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WEAK REDUCIBILITY OF COMPUTABLE AND GENERALIZED COMPUTABLE NUMBERINGS

Z.K. IVANOVA, M.KH. FAIZRAHMANOV

Abstract. We consider universal and minimal computable numberings with respect to weak reducibility. A family of total functions that have a universal numbering and two non-weakly equivalent computable numberings is constructed. A sufficient condition for the non-existence of minimal A-computable numberings of families with respect to weak reducibility is found for every oracle A.

Keywords: computable numbering, w-reducibility, A-computable numbering, Rogers semilattice.

1. Introduction

The goal of this article is to carry out a comparative analysis of elementary theories of classes of Rogers semilattices of computable and generalized computable families with respect to classic and *weak* reducibilities of numberings. The weak or *w-reducibility* of numberings was first introduced and studied in work [1]. In the cited paper, it emerged in relation to a research of universal minimal coverings in Rogers semilattices of families of arithmetical sets.

Definition 1. We say that a numbering μ of a set S is*weakly reducible* to a numbering ν of the same set (in this case, a designation $\mu \leqslant_w \nu$ is used), if there exists a computable function f such that for every x there is $y \in D_{f(x)}$, for which $\mu(x) = \nu(y)$, where by D_n a finite set with a canonical index n is designated.

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In this paper, apart from a classical notion of computable numbering we will also encounter *A-computable numberings.*

Definition 2 ([2, 3]). A numbering ν of a countable family of subsets $\mathbb{N}$ is called *A-computable*, if a set of pairs

$$G_\nu = \{\langle x, y\rangle : x, y \in \mathbb{N},\ y \in \nu(x)\}$$

is A-computably enumerable (A-c.e.).

When $A = \emptyset$, we come to a classical notion of *computable* numberings (see [4]), and when $A = \emptyset^{(n+1)}$, we have a notion of Σ^0_{n+2}-*computable* numberings (see, for example, [1, 2, 5, 6]). We call a family *computable* (*A-computable*, Σ^0_{n+2}-*computable*), if it possesses a computable (A-computable, Σ^0_{n+2}-computable) numbering.

We will provide some preliminary information from general theory of numberings. If μ and ν are two numberings of a set S, we say that μ *is reducible to* to ν, if there exists a computable function f, such that $\mu = \nu \circ f$. It is easy to see that classical reducibility $\mu \leqslant \nu$ also yields reducibility $\mu \leqslant_w \nu$. The introduced relations $\leqslant$ and $\leqslant_w$ are preorders on a set of all A-computable numberings of the A-computable family $\mathcal{S}$. That means that they set the following equivalence relations

$$\mu \equiv \nu \Leftrightarrow \mu \leqslant \nu \,\&\, \nu \leqslant \mu,\ \ \mu \equiv_w \nu \Leftrightarrow \mu \leqslant_w \nu \,\&\, \nu \leqslant_w \mu.$$

A set of all classes of $\equiv$-equivalent A-computable numberings of the family $\mathcal{S}$ with respect to an order, given by a reducibility relation, forms an upper semilattice $\mathcal{R}^A(\mathcal{S})$, referred to as *Rogers semilattice* of A-comtutable numberings of the family $\mathcal{S}$ (with respect to classical reducibility). By replacing the reducibility $\leqslant$ with $\leqslant_w$, in a similar way we can come to a notion of the *Rogers semilattice* of A-computable numberings of the family $\mathcal{S}$ with respect to weak reducibility. In both semilattices, the exact upper bound of equivalency classes of numberings ν_0 and ν_1 is the class of their *direct sum* $(\nu_0 \oplus \nu_1)(2x + i) = \nu_i(x)$, $i = 0, 1$.

2. Weak reducibility and minimal coverings

An element a of a partially ordered set L is called a *minimal covering* of element $b \in L$, if there is no element $x \in L$ for which $a < x < b$.

In [1], it was established that if the largest element $[\nu]$ of a semilattice $\mathcal{R}^{\emptyset^{(n+1)}}(\mathcal{S})$ is a minimal covering of the element $[\mu] \in \mathcal{R}^{\emptyset^{(n+1)}}(\mathcal{S})$, then $\nu \leqslant_w \mu$. On the other hand, in [7] it was proved that the largest elements of these semilattices cannot be minimal coverings. However, the initial theorem admits the following generalization while the proof retains the main ideas from [1].

Theorem 1. *Let A be a high set and $\mathcal{S}$ be a A-computable family that possesses a universal A-computable numbering ν. If $[\nu]$ is a minimal covering $[\mu]$ in $\mathcal{R}^A(\mathcal{S})$, then $\nu \leqslant_w \mu$.*

Proof. First, we will establish the existence of an A-computable sequence of the family of all injective computable functions. According to [8], there exists a A-computable sequence $\{g_n\}_{n\in\mathbb{N}}$, consisting of all computable functions. For all n and x, we put

$$f_n(x) = \begin{cases} g(x), & \text{if } \forall y \leqslant x \forall z \leqslant x\, [y \neq z \Rightarrow g(y) \neq g(z)], \\ \sum\limits_{i=0}^{x} (g(i) + 1), & \text{otherwise.} \end{cases}$$

It is clear that for every n the function f_n is injective. Moreover, if g_n is injective, then $f_n = g_n$. Therefore, $\{f_n\}_{n\in\mathbb{N}}$ is the required sequence. We will show that for every immune set $X \leqslant_T A$ there exists a computable function h, such that $\nu(x) = \mu(h(x))$ for every $x \in X$.

Assume, on the contrary, that there exists an immune set $X \leqslant_T A$, such that $\nu \restriction X \neq \mu \circ h \restriction X$ for every computable function h. As it was proved in [9], if the set A is high, then every universal A-computable numbering is precomplete and, hence, also cylindrical (see [4]). Therefore, we can choose a computable function p, such that for every x, s, t an inequality $p(x, s) \neq p(x, t)$ holds when $s \neq t$, and $\nu(x) = \nu(p(x, s))$ for all x, s. We will construct an A-computable numbering γ of the family $\mathcal{S}$, for which $\mu < \mu \oplus \gamma < \nu$, thus obtaining a contradiction to the choice of $[\mu]$ and $[\nu]$.

Construction of γ

Step 0. We put $\gamma(x) = \nu(x)$ for all $x \in X$.

Step $2s + 1$. Suppose that $s = c(n, x)$. Due to the immunity of X, we can choose the smallest t such that the value of $\gamma(f_n(p(x, t)))$ is not defined. We put $\gamma(f_n(p(x, t))) = \mu(x)$.

Step $2s+2$. We will choose the smallest z, on which the value of $\gamma(z)$ is not defined, and put $\gamma(z) = \nu(s)$.

With that, the construction is finished. From the existence of even steps it follows that γ enumerates the whole family $\mathcal{S}$. Since $\gamma(x) = \nu(x)$ for all $x \in X$, we have that $\gamma \nleqslant \mu$. Assume that $\nu \leqslant \gamma$. Then γ is also cylindrical. Hence, $\nu = \gamma \circ f_n$ for some n. Directly from construction it follows that for all x there exists t, for which

$$\mu(x) = \gamma(f_n(p(x, t))) = \nu(p(x, t)) = \nu(x).$$

On the other hand, $\nu \nleqslant \mu$. We have arrived at a contradiction. Therefore, $\nu \nleqslant \gamma$, and due to indecomposability of ν (see [4]), we have that $\nu \nleqslant \mu \oplus \gamma$ also holds.

As it was proved in [10], every hyperimmune degree, especially a turing one, contains a bi-immune set. We fix the bi-immune set $X \leqslant_T A$ and computable functions h_1, h_2, such that for all x

$$\nu(x) = \begin{cases} \mu(h_1(x)), & \text{if } x \in X, \\ \mu(h_2(x)), & \text{if } x \notin X. \end{cases}$$

Therefore, $\nu(x) \in \{\mu(h_1(x)), \mu(h_2(x))\}$. Hence, $\nu \leqslant_w \mu$. □

3. Universal numberings with respect to weak reducibility

It is well known [11] that non-trivial Rogers semilattices of families of total functions do not have any largest elements. We will show that when transitioning from classical reducibility to the weak one, the formulated statement turns to be incorrect.

Definition 3. A computable numbering ν of the family $\mathcal{S}$ *w-universal*, if every computable numbering μ of the same family is weakly reducible to ν.

It is clear that w-universal numberings form the largest elements of Rogers semilattices of families under numbering.

Theorem 2. *There exists a family of total functions $\mathcal{F}$ with a w-universal numbering, which possesses two non w-equivalent computable numberings.*

Proof. To present the proof, we will introduce some additional designations. Let h be a binary function. For every $z \in \mathbb{N}$, we will define a function $h[z]$ by putting $h[z] = \lambda y[h(z, y)]$. We designate by $\mathcal{F}_h$ the following family of unary functions

$$\mathcal{F}_h = \{h[z] : z \in \mathbb{N}\}.$$

To define the required family $\mathcal{F}$, it is said to be w-universal numbering ν, and a computable numbering μ, such that $\nu \not\leqslant_w \mu$, we define binary computable functions f and g which satisfy the conditions:

1) if a binary partially computable function $\varphi_e^{(2)}$ with a Gödel number e is total and $\mathcal{F}_{\varphi_e^{(2)}} = \mathcal{F}_f$, then there exists a computable function h_e for which
$$\forall z \exists z_1 \, [z_1 \in D_{h_e(z)} \,\&\, \varphi_e^{(2)}[z] = f[z_1]];$$
2) $\mathcal{F}_f = \mathcal{F}_g$;
3) if a unary partially computable funtion φ_e is total, then

$$\exists k \forall l \, [l \in D_{\varphi_e(k)} \Rightarrow f[k] \neq g[l]]. \tag{1}$$

It is easy to see that in this case the family $\mathcal{F} = \mathcal{F}_f$ and the numberings $\nu : z \mapsto f[z]$, $\mu : z \mapsto g[z]$ will be the required ones.

Construction of computable functions f, g, and a sequence of partial functions $\{h_e\}_{e \in \mathbb{N}}$

Step 0. For all x, y, t, we put

$$f(2c(x, t), y) = f(2c(x, t) + 1, 0) = g(2x, y) = 2x.$$

Suppose that for all e, z the value of $h_e(z)$ is not defined.

Step $2s + 1 = 2c(e, u) + 1$. For all $x, t \in \mathbb{N}$ and the smallest y for which the value of $f(2c(x, t) + 1, y)$ is not defined, we put $f(2c(x, t) + 1, y) = 2x$. We fix the first z for which the value of $h_e(z)$ is not defined. If $\varphi_{e,s}^{(2)}(z, 0) = 2x$ for some x, then we define the value of $h_e(z)$ as being equal to the canonical index of the set $\{2c(x, s), 2c(x, s) + 1\}$. But if there is no suitable x, then we just proceed to the next step.

Step $2s + 2 = 2c(e, u) + 2$. If the value of $\varphi_{e,s}(2e + 1)$ is defined and $2e \in D_{\varphi_e(2e+1)}$, then we choose the smallest y_0 for which the value of $f(2e + 1, y_0)$ is not defined, and put that $f(2e + 1, y) = 2e + 1$ for all $y \geqslant y_0$. We fix the first z such that $2z + 1 \notin D_{\varphi_e(2e+1)}$ and $g[2z + 1]$ is a nowhere-defined function, and define $g(2z + 1, y) = f(2e + 1, y)$ for all y. Next, for all v such that $2v + 1 \in D_{\varphi_e(2e+1)}$ and the function $g[2v + 1]$ is defined nowhere, we put $g(2v + 1, y) = 2e$ for every y.

But if the value of $\varphi_{e,s}(2e + 1)$ is not defined, or $2e \notin D_{\varphi_e(2e+1)}$, or $f[2e + 1]$ is total, then we proceed to the next step.

The construction is finished. First, we will show that the function g is total. During the zero step of the construction, we define for all x, y the values of $g(2x, y)$. We will fix x arbitrarily and prove that the function $g[2x + 1]$ is also total. To do that, we define the computable function d, putting for all i the value of the function $\varphi_{d(e)}(i)$ as equal to the canonical index of the set $\{2e, 2x + 1\}$. Let e be a fixed point of the function d: $\varphi_{d(e)} = \varphi_e$. We fix the first step $s = c(e, u)$ on which the value $\varphi_{e,s}(2e+1)$ is defined. Note that $\{2e, 2x+1\} \subseteq D_{\varphi_e(2e+1)}$. Then the function $g[2x+1]$ becomes total not later than on the step $2s+2 = 2c(e,u)+2$.

Now we will prove that conditions 1)–3) hold.

We choose an arbitrary e such that the function $\varphi_e^{(2)}$ is total and $\mathcal{F}_{\varphi_e^{(2)}} = \mathcal{F}_f$. For an arbitrary z, we choose the smallest u and also x such that $\varphi_{e,s}^{(2)}(z,0) = 2x$ when $s = c(e,u)$. Then the value of $h_e(z)$ is defined and

$$D_{h_e(z)} = \{2c(x,s), 2c(x,s)+1\}.$$

Moreover, directly from construction it follows that either $\varphi_e^{(2)}[z] = f[2c(x,s)]$, or $\varphi_e^{(2)}[z] = f[2c(x,s)+1]$.

We will check whether the second condition holds. Directly from construction it follows that $\mathcal{F}_g \subseteq \mathcal{F}_f$. We will justify the reverse. For all x and t, we have that $f[2\langle x,t\rangle] = g[2x]$. Suppose that for a given e, the inequality

$$f[2e+1] \neq f[2e]$$

holds. According to the actions performed during the steps of the form $2s+1 = 2c(e,u)+2$ of the construction, we have that the value of $\varphi_{e,s}(2e+1)$ is defined and $f[2e+1] = g[2z+1]$, where z is the smallest number for which $2z+1 \notin D_{\varphi_e(2e+1)}$.

Finally, we check whether the third condition holds. Let φ_e be total. Assume that $2e \notin D_{\varphi_e(2e+1)}$. Then, according to construction, if $f[2e+1] = g[l]$, then $l = 2e$. Therefore, (1) holds when $k = 2e+1$. Now suppose that $2e \in D_{\varphi_e(2e+1)}$. Then the only l, for which $f[2e+1] = g[l]$, does not belong to $D_{\varphi_e(2e+1)}$.

This concludes the proof of the theorem. □

It is worth noting that the conclusion of the proven theorem is not absolute with respect to every oracle. Indeed, we choose an arbitrary set A such that $\emptyset' \leqslant_T A$. Then every finite A-computable family $\mathcal{F}$ of everywhere defined functions does not have any w-universal numberings. This follows from the fact that for every A-computable numbering ν of the family $\mathcal{F}$ we can define an A-computable numbering $\mu \nleqslant_w \nu$ of some subfamily $\mathcal{F}$, having set, for example, for every $e \in K = \{e : \varphi_e(e) \text{ defined}\}$, the value of $\mu(e)$ to be equal to the function of the family $\mathcal{F}$, distinct from all functions of the finite set $\{\nu(x) : x \in D_{\varphi_e(e)}\}$ (when $e \notin K$, we can put $\mu(e) = \nu(0)$).

Corollary 1. *Let K_m and K_w be the classes of Rogers semilattices of all computable families of total functions, defined by a classical and weak redicibilities of numberings respectively. Then* $\mathrm{Th}(K_m) \neq \mathrm{Th}(K_w)$.

Proof. As it has been noted above, according to [11], every semilattice from K_m containing more than one element, does not have the largest element. As it follows from Theorem 2, the corresponding sentence does not belong to $\mathrm{Th}(K_w)$. □

4. Minimal numberings with respect to a weak reducibility

In this paragraph, a dependence of one well-known sufficient condition of existence of positive computable numberings (see [12]) on the absence of w-minimal computable numberings of the given families is described. The corresponding result is absolute (does not depend on the choice of an oracle).

Definition 4. We say that an A-computable numbering μ of a family $\mathcal{S}$ is *w-minimal*, if for every A-computable numbering $\alpha \leqslant_w \mu$ of the same family, the reducibility $\mu \leqslant_w \alpha$ holds.

It is trivial that w-minimal numberings define the minimal elements of Rogers semilattices of the families under numbering.

Theorem 3. *Suppose that A is an arbitrary set and $\mathcal{S}$ is an infinite A-computable family that satisfies the following conditions:*

1) $\mathcal{S}_0 = \{R \in \mathcal{S} : R$ *is maximal in* $\mathcal{S}$ *with respect to* $\subseteq\}$ *is finite;*
2) *every set* $Q \in \mathcal{S}$ *contains in some set* $R \in \mathcal{S}_0$.

Then S does not have any w-minimal A-computable numberings.

Proof. Suppose that $\mathcal{S}_0 = \{R_0, R_1, \ldots, R_n\}$ and ν is an arbitrary A-computable numbering of the family $\mathcal{S}$. Without loss of generality, we will consider that $\nu(i) = R_i$ when $i \leqslant n$. We will show that ν is not w-minimal.

To do that, we will need the notion of a *coding sequence*. For a pair of numbers $k < l$, we designate by $C_{k,l}$ a set of all triples $\langle u, v, w\rangle$ satisfying the following conditions:

(1) $D_u \cup D_v$ is the initial segment of natural numbers;
(2) $D_u \cap D_v = \emptyset$;
(3) $w \in \{k, l\}$.

Then we will call every finite sequence $\{\langle u_k, v_k, w_k\rangle\}_{k<l}$, where $\langle u_k, v_k, w_k\rangle \in C_{k,l}$, a *coding sequence*. We enumerate effectively and without repetitions all coding sequences using natural numbers, and further we will identify a coding sequence with its number. Let $\mathrm{lh}(y)$ be the length (that is, the number of elements) of a coding sequence y. Coding sequences of length l will be used for checking of inequalities $\nu(l) \neq \nu(k)$, $k < l$. Let $\{\nu_s(x)\}_{s,x\in\mathbb{N}}$ be a double strongly A-computable sequence monotonic in s, such that $\nu(x) = \bigcup_s \nu_s(x)$ for all x. We say that the coding sequence $\{\langle u_k, v_k, w_k\rangle\}_{k<l}$ *satisfies the positive conditions in the step* s, if for all $k < l$

$$D_{u_k} \subseteq \nu_s(k) \cap \nu_s(l) \,\&\, m_k = \max(D_{u_k} \cup D_{v_k}) + 1 \in \nu_s(w_k) \ (\max\emptyset = -1),$$

and *the negative ones*, if for all $k < l$

$$D_{v_k} \subseteq \mathbb{N} \setminus (\nu_s(k) \cup \nu_s(l)) \,\&\, m_k \notin \nu_s(\bar{w}_k) \ (\bar{w}_k \in \{k, l\} \setminus \{w_k\}).$$

We will say that a coding sequence *satisfies the positive (negative) conditions*, if it satisfies them in some step (all steps). It is easy to see that for every $X \in \mathcal{S}$ there exists a unique coding sequence y that satisfies both positive and negative conditions, such that $\nu(\mathrm{lh}(y)) = X$.

Now we will define an A-computable numbering $\alpha \leqslant_w \nu$ of the family $\mathcal{S}$, for which $\nu \not\leqslant_w \alpha$. We fix some A-computable function

$$f : \mathbb{N} \times \mathbb{N} \to \{0, \ldots, n\},$$

such that $\nu_s(x) \subseteq R_{f(x,s)}$ for all x and s.

Construction of α

STEP 0. For all y and t we put

$$\alpha_0(c(y,t)) = \begin{cases} \nu_t(l), & \text{if } y \text{ is a coding sequence of length } l > n, \\ & \text{satisfying the positive conditions in the step } t, \\ R_{\mathrm{lh}(y)}, & \text{if } y \text{ is a coding sequence of length} \\ & \text{not exceeding } n, \\ R_0, & \text{in the rest of the cases.} \end{cases}$$

Suppose that

$M_0 = \{c(y,t)$: the length of the coding sequence$y$ does not exceed n or y does not satisfy the positive conditions in the step $t\}$.

STEP $2s+1 = 2c(e,u)+1$. We choose the smallest $x > \max\{e,n\}$, such that:

- there exists a coding sequence y of length x, satisfying in some step $t < s$ the positive conditions (we fix the first of such t) and the negative conditions in step s;
- the value of $\varphi_{e,s}(x)$ is defined (suppose that $\varphi_e(x) = n$);
- there exists v for which

$$t \leqslant v \leqslant s \,\&\, c(y,v) \in D_n \setminus M_{2s}. \tag{2}$$

We put $\alpha_{2s+1}(c(y,v)) = R_{f(x,s)}$ for all v satisfying (2), and

$$M_{2s+1} = M_{2s} \cup \{c(y,v) : v \text{ satisfies } (2)\}.$$

If there is no such x, then we proceed to the next step, leaving all definable values without any changes.

STEP $2s+2$. For all y and all $t < s$ we put that

$$\alpha(c(y,t)) = \begin{cases} \nu_s(\mathrm{lh}(y)), & \text{if } y \text{ in step } t \text{ satisfies the positive} \\ & \text{conditions and in step } s \text{ satisfies the negative conditions,} \\ R_{f(\mathrm{lh}(y),s)}, & \text{in the opposite case.} \end{cases}$$

Suppose that

$$M_{2s+2} = M_{2s+1} \cup \{c(y,t) : t < s \text{ and } y \text{ does not satisfy the negative conditions in step } s\}.$$

In the end of the construction, we put $\alpha(x) = \bigcup_s \alpha_s(x)$ for all x. It is easy to see that $\alpha \leqslant_w \nu$. Indeed, for all y, t, we have that

$$\alpha(c(y,t)) \in \{R_0, \ldots, R_n, \nu(\mathrm{lh}(y))\}.$$

We will show that $\nu \not\leqslant_w \alpha$. To do that, we fix an arbitrary e, such that φ_e is total. We choose the smallest $x > \max\{e,n\}$, such that some coding sequence y of length x satisfies both positive and negative conditions. Then, according to the description of the steps $2s+1 = 2c(e,u)+1$, we have that $\nu(x) \neq \alpha(z)$ for all $z \in D_{\varphi_e(x)}$.

Finally, note that if the coding sequence y of length larger than n satisfies both positive and negative requirements, then there only exists a finite number of v for which $\alpha(c(y,v)) \neq \nu(\mathrm{lh}(y))$ (or $\alpha(c(y,v)) \in \mathcal{S}_0$, which is the same). Therefore, α enumerates the whole family $\mathcal{S}$. This concludes the proof of the theorem. □

Corollary 2. *Let A be a high set and let K_m^A and K_w^A be classes of Rogers semilattices of all A-computable families given by a classical and weak reducibilities of numberings respectively. Then* $\mathrm{Th}(K_m^A) \neq \mathrm{Th}(K_w^A)$.

Proof. According to [13], every infinite A-computable family has an infinite number of pairwise non-equivalent minimal A-computable numberings. By theorem 3, there exists an A-computable family without any w-minimal A-computable numberings. □

Corollary 3. *A family of all c.e. sets does not have any w-minimal computable numberings.*

Finally, we will provide some examples of nontrivial computable families that possess w-minimal numberings.

Proposition 1. *Let ν be a single-valued computable numbering of some family of computable functions $\mathcal{F}$. Then for every computable numbering $\alpha \leqslant_w \nu$ of the family $\mathcal{F}$, a reducibility $\nu \leqslant \alpha$ is valid.*

Proof. Suppose that α is weakly reducible ν via a computable function f. We choose an arbitrary x. Due to the fact that the function $\nu(x)$ has a unique ν-number, for all y we have that

$$\alpha(y) = \nu(x) \Leftrightarrow \tag{3}$$

$$x \in D_{f(y)} \,\&\, \forall z \in D_{f(y)}\, [z \neq x \Rightarrow \nu(z) \neq \alpha(y)]. \tag{4}$$

Therefore, for all x we can effectively find y, satisfying (4) and, hence, also the equality $\nu(x) = \alpha(y)$. Thus, $\nu \leqslant \alpha$. □

Due to the fact that every infinite family of functions has a single-valued computable numbering (see [4]), we come to a corollary.

Corollary 4. *Every computable family of total functions has a w-minimal computable numbering.*

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Zlata Konstantinovna Ivanova
Kazan (Volga Region) Federal University,
18, Kremlyovskaya str.,
Kazan, 420008, Russia
Email address: ivanova_zk@mail.ru

Marat Khaidarovich Faizrahmanov
Kazan (Volga Region) Federal University,
18, Kremlyovskaya str.,
Kazan, 420008, Russia
Email address: marat.faizrahmanov@gmail.com