Q(x)+cxy+d)}{\alpha x}dx$
$= \frac{1}{\alpha}\left(\oint_\Gamma\left(ax + \frac{P(y)}{y} + \frac{b}{y}\right)dy - \oint_\Gamma\left(cy + \frac{Q(x)}{x} + \frac{d}{x}\right)dx\right).$

By applying the GREEN formula we obtain

$\oint_\Gamma\left(ax + \frac{P(y)}{y} + \frac{b}{y}\right)dy - \oint_\Gamma\left(cy + \frac{Q(x)}{x} + \frac{d}{x}\right)dx$
$= \int\int_{int(\Gamma)}\left(\frac{\partial\left(ax + \frac{P(y)}{y} + \frac{b}{y}\right)}{\partial x} + \frac{\partial\left(cy + \frac{Q(x)}{x} + \frac{d}{x}\right)}{\partial y}\right)dxdy$
$= \int\int_{int(\Gamma)}(a+c)\,dxdy,$

where $int(\Gamma)$ denotes the interior of Γ.

As $a + c \neq 0$, and $\alpha \neq 0$, then $\int_0^T K(x(t),y(t))dt$ is nonzero. □

Example 2. *Septic Kolmogorov system with two limit cycles.*
Let $a = b = c = d = \alpha = 1$, $P(y) = y^2$, $Q(x) = x^2$,

$U(x,y) = \left((x-2)^2 + (y-2)^2 - 1\right)^2 - (x-1)^2(y-1) + (x-1)(y-1)^2.$

The differential system

(3.4)

$$\dot{x} = x\left(\begin{array}{c} \left(y^2+xy+1\right)\left(\left((x-2)^2+(y-3)^2-1\right)^2-(x-2)^2(y-3)+4(x-2)(y-3)\right) \\ -y\left(4x^2y-13x^2-16xy+56x+4y^3-36y^2+120y-156\right) \end{array}\right),$$
$$\dot{y} = y\left(\begin{array}{c} \left(x^2+xy+1\right)\left(\left((x-2)^2+(y-3)^2-1\right)^2-(x-2)^2(y-3)+4(x-2)(y-3)\right) \\ +x\left(4x^3-24x^2+4xy^2-26xy+86x-8y^2+56y-120\right) \end{array}\right)$$

admits two limit cycles represented by the curve $\left((x-2)^2+(y-3)^2-1\right)^2-(x-2)^2(y-3)+4(x-2)(y-3)=0$. *See Figure 1.*

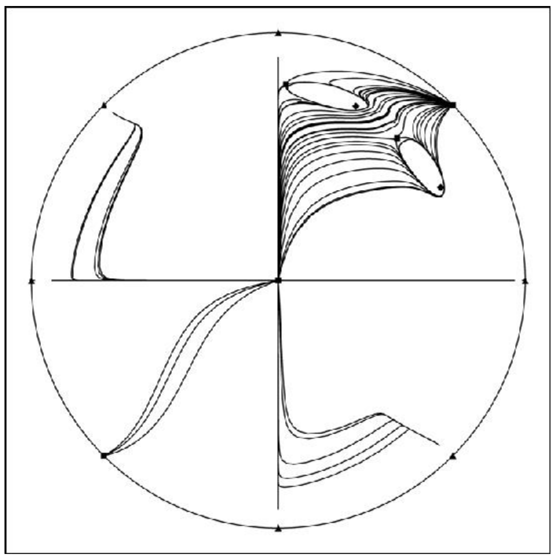

FIG. 1. The phase portrait in the Poincar disc of the polynomial differential system (3.4)

Theorem 3. *Let U, C^1 function in open subset $V=\left\{(x,y)\in\mathbb{R}^2,\ x>0, y>0\right\}$.*
We consider a polynomial differential system

$$\begin{array}{l} \dot{x} = x\left(aU-\alpha xy^2U_y\right), \\ \dot{y} = y\left(bU+\alpha yx^2U_x\right), \end{array} \tag{3.5}$$

where $\alpha\neq 0$ and $a+b\neq 0$, then the system(3.5) admits all the bounded components of $U=0$ as hyperbolic limit cycles.

Proof. Let Γ the curve of $U=0$.
i) Γ is an invariant curve:
$\frac{dU}{dt}=U_x\left(x\left(aU-\alpha xy^2U_y\right)\right)+U_y\left(y\left(bU+\alpha yx^2U_x\right)\right)$
$=\left(axU_x+byU_y\right)U.$
The cofactor is $K(x,y)=axU_x+byU_y$.
ii) $\int_0^T div(\Gamma)dt$ is nonzero.
$\int_0^T div(\Gamma)dt=\int_0^T K(x(t),y(t))dt$

$$= \oint_\Gamma \frac{axU_x}{\alpha y^2 x^2 U_x} dy + \oint_\Gamma \frac{byU_y}{-\alpha x^2 y^2 U_y} dx$$
$$= \oint_\Gamma \frac{a}{\alpha x y^2} dy - \oint_\Gamma \frac{b}{\alpha x^2 y} dx$$
$$= \int\int_{int(\Gamma)} \left(\frac{a}{\alpha y^2} \frac{\partial\left(\frac{1}{x}\right)}{\partial x} + \frac{b}{\alpha x^2} \frac{\partial\left(\frac{1}{y}\right)}{\partial y} \right) dxdy$$
$$= -\frac{a+b}{\alpha} \int\int_{int(\Gamma)} \frac{1}{x^2y^2} dxdy.$$

As $\alpha \neq 0$ and $a + b \neq 0$ then $\int_0^T div(\Gamma)dt$ is nonzero. □

Example 4. *A septic Kolmogorov system with three limit cycles.*

Let $\alpha = a = b = 1$, $U(x,y) = \left((x-2)^2 + (y-2)^2 - 1\right)^2 - (x-2)^2(y-2) - (x-2)(y-2)^2$.

The differential system

(3.6)
$$\dot{x} = x \left(\begin{array}{c} \left(\left((x-2)^2 + (y-2)^2 - 1\right)^2 - (x-2)^2(y-2) - (x-2)(y-2)^2\right) \\ -xy^2\left(4x^2y - 9x^2 - 18xy + 40x + 4y^3 - 24y^2 + 64y - 68\right) \end{array} \right),$$
$$\dot{y} = y \left(\begin{array}{c} \left(\left((x-2)^2 + (y-2)^2 - 1\right)^2 - (x-2)^2(y-2) - (x-2)(y-2)^2\right) \\ +yx^2\left(4x^3 - 24x^2 + 4xy^2 - 18xy + 64x - 9y^2 + 40y - 68\right) \end{array} \right)$$

admits three limit cycles represented by the curve $\left((x-2)^2 + (y-2)^2 - 1\right)^2 - (x-2)^2(y-2) - (x-2)(y-2)^2 = 0$. *See Figure 2.*

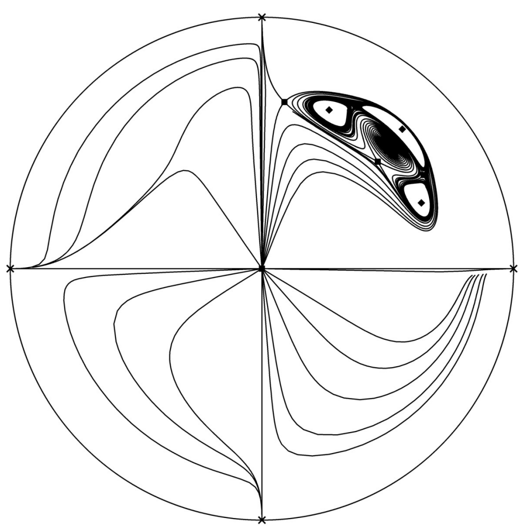

FIG. 2. The phase portrait in the Poincar disc of the polynomial differential system (3.6)

Now, we consider a differential system

$$\begin{aligned} \dot{x} &= x\left(aU - y\Phi(x,y)U_y\right), \\ \dot{y} &= y\left(bU + x\Phi(x,y)U_x\right), \end{aligned} \tag{3.7}$$

where $\Phi(x,y)$ can be non-polynomial.

Theorem 5. *Let U and Φ, C^1 functions in open subset*

$$V=\left\{(x,y)\in\mathbb{R}^2,\ x>0,y>0\right\},$$

$\Phi(x,y)=\frac{c}{a+b}xy+\frac{y^a}{x^b}$, *where a,b,c are nonzero reals and $a+b\neq 0$. If the curve $\Phi(x,y)=0$ lies outside all bounded components of the non-singular curve $U=0$, then the differential system*

$$\begin{aligned}\dot{x}&=x\left(aU-y\Phi U_y\right),\\ \dot{y}&=y\left(bU+x\Phi U_x\right)\end{aligned}\tag{3.8}$$

admits all the bounded components of $U=0$ as hyperbolic limit cycles.

Proof. Let Γ the curve of $U=0$.

i) Γ is an invariant curve:

$\frac{dU}{dt}=U_x\left(x\left(aU-y\Phi U_y\right)\right)+U_y\left(y\left(bU+x\Phi U_x\right)\right)$

$\quad=\left(axU_x+byU_y\right)U.$

The cofactor is $K(x,y)=axU_x+byU_y$.

ii) $\int_0^T div(\Gamma)dt$ is nonzero.

$\int_0^T div(\Gamma)dt=\int_0^T K(x(t),y(t))dt$

$\quad=\oint_\Gamma\frac{axU_x}{yx\Phi U_x}dy+\oint_\Gamma\frac{byU_y}{-xy\Phi U_y}dx$

$\quad=\oint_\Gamma\frac{a}{y\Phi}dy-\oint_\Gamma\frac{b}{x\Phi}dx$

$\quad=\int\int_{int(\Gamma)}\left(\frac{a}{y}\frac{\partial\left(\frac{1}{\Phi}\right)}{\partial x}+\frac{b}{x}\frac{\partial\left(\frac{1}{\Phi}\right)}{\partial y}\right)dxdy$

$\quad=-\int\int_{int(\Gamma)}\frac{\frac{a}{y}\frac{\partial\Phi}{\partial x}+\frac{b}{x}\frac{\partial\Phi}{\partial y}}{\Phi^2}dxdy.$

$\Phi(x,y)=\frac{c}{a+b}xy+\frac{y^a}{x^b}\Rightarrow$

$\frac{a}{y}\frac{\partial\Phi}{\partial x}+\frac{b}{x}\frac{\partial\Phi}{\partial y}=\frac{a}{y}\left(-\frac{1}{x^{b+1}(a+b)}\left(y^ab^2+y^aab-cx^{b+1}y\right)\right)$

$+\frac{b}{x}\left(\frac{1}{x^by(a+b)}\left(y^aa^2+y^aab+cx^{b+1}y\right)\right)=c.$

So $\int_0^T div(\Gamma)dt=-\int\int_{int(\Gamma)}\frac{c}{\Phi^2}dxdy.$

As $c\neq 0$, $\int_0^T div(\Gamma)dt$ is nonzero. □

Remark 6. *If $a=b=1$, $\Phi(x,y)=\frac{c}{2}xy+\frac{y}{x}$. When U is algebraic the differential system (3.7) is polynomial and it is writing as*

$$\begin{aligned}\dot{x}&=xU-\left(\tfrac{c}{2}x^2y^2+y^2\right)U_y,\\ \dot{y}&=yU+\left(\tfrac{c}{2}x^2y^2+y^2\right)U_x.\end{aligned}\tag{3.9}$$

Example 7. *A quintic differential system with one limit cycle*

Let $a=b=1$, $c=2$, $U(x,y)=2x^2-10x+3y^2-12y+20$.

The differential system

$$\begin{aligned}\dot{x}&=x\left(2x^2-10x+3y^2-12y+20\right)-y^2\left(x^2+1\right)(6y-12),\\ \dot{y}&=y\left(2x^2-10x+3y^2-12y+20\right)+y^2\left(x^2+1\right)(4x-10)\end{aligned}\tag{3.10}$$

admits one limit cycle represented by the curve $2x^2-10x+3y^2-12y+20=0$. See Figure 3.

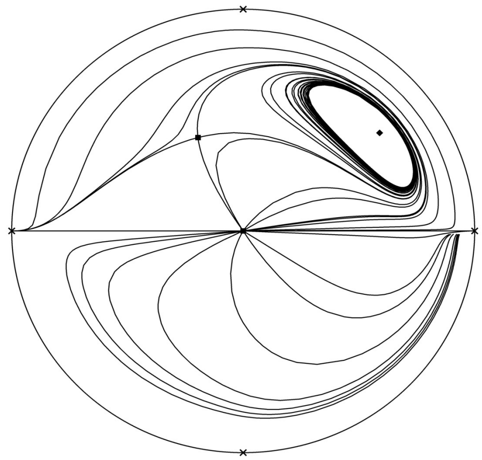

FIG. 3. The phase portrait in the Poincar disc of the polynomial differential system (3.10)

4. Conclusion

For a choice of algebraic curves of degree n, which present bounded components, we have proposed classes of Kolmogorov differential systems of degree greater than n, where it suffices to check certain conditions on the parameters to directly conclude the existence and the number of limit cycles. In addition we have given the algebraic expression of these limit cycles. In the future, we hope to extend this work for non-algebraic limit cycles.

Note: All figures are plotted on the Poincaré disc by using polynomial planar phase portraits program, see for instance [7, pages 233-257].

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