REMOVABLE SETS FOR SOBOLEV SPACES WITH MUCKENHOUPT $A_1$-WEIGHT

V.A. SHLYK

ABSTRACT. Let $\Omega$ be an open set in $\mathbb{R}^n$, $n \geq 2$, and $E$ be a relatively closed subset of $\Omega$. In this paper we obtain a criterion of equality $L^1_{1,\omega}(\Omega \setminus E) = L^1_{1,\omega}(\Omega)$ in terms of $E$ as an $NC_{1,\omega}$-set in $\Omega$ with $A_1$-weight $\omega$. In addition, we establish exact characterizations of $NC_{1,\omega}$-sets in terms of $NED_{1,\omega}$-sets and of the $(1, \omega)$-girth condition. In the case $\omega \equiv 1$, these results complete the studies of Vodop’yanov and Gol’dstein on removable sets for $L^1_p(\Omega)$, $p \in (1, +\infty)$.

Keywords: Sobolev space, capacity and modulus of condenser, Muckenhoupt weight, removable set.

1. INTRODUCTION

In [18] Vodop’yanov and Gol’dstein gave a criterion of removable singularities for $L^1_p(\Omega), W^1_p(\Omega)$ in terms of $NC_p$-sets, $1 < p < \infty$. An $NC_p$-set can be considered as a $p$-analog of an $NED$-set, earlier introduced by V"ais"al"a [16] as result of generalizing the concept of $NED$-set in $\mathbb{R}^2$ [1] to $\mathbb{R}^n$, $n \geq 2$. Also note the definitions of an $NC_p$-set in $\Omega$ or an $NED$-set in $\mathbb{R}^n$ are based on condensers whose plates are the pair of arbitrary disjoint continua located outside this set in $\Omega$ or in $\mathbb{R}^n$, respectively.

Latter [6] Dynchenko and Shlyk obtained similar assertions about removable singularities for the space $L^1_{p,\omega}(\Omega)$ in terms of $NC_{p,\omega}$-sets in $\Omega$, where $\omega$ is a Muckenhoupt $A_p$-weight, $1 < p < \infty$. Their definition of $NC_{p,\omega}$-set in $\Omega$ (as well as the initial definition of $NED$-set by Ahlfors–Beurling [1] in $\mathbb{R}^2$) is based on condensers formed by an arbitrary coordinate rectangles $\Pi, \bar{\Pi} \subset \Omega$, and by any pair of its opposite facets.
The main aim of this paper is to define $NC_1,\omega$-sets, $NED_1,\omega$-sets in $\Omega$ with Muckenhoupt $A_1$-weight, using proper coordinate rectangular condensers, and to give criteria of equality $L^1_1,\omega(\Omega \setminus E) = L^1_1,\omega(\Omega)$ in terms of $E$ as an $NC_1,\omega$-set, $NED_1,\omega$-set or with the $(1,\omega)$-girth condition (see Theorem 3).

Here we note the difficulties in proving the main results of the paper, which are not present at $p > 1$. Namely, the equality of $(1,\omega)$-modulus and $(1,\omega)$-capacity of the condenser is not known in general, even for $\omega \equiv 1$. Similarly, the equality $M_{1,\omega}(\bigcup_j \Gamma_j) = \lim_{j \to \infty} M_{1,\omega}(\Gamma_j)$ is unknown if $\Gamma_j \subset \Gamma_{j+1}$, $j \geq 1$, and $M_{1,\omega}(\cdot)$ is a $(1,\omega)$-modulus of a curve family in $R^n$.

2. Preliminaries

2.1. Some definitions and notations. Throughout the text the symbol $\Omega$ denotes a non-empty open set in Euclidean space $R^n = \{x = (x_1, \ldots, x_n)\}$, where $n \geq 2$. Respectively, $E$ denotes a relatively closed subset of $\Omega$, the norm of a point $x = (x_1, \ldots, x_n)$ is given by $|x| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$. We put $N = \{1, 2, \ldots\}, R = (-\infty, +\infty)$.

If $F \subset R^n$ then $\partial F$, $\bar{F}$ stand for the boundary and the closure of $F$ in $R^n$, respectively. The distance between two sets $A, B \subset R^n$ is denoted by $\text{dist}(A, B)$. For an open set $U \subset R^n$, we use the notation $U \Subset \Omega$ in order to indicate that $U$ is bounded and $\bar{U} \subset \Omega$. The restriction of a function $f$ to a set $F$ is denoted by $f|_F$. Given $x \in R^n$ and $r > 0$, let $B(x, r) = B_r(x) = \{y \in R^n: |y - x| < r\}$. If $a > 0$ then $aB_r(x) = B_{ar}(x)$. The symbol $H^s$ stands for the usual $s$-dimensional Hausdorff measure in $R^n$; $m_n$ is a Lebesgue measure in $R^n$ and put $|F| = m_n(F)$ for $m_n$-measurable set $F \subset R^n$.

Let $C^\infty(\Omega)$ be the space of infinitely differentiable functions in $\Omega$; the space of functions in $C^\infty(R^n)$ with a compact support in $\Omega$ is denoted by $C^\infty_0(\Omega)$.

We will use the abbreviation “a.e.” for “almost everywhere” with respect to $m_n$-measure. Similarly, “measurable” and “locally integrable” always mean Lebesgue measurable and locally integrable with respect to $m_n$-measure.

Let $F$ be a measurable subset of $R^n$, and $u$ is a measurable real-valued function on $F$. For $1 \leq p < \infty$ let

$$\|u\|_{L_p(F)} = \left(\int_F |u(x)|^p \, dx\right)^{1/p}.$$ 

Assume that $u(x)$ is a measurable function defined on $\Omega$. We say that $u$ is locally integrable to the power $p \in [1, +\infty)$ on $\Omega$ (and write $u \in L_p(\Omega, \text{loc})$) if $\|u\|_{L_p(F)} < \infty$ for every compact set $F \subset \Omega$. The class of all functions $u$ such that $\|u\|_{L_p(\Omega)} < \infty$ is denoted by $L_p(\Omega)$.

If $\Omega = R^n$ we shall often omit $\Omega$ in notations of spaces and norm. Integration without indication of limits extends over $R^n$.

Let $C, C_1, C_2, \ldots$ denote positive constants that depend on ”dimensionless” parameters $n, p, m$ and the like.

We call the quantities $a$ and $b$ equivalent and write $a \sim b$ if $C_1 a \leq b \leq C_2 a$.

If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an $n$-tuple of nonnegative integers $\alpha_i$, we call $\alpha$ a multi-index and denote by $x^\alpha$ the monomial $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ which has degree $|\alpha| = \sum_{i=1}^n \alpha_i$. 

...
Similarly, if \( D_j = \frac{\partial}{\partial x_j} \) then \( D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n} \) denotes a differential operator of order \(|\alpha|\). Note that \( D^{(0,\ldots,0)}u = u \) for real-valued functions \( u \). We use the notations \( \nabla_m = \{ D^\alpha : |\alpha| = m \} \), \( \nabla = \nabla_1 \).

By a weight we shall mean a locally integrable function \( \omega \) on \( \mathbb{R}^n \) such that \( \omega > 0 \) for a.e. \( x \in \mathbb{R}^n \).

Then for \( 1 \leq p < \infty \) we define \( L_{p,\omega}(\Omega) \) as the set of measurable functions \( f \) on \( \Omega \) such that

\[
\|f\|_{L_{p,\omega}(\Omega)} = \left( \int_{\Omega} |f|^p \omega \, dx \right)^{1/p} < \infty.
\]

As usual, any two functions \( f \) and \( g \) in \( L_{p,\omega}(\Omega) \) that are equal a.e. on \( \Omega \) will be identified.

Let \( \mathcal{F}_1 \) be a class of functions given on \( \Omega \), and \( \mathcal{F}_2 \) be another class of functions given on \( \Omega' \), where \( \Omega' \subset \Omega \). Below if \( f \in \mathcal{F}_1 \) then \( f \in \mathcal{F}_2 \) means \( f|_{\Omega'} \in \mathcal{F}_2 \).

Denote by \( L_{p,\omega}(\Omega, \text{loc}) \) the class of all measurable functions \( f \) on \( \Omega \) such that \( f \in L_{p,\omega}(\Omega') \) for all open sets \( \Omega' \subset \Omega \).

### 2.2. \( A_1 \)-weights

Following B. Muckenhoupt [12] a weight \( \omega \) is called an \( A_1 \)-weight, if there exists a positive constant \( A \) such that for every ball \( B \subset \mathbb{R}^n \),

\[
\left( \frac{1}{|B|} \int_B \omega \, dx \right) \sup_{x \in B} \frac{1}{\omega(x)} \leq A.
\]

The infimum over all such constants \( A \) is called the \( A_1 \)-constant of \( \omega \). Denote by \( A_1 \) the class of \( A_1 \)-weights. Throughout the text let \( m \in \mathbb{N} \), \( \omega \in A_1 \) unless otherwise stated.

Set

\[
M\omega(x) = \sup_{r > 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} \omega(y) \, dy.
\]

We mention two assertions concerning \( A_1 \)-weight.

**Proposition 1** ([15, Remark 1.2.4],[10, Theorem 2.7]). If \( \omega \in A_1 \) then \( L_{1,\omega}(\Omega) \) is complete space in norm \( \| \cdot \|_{L_{1,\omega}(\Omega)} \) and \( L_{1,\omega}(\Omega) \subset L_1(\Omega, \text{loc}) \). In addition, \( L_{1,\omega}(\Omega) \subset L_1(\Omega) \) in the case of a bounded \( \Omega \subset \mathbb{R}^n \).

**Proposition 2** ([15, Remark 1.2.4, Properties 7 and 8]). If \( \omega \in A_1 \) then there exist constants \( C, C_1 \) such that \( \omega(x) \geq \frac{C}{(1 + |x|)^n} \) and \( M\omega(x) \leq C_1 \omega(x) \) for a.e. \( x \in \mathbb{R}^n \).

### 2.3. Weighted Sobolev spaces

Suppose that \( u : \Omega \to R \) is a function in \( L_1(\Omega, \text{loc}) \). This function \( u \) on \( \Omega \) has a weak derivative of order \(|\alpha|\) if there is a locally integrable function (denoted by \( D^\alpha u \)) such that

\[
\int_{\Omega} u \cdot D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_{\Omega} D^\alpha u \cdot \varphi \, dx
\]

for all \( \varphi \in C_0^\infty(\Omega) \). For \( m \in \mathbb{N} \) and any \( \omega \in A_1 \), \( L_{m,\omega}^n(\Omega) \) is the space of functions \( u \) having on \( \Omega \) weak derivatives \( D^\alpha u \) for all orders \(|\alpha|, |\alpha| \leq m \), and satisfying

\[
\|u\|_{L_{m,\omega}^n(\Omega)} = \int_{\Omega} |\nabla_m u| \omega \, dx < \infty.
\]
where \( |\nabla_m u| = \left( \sum_{|\alpha|=m} (D^\alpha u)^2 \right)^{1/2} \). For \( m = 0 \) set \( L^0_{1,\omega}(\Omega) = L_{1,\omega}(\Omega), \nabla_0 u = u \).

Introduce the spaces
\[
W^m_{1,\omega}(\Omega) = L^m_{1,\omega}(\Omega) \cap L_{1,\omega}(\Omega), \quad W^{m,1}_{\omega}(\Omega) = \bigcap_{k=0}^{m} L^k_{1,\omega}(\Omega),
\]
equipped with the norms
\[
\|u\|_{W^m_{1,\omega}(\Omega)} = \|u\|_{L^m_{1,\omega}(\Omega)} + \|u\|_{L_{1,\omega}(\Omega)}, \quad \|u\|_{W^{m,1}_{\omega}(\Omega)} = \sum_{k=0}^{m} \|\nabla_k u\|_{L_{1,\omega}(\Omega)}.
\]

Next we let \( P_{m-1} \) be the collection of all polynomials of degree \( \leq m - 1 \). Let us consider the factor-space \( \tilde{L}^m_{1,\omega}(\Omega) = L^m_{1,\omega}(\Omega)/P_{m-1} \) (with norm \( \| \cdot \|_{L^m_{1,\omega}(\Omega)} \)). Elements of the space \( L^m_{1,\omega}(\Omega) \) on each connected component \( D \) of the set \( \Omega \) are classes \( \{u + P_D\} \) where \( u \in L^m_{1,\omega}(\Omega) \) and \( P_D \in P_{m-1} \).

In the case \( \omega \equiv 1 \), the weight spaces considered above with the weight \( \omega \) will be written below without the symbol \( \omega \).

Note that a number of important properties of spaces \( W^{m,1}_{\omega}(\Omega) \), \( L^m_{1,\omega}(\Omega) \) (in other notations and with equivalent norms) were obtained in \([5, 15]\). We use the following ones below.

**Proposition 3** ([5, Theorem 4.9]). If \( \Omega \) is an open connected set and \( \omega \in A_1 \) then \( \tilde{L}^m_{1,\omega}(\Omega) \) is a Banach space. In particular, if \( \{u_j\} \) is a Cauchy sequence in \( L^m_{1,\omega}(\Omega) \) then there exists \( u_0 \in \tilde{L}^m_{1,\omega}(\Omega) \) such that \( \nabla_m u_j \to \nabla_m u_0 \) in \( L_{1,\omega}(\Omega) \) as \( j \to \infty \).

**Proposition 4** ([5, Corollary 4.10]). Let \( \Omega \) be an open connected set, let \( \{u_j\} \) be a Cauchy sequence in \( L^m_{1,\omega}(\Omega) \), and let \( u \) be a function in \( L^m_{1,\omega}(\Omega) \) such that \( \|\nabla_m (u_j - u)\|_{L_{1,\omega}(\Omega)} \to 0 \). Then there exists a sequence of polynomials \( \{P_j\} \subset P_{m-1} \) with \( u_j - P_j \to u \) in \( L_{1,\omega}(K) \) for all compact sets \( K \subset \Omega \).

**Proposition 5** ([5, Theorem 4.2]). Let \( \omega \in A_1 \). If \( u \in L^m_{1,\omega}(\Omega) \) then
\[
(2) \quad \int_K |D^\alpha u(\omega) dx < \infty
\]
for all compact \( K \subset \Omega, \ 0 \leq |\alpha| \leq m \).

**Definition 1.** If the restriction operator \( \theta : L^m_{1,\omega}(\Omega) \to L^m_{1,\omega}(\Omega \setminus E) \) \( (\theta u = u|_{\Omega \setminus E}) \) induces the isometric isomorphism of the normed spaces \( \tilde{L}^m_{1,\omega}(\Omega) \) and \( L^m_{1,\omega}(\Omega \setminus E) \), then we write \( L^m_{1,\omega}(\Omega) = L^m_{1,\omega}(\Omega \setminus E) \). In other words, this means that \( |E| = 0 \) and for every function \( u \in L^m_{1,\omega}(\Omega \setminus E) \) there is a function \( v \in L^m_{1,\omega}(\Omega) \) for which \( v|_{\Omega \setminus E} = u \). In this case, the function \( v \) is called an extension of the function \( u \) in \( L^m_{1,\omega}(\Omega) \) and \( E \) is called a removable set for \( L^m_{1,\omega}(\Omega) \).

Similarly we define removable sets for \( W^{m,1}_{\omega}(\Omega) \) and \( W^m_{1,\omega}(\Omega) \).

### 2.4. Mollifications

Let \( \psi \in C^\infty_0(R^n) \) be nonnegative function such that \( \supp \psi \subset B_1(0) \) and \( \int \psi(x) dx = 1 \). For any function \( u \in L_1(\Omega) \), extended by zero on \( R^n \setminus \Omega \), we define the family of its mollifications
\[
(M_\varepsilon u)(x) = \varepsilon^{-n} \int u(y) \psi \left( \frac{y - x}{\varepsilon} \right) dy = \int_{|\xi|<1} u(x + \varepsilon \xi) \psi(\xi) d\xi, \quad 0 < \varepsilon \leq 1.
\]
The number \( \varepsilon \) shall be called a radius of mollification.

The following result is known.

**Proposition 6** ([15, Theorem 2.1.4, Corollary 2.1.5]). Suppose that \( u \in W^{m,1}_{\omega}(\Omega) \) and let \( \Omega' \) be an open set, \( \Omega' \Subset \Omega \). Then \( (M_{\varepsilon}u)(x) \in C^{\infty}(\Omega) \cap L_{1,\omega}(\Omega) \) and for \( 0 < \varepsilon < \min(\text{dist}(\Omega, \partial \Omega), 1) \) the equality \( D^\alpha M_{\varepsilon}u = M_{\varepsilon}D^\alpha u \) is true on \( \Omega' \), \( 1 \leq |\alpha| \leq m; M_{\varepsilon}u \to u \) in \( W^{m,1}_{\omega}(\Omega') \) as \( \varepsilon \to 0 \). In the case \( \Omega = \mathbb{R}^n \) we have convergence \( M_{\varepsilon}u \to u \) in \( W^{m,1}_{\omega}(\mathbb{R}^n) \) as \( \varepsilon \to 0 \).

Using Proposition 6 and an approach due to Maz'ya [11, Sec. 1.1.5, Theorem 1] (see proof of Theorem 2, Sec. 4), we obtain another assertion.

**Proposition 7.** Let \( u \in L^1_{1,\omega}(\Omega) \) and \( \{\Omega_j\} \) be some sequence of open sets \( \Omega_j \) such that \( \Omega_j \Subset \Omega_{j+1} \subset \Omega \) and \( \bigcup \Omega_j = \Omega \). Then there exists a sequence of bounded functions \( u_j \in L^1_{1,\omega}(\Omega) \cap C^\infty(\Omega) \), \( j \geq 1 \), such that

\[
\int_{\Omega_j} |u - u_j| \omega \, dx < \frac{1}{j}, \quad \lim_{j \to \infty} \|u - u_j\|_{L^1_{1,\omega}(\Omega)} = 0.
\]

2.5. \((1,\omega)\)-modulus and \((1,\omega)\)-capacity. Let \( \Gamma \) be a family of locally rectifiable curves in \( \mathbb{R}^n \). We denote by \( \text{adm} \, \Gamma \) the set of Borel functions \( \rho : \mathbb{R}^n \to [0; +\infty] \) satisfying the condition: for every \( \gamma \in \Gamma \) we have \( \int_\gamma \rho \, ds \geq 1 \). In the case \( \Gamma = \emptyset \) we assume that \( \text{adm} \, \Gamma \) contains the function \( \rho \equiv 0 \). The \((1,\omega)\)-modulus of \( \Gamma \), denoted by \( M_{1,\omega}(\Gamma) \), is defined as

\[
M_{1,\omega}(\Gamma) = \inf \int_{\mathbb{R}^n} \rho \, dx,
\]

where the infimum is taken over all \( \rho \in \text{adm} \, \Gamma \). For the basic facts about the \((p,\omega)\)-modulus, \( 1 \leq p < \infty \), see [13]. Now let \( F_0, F_1 \) be compact disjoint sets in \( \mathbb{R}^n \). Then a triple of sets \((F_0, F_1, \Omega)\) is called a condenser in \( \Omega \). Let \( \Gamma(F_0, F_1, \Omega) \) be the family of all locally rectifiable curves connecting \( F_0 \cap \overline{\Omega} \) and \( F_1 \cap \overline{\Omega} \) in \( \Omega \). More precisely, if \( \gamma \in \Gamma(F_0, F_1, \Omega) \) then there exists a representation \( x(s) : I \to \Omega \) of curve \( \gamma \) in terms of arc length (see [13, Sec. 2.1]), where \( I \) is an open interval, \( \overline{x(I)} \cap F_0 \) and \( \overline{x(I)} \cap F_1 \) are both non-empty.

We write \( M_{1,\omega}(F_0, F_1, \Omega) \) for the \((1,\omega)\)-modulus of \( \Gamma(F_0, F_1, \Omega) \). By definition, \( M_{1,\omega}(F_0, F_1, \Omega) = M_{1,\omega}(F_0 \cap \overline{\Omega}, F_1 \cap \overline{\Omega}, \Omega) \) and \( M_{1,\omega}(F_0, F_1, \Omega) = 0 \) if at least \( F_0 \cap \overline{\Omega} = \emptyset \) or \( F_1 \cap \overline{\Omega} = \emptyset \). The number \( M_{1,\omega}(F_0, F_1, \Omega) \) will also be called the \((1,\omega)\)-modulus of condenser \((F_0, F_1, \Omega)\).

Now let's define \((1,\omega)\)-capacity \( C_{1,\omega}(F_0, F_1, \Omega) \) of the condenser \((F_0, F_1, \Omega)\). Suppose that \( F_0 \cup F_1 \subset \Omega \). Then we set \( C_{1,\omega}(F_0, F_1, \Omega) = 0 \) if, at least, \( F_0 = \emptyset \) or \( F_1 = \emptyset \). If \( F_0 \) and \( F_1 \) are non-empty sets then

\[
C_{1,\omega}(F_0, F_1, \Omega) = \inf \int_{\Omega} |\nabla u| \omega \, dx,
\]

where the infimum is taken over all real-valued functions \( u \) such that \( u|_{\Omega} \) satisfies locally the Lipschitz condition and \( u = j \) in some neighborhood of \( F_j, j = 0, 1 \).

Denote the set of all admissible functions of this kind by \( \text{Adm}(F_0, F_1, \Omega) \). In general, we define \((1,\omega)\)-capacity of a condenser \((F_0, F_1, \Omega)\) as \( C_{1,\omega}(F_0, F_1, \Omega) = C_{1,\omega}(F_0 \cap \overline{\Omega}, F_1 \cap \overline{\Omega}, \Omega) \).
By Rademacher’s theorem, the function \( u \in \text{Adm}(F_0, F_1, \Omega) \) is differentiable a.e. on \( \Omega \). Set for \( x \in \Omega \)

\[
L(x, u) = \lim_{h \to 0} \sup \frac{|u(x + h) - u(x)|}{|h|}.
\]

Then \( L(x, u) \) is a Borel function on \( \Omega \) and \( |\nabla u(x)| = L(x, u) \) at differentiability points of \( u \). If \( u \) is not differentiable at \( x \in \Omega \), we set \( |\nabla u(x)| = L(x, u) \) (see [17, Theorem 5.1]).

It follows from the Vitali-Carathéodory theorem (see [14, p.37, Theorem 2.24]) that given \( f : \mathbb{R}^n \to [0, +\infty] \), \( f \in L_{1,\omega} \), there exists a lower semi-continuous function \( g \geq f \) with \( \|g\|_{L_{1,\omega}} \) arbitrarily closed to \( \|f\|_{L_{1,\omega}} \). We shall apply below the following assertion.

**Proposition 8** ([13, Sec. 2.2, p.19]).

\[
M_{1,\omega}(\Gamma) = \inf_{\rho} \left\{ \int \rho \omega dx : \rho \text{ is lower semi-continuous and } \rho \in \text{adm} \Gamma \right\}.
\]

2.6. **Removable sets.** Here we define three types of sets \( E \in \Omega \) (recall that \( E \) is a relatively closed subset of \( \Omega \)) which will be removable singularities for \( L_{1,\omega}^{1} \). Let here and further \( \Pi \) be any coordinate rectangle

\[
\{x = (x_1, \ldots, x_n) : a_i < x_i < b_i, i = 1, \ldots, n\},
\]

where \( a_i, b_i \in \mathbb{R} \). Denote the facets of this rectangle, parallel to the hyperplane \( x_i = 0 \), by \( \sigma_{0i} \subset \{x : x_i = a_i\} \) and \( \sigma_{1i} \subset \{x : x_i = b_i\} \), \( i = 1, \ldots, n \). If

\[
C_{1,\omega}(\sigma_{0i}, \sigma_{1i}, \Pi \setminus E) = C_{1,\omega}(\sigma_{0i}, \sigma_{1i}, \Pi), \quad i = 1, \ldots, n,
\]

for every coordinate rectangle \( \Pi \) with \( \bar{\Pi} \subset \Omega \), then \( E \) is called \( NC_{1,\omega} \)-set in \( \Omega \).

Now let \( m_n(E) = 0 \), and let \( e \subset E \) be an arbitrary compact. Set \( K_j(e, \Omega) = \{\Pi : \text{dist}(\sigma_{0j} \cup \sigma_{1j}, e) > 0 \}, i = 1, \ldots, n \). In addition, for \( \Pi \in K_j(e, \Omega) \) put \( \Pi_{j,\delta} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : a_j < x_j < b_j, a_i + \delta < x_i < b_i - \delta, i \neq j\} \), where

\[
0 < \delta < \min_{1 \leq i \leq n} \frac{b_i - a_i}{2}.
\]

If, regardless of choice of compact set \( e \subset E \) the estimate

\[
C_1 M_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi \setminus e) \geq \lim_{\delta \to 0} M_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi_{j,\delta})
\]

is valid for every coordinate rectangle \( \Pi \in K_j(e, \Omega) \) and all \( j = 1, \ldots, n \), then \( E \) is called a \( N\overline{E}D_{1,\omega} \)-set in \( \Omega \). In the definition \( C_1 \) is a constant from Proposition 2. Observe that in the case \( \omega \equiv 1 \), inequality (5) is equivalent to equality (see Corollary 1 from Sec. 4)

\[
M_1(\sigma_{0j}, \sigma_{1j}, \Pi \setminus e) = M_1(\sigma_{0j}, \sigma_{1j}, \Pi)
\]

for every compact \( e \subset E \) and coordinate rectangle \( \Pi \in K_j(e, \Omega), j = 1, \ldots, n \).

In order to define another type of removable sets for \( L_{1,\omega}^{1} \), we introduce the following concepts.

Let \( X_i \) be the family of straight lines in \( \mathbb{R}^n \), parallel to the coordinate \( x_i \)-axis, \( i = 1, \ldots, n \). Index every line \( l \in X_i \) by the point \( a \in l \cap H_i \), where \( H_i = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i = 0\} \). Then say that some property holds for \( m_{n-1} \)-almost all lines in \( X_i \) (or segments on these lines), whenever the corresponding set of points \( a \) on \( H_i \) for lines (or their segments) in \( X_i \) violating this property, is of \( m_{n-1} \)-measure zero.
We shall say that $\Omega$ does not partitioned locally by $E$ if $B \setminus E$ is a domain for all balls $B \subseteq \Omega$. It is easily to see that in this definition the balls $B \subseteq \Omega$ can be replaced with coordinate rectangles $\Pi \subseteq \Omega$.

**Definition 2.** Take a compact set $e \subset \mathbb{R}^n$ with $m_n(e) = 0$ so that $\mathbb{R}^n$ does not partitioned locally by $e$. We say that $e$ satisfies the $(1, \omega)$-girth condition respect to $X_i$ if every Borel function $\rho : \mathbb{R}^n \setminus e \to [0, +\infty)$, $\rho \in L_{1,\omega}(\mathbb{R}^n \setminus e)$, locally bounded on $\mathbb{R}^n \setminus e$, satisfies the following $\varepsilon$-girth condition for given $\varepsilon > 0$.

Let $\Pi$ be some coordinate rectangle in $\mathbb{R}^n$ and $e \subseteq \Pi$. Let $\Gamma_i(\Pi) = \{l \cap \Pi : l \in X_i, l \cap e \neq \emptyset\}$. Then for $m_{n-1}$-almost all segments $\tau \in \Gamma_i(\Pi)$ we can indicate a finite sequence of mutually disjoint intervals $(c_k, d_k) \subset \tau$ and rectifiable curves $\gamma_k \subset \Pi \setminus e$, joining $c_k$ with $d_k$, $k = 1, \ldots, k_1$, such that

$$\sum_{k=1}^{k_1} \int_{c_k}^{d_k} \rho \, ds < \varepsilon, \quad \sum_{k=1}^{k_1} \int_{c_k}^{d_k} ds < \varepsilon, \quad \bigcup_{k=1}^{k_1} (c_k, d_k) \supset \tau \cap e.$$

Refer to the last requirement as the $\varepsilon$-girth condition on $\tau$ for the $\rho$.

**Remark.** Let’s denote the family of all such segments $\tau$ for the function $\rho$ by $\Gamma_i(e, \Pi, \varepsilon)$. Take $\varepsilon = \frac{1}{k}$, $k \in \mathbb{N}$ and set $\Gamma_0(e, \Pi) = \bigcap_k \Gamma_i(e, \Pi, \frac{1}{k})$, $F_k = \{l \cap H_i : l \in X_i \text{ and there exists a segment } \tau \in \Gamma_i(e, \Pi, \frac{1}{k}), \tau \subset l\}$, $F_0 = \{l \cap H_i : l \in X_i \text{ and there exists a segment } \tau \in \Gamma_0(e, \Pi), \tau \subset l\}$. It is easy to see that $F_0 \subset F_{k+1} \subset F_k$ and $m_{n-1}(F_k) = m_{n-1}(F_0)$ for all $k \geq 1$ and $\rho$ satisfies the $\varepsilon$-girth condition on all $\tau \in \Gamma_0(e, \Pi)$ for any $\varepsilon > 0$.

Moreover, the function $\rho$ satisfies the $\varepsilon$-girth condition on any segment $[a, b] \subset \tau, \tau \in \Gamma_0(e, \Pi)$, if $a$ and $b \notin e$, for arbitrary $\varepsilon > 0$.

**Definition 3.** If $e$ satisfies the $(1, \omega)$-girth condition with respect to $X_i$ for all $i = 1, \ldots, n$, then we say that $e$ satisfies the $(1, \omega)$-girth condition in $\mathbb{R}^n$.

**Definition 4.** We say that $E$, $m_n(E) = 0$ and $\Omega$ does not partitioned locally by $E$, satisfies the $(1, \omega)$-girth condition in $\Omega$, if any compact set $e \subset E$ satisfies the $(1, \omega)$-girth condition in $\mathbb{R}^n$.

The family of all such sets $E \subset \Omega$ is denoted by $G_{1,\omega}(\Omega)$. Further $\omega \equiv 1$ we shall often omit in notations of spaces, norms, families. For example, in the case $\omega \equiv 1$ we write $G_{1,\omega}$ as $G_1$.

### 3. About One Class of Admissible Metrics

In this section $F_0$, $F_1$ will be compact non-empty sets in $\Omega$. Any function $\rho \in \text{adm } \Gamma(F_0, F_1, \Omega)$ will also be called an admissible metric for $\Gamma(F_0, F_1, \Omega)$. If $\Gamma(F_0, F_1, \Omega) = \emptyset$ then, by definition, the admissible metric for $\Gamma(F_0, F_1, \Omega)$ is an arbitrary Borel function $\rho : \mathbb{R}^n \to [0, +\infty]$. We let $d : \mathbb{R}^n \to [0, +\infty]$ be the function defined by $d(x) = \text{dist}(x, (\mathbb{R}^n \setminus \Omega) \cup F_0 \cup F_1)$. It is well-known that $d(x)$ satisfies the Lipschitz condition with the Lipschitz constant $Lip(d) \leq 1$. In the case where $d(x)$ is differentiable at $x \in \Omega \setminus (F_0 \cup F_1)$, it follows that $|\nabla d| = 1$ (see [7, Sec. 3.2.34]). Further we use the following technical results.

**Lemma 1.** $M_{1,\omega}(F_0, F_1, \Omega)$ and $C_{1,\omega}(F_0, F_1, \Omega)$ are finite.

**Proof.** Let $U_0$ and $U_1$ be open bounded sets in $\mathbb{R}^n$ such that $F_0 \subseteq U_0, F_1 \subseteq U_1$, $\overline{U_0} \cap \overline{U_1} = \emptyset$. In addition, take the ball $B_0 = B(0, r)$ so that $\overline{U_0} \cup \overline{U_1} \subseteq B_0$. Then...
Set $\varphi \in C^\infty_0(B_0)$, $0 \leq \varphi \leq 1$, $\varphi = 0$ on $U_0$ and $\varphi = 1$ on $\overline{U_1}$ (see [8, Chapter 1, Theorem 2.6]). Because of choice of $\varphi$ and Proposition 1 it follows that $\varphi \in \text{Adm}(F_0, F_1, R^n)$ and $|\nabla \varphi| \in \text{adm}(F_0, F_1, R^n)$ and $\int_{R^n} |\nabla \varphi| \omega \, dx < \infty$. This implies $M_{1, \omega}(F_0, F_1, \Omega) < \infty$, $C_{1, \omega}(F_0, F_1, \Omega) < \infty$. The Lemma is proved.

**Lemma 2.** For every $\varepsilon > 0$, there is a function $\rho \in \text{adm}(F_0, F_1, \Omega) \cap L_{1, \omega}(\Omega)$, for which the following conditions are realized:

1. $\rho$ is lower semi-continuous on $R^n$;
2. $\rho$ is continuous on $\Omega \setminus (F_0 \cup F_1)$;
3. $\rho$ is a positive function on $R^n$ such that for any compact set $K \subset R^n$
   \[ \inf_{K} \rho > 0 \]
   
where the constant $C_1$ does not depend on $\varepsilon$. In the case $\omega \equiv 1$, put in (7) $C_1 = 1$.

**Proof.** For $\varepsilon > 0$, by Proposition 8 and Lemma 1, let $\rho_1$ be some admissible metric for $\Gamma(F_0, F_1, \Omega)$, $\rho_1 \in L_{1, \omega}(R^n)$ and $\rho_1$ be a lower semi-continuous on $R^n$ with

\[ M_{1, \omega}(F_0, F_1, \Omega) \leq \int_{R^n} \rho_1 \omega \, dx \leq C_1 M_{1, \omega}(F_0, F_1, \Omega) + \varepsilon, \]

where $\omega$ is the constant from Proposition 2.

Define $\rho_2$ as

\[ \rho_2(x) = \frac{1}{|B(0, 1)|} \int_{B(0, 1)} \rho_1 \left( \frac{d(x) \, y}{2k} \right) dy, \]

where $T_k$ is the averaging operator used in [3] and studied in details in [9, Lemma 4.3]. Due to the known properties of the operator the function $\rho_2$ is lower semi-continuous on $R^n$ and continuous on $\Omega \setminus (F_0 \cup F_1)$. Here note that to prove these properties, only local integrability of $\rho_1$ is required in addition.

By integration (8), we get

\[ \int_{R^n} \rho_2(x) \omega \, dx = \int_{R^n} \left[ \frac{1}{|B(0, 1)|} \int_{B(0, 1)} \rho_1 \left( x + \frac{d(x) \, y}{2k} \right) dy \right] \omega(x) \, dx. \]

Interchanging the order of integration gives

\[ \|\rho_2\|_{L_{1, \omega}} = \frac{1}{|B(0, 1)|} \int_{B(0, 1)} \int_{R^n} \rho_1 \left( x + \frac{d(x) \, y}{2k} \right) \omega(x) \, dy. \]

Define for $y \in B(0, 1)$, $\theta_{y,k} : R^n \to R^n$ by $z = \theta_{y,k}(x) = x + \frac{d(x) \, y}{2k}$. By construction, for all $x, x' \in R^n$ (see [9, Theorem 2.1])

\[ \left( 1 - \frac{1}{2k} \right) |x - x'| \leq |\theta_{y,k}(x) - \theta_{y,k}(x')| \leq \left( 1 + \frac{1}{2k} \right) |x - x'|. \]
Thus, the mapping $\theta_{y,k}$ is a Lipschitz homeomorphism and $\theta_{y,k}(\Omega \setminus (F_0 \cup F_1)) = \Omega \setminus (F_0 \cup F_1)$. In addition, Jacobian of the mapping $\theta_{y,k}$ is equal to $1 + \frac{y}{2k} \nabla d(x)$ a.e. on $R^n$. Hence [9, Lemma 4.3], by the changing of variables formula with $z = \theta_{y,k}$ as the mapping function, we obtain in (9)

$$\|\rho_2\|_{L_1,\omega} \leq \frac{1}{|B(0,1)|(1 - \frac{1}{2k})} \int_{B(0,1)} \int_{R^n} \rho_1(z)\omega(x(z)) \, dz \, dy.$$ 

Repeated interchanging of order of integration gives

$$\|\rho_2\|_{L_1,\omega} \leq \frac{1}{(1 - \frac{1}{2k})} \int_{R^n} \left( \frac{1}{|B(0,1)|} \int_{B(0,1)} w(x(z)) \, dy \right) \rho_1(z) \, dz.$$ 

By Proposition 2 and $x = z - \frac{1}{2k} d(x(z)) y$, we deduce

$$\frac{1}{|B(0,1)|} \int_{B(0,1)} \omega \left( z - \frac{1}{2k} d(x(z)) y \right) \, dy = \frac{1}{|B \left( z, \frac{d(x(z))}{2k} \right)|} \int_{B \left( z, \frac{d(x(z))}{2k} \right)} \omega(y) \, dy \leq$$

(10) $M\omega(z) \leq C_1 \omega(z)$

a.e. on $R^n$. Hence

$$\|\rho_2\|_{L_1,\omega} \leq \frac{C_1}{(1 - \frac{1}{2k})} \frac{1}{R^n} \int \rho_1(z)\omega(z) \, dz = \frac{C_1}{(1 - \frac{1}{2k})} \|\rho_1\|_{L_1,\omega}.$$ 

Here note that for $\omega \equiv 1$ we have the equality $M\omega(x) = 1$ on $R^n$. In other words, we can assume $C_1 = 1$ for $\omega \equiv 1$ in (10).

Moreover, using standard arguments (see proofs [9, Lemma 4.3], [4, Theorem 2.1]), we have $g_k = (1 + \frac{1}{2k}) \rho_2 \in \text{adm}(F_0, F_1, \Omega)$. This implies

$$M_{1,\omega}(F_0, F_1, \Omega) \leq \int_{R^n} g_k \omega \, dx \leq$$

$$\frac{C_1 \left( 1 + \frac{1}{2k} \right)}{(1 - \frac{1}{2k})} \int_{R^n} \rho_1 \omega \, dx \leq C_1 \left( 1 + \frac{1}{2k} \right) M_{1,\omega}(F_0, F_1, \Omega) + \frac{\varepsilon}{3}.$$ 

Let $g(x)$ be a positive continuous function on $R^n$ with $\int_{R^n} g(x)\omega(x) \, dx < \frac{\varepsilon}{3}$. Its construction is similar to the construction of a positive function $\alpha(x) \in C^\infty$ in [13, Lemma 2.4.1]. Then it is clear that for large $k \in \mathbb{N}$ the function $\rho = g_k + g$ satisfies the conditions of the Lemma. Thus, the Lemma is proved. 

4. Some properties of $NED_{1,\omega}$, $NC_{1,\omega}$-sets

In this section, we will establish a number of properties of sets that will be removable for $L^1_{1,\omega}(\Omega)$ and use these properties in proving the main results of our paper.

Property 1. For any coordinate rectangle $\Pi = \{ x \in R^n : a_i < x_i < b_i, i = 1, \ldots, n \}$ we have (see (4))

(11) $M_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi) \text{ ess sup} \frac{1}{\omega} \geq m_{n-1}(\sigma_{0j}), j = 1, \ldots, n.$
Proof. By Lemma 1, it follows that $M_{1,\omega}(\sigma_{0j},\sigma_{1j},\Pi) < \infty$ for all $j = 1, \ldots, n$. Fix $j \in \{1, \ldots, n\}$. Then for each $\varepsilon > 0$ we can find $\rho \in \text{adm}(\sigma_{0j},\sigma_{1j},\Pi)$ such that

$$M_{1,\omega}(\sigma_{0j},\sigma_{1j},\Pi) \leq \int_{R^n} \rho \omega \, dx < M_{1,\omega}(\sigma_{0j},\sigma_{1j},\Pi) + \varepsilon.$$  

In addition, $\int_{a_j}^{b_j} \rho(x) \, dx \geq 1$ for all $x'$, where $x' \in \Pi' = \{x' = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \in R^{n-1} : a_i < x_i < b_i, i \neq j\}$. This implies

$$\int_{\Pi} \left( \int_{a_j}^{b_j} \rho \omega \, dx_j \right) \, dx' \geq m_{n-1}(\sigma_{0j}).$$

Hence, $\int \rho \omega \, dx \cdot \text{ess sup}_{\Pi} \frac{1}{\omega} \geq m_{n-1}(\sigma_{0j})$. Thus, the Property is proved. □

Set $\omega \equiv 1$ in (11) and let $\rho_0 = \frac{1}{b_j - a_j}$ on $\Pi$ and $\rho_0 = 0$ on $R^n \setminus \Pi$. By choice, $\text{ess sup}_{\Pi} \frac{1}{\omega} = 1$, $\rho_0 \in \text{adm}(\sigma_{0j},\sigma_{1j},\Pi)$ and $\int R^n \rho_0 \, dx = m_{n-1}(\sigma_{0j})$. It follows from (11) another assertion.

Corollary 1. In the case of $\omega \equiv 1$ we see

$$M_1(\sigma_{0j},\sigma_{1j},\Pi) = m_{n-1}(\sigma_{0j}), \quad j = 1, \ldots, n,$$

and, by (5),

$$\lim_{\delta \to +0} M_1(\sigma_{0j},\sigma_{1j},\Pi,\delta) = \lim_{\delta \to +0} M_1(\sigma_{0j} \cap \Pi_{j,\delta},\sigma_{1j} \cap \Pi_{j,\delta},\Pi_j,\delta) =$$

$$\lim_{\delta \to +0} m_{n-1}(\sigma_{0j} \cap \Pi_{j,\delta}) = m_{n-1}(\sigma_{0j}) = M_1(\sigma_{0j},\sigma_{1j},\Pi).$$

Moreover, in view of Proposition 2, we have also the estimate

$$M_{1,\omega}(\sigma_{0j},\sigma_{1j},\Pi) \geq C_1 \cdot \frac{1}{\text{sup}_{\Pi}(1 + |x|)^\alpha} m_{n-1}(\sigma_{0j})$$

for all $j = 1, \ldots, n$.

Property 2.

$$C_{1,\omega}(\sigma_{0j},\sigma_{1j},\Pi) \geq M_{1,\omega}(\sigma_{0j},\sigma_{1j},\Pi).$$

Proof. For given $\varepsilon > 0$ we can find $u_\varepsilon \in \text{Adm}(\sigma_{0j},\sigma_{1j},\Pi)$ such that

$$C_{1,\omega}(\sigma_{0j},\sigma_{1j},\Pi) \leq \int_{\Pi} |\nabla u_\varepsilon| \omega \, dx < C_{1,\omega}(\sigma_{0j},\sigma_{1j},\Pi) + \varepsilon.$$

In addition, we see

$$1 \leq \left| \int_{\gamma} \frac{\partial u_\varepsilon}{\partial s} \, ds \right| \leq \int_{\gamma} |\nabla u_\varepsilon| \, ds.$$
for all \( \gamma \in \Gamma(\sigma_{0j}, \sigma_{1j}, \Pi) \). Hence \( \rho \in \text{adm}(\sigma_{0j}, \sigma_{1j}, \Pi) \) if \( \rho = |\nabla u_\varepsilon| \) on \( \Pi \) and \( \rho = 0 \) on \( \mathbb{R}^n \setminus \Pi \). This implies
\[
M_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi) \leq \int_\Pi \rho \omega \, dx \leq C_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi) + \varepsilon.
\]
The arbitrariness of \( \varepsilon \) yields the inequality required in the Property. \( \square \)

**Property 3.** If \( E \) is an \( NC_{1,\omega} \)-set in \( \Omega \) then \( \sigma_{0j} \cap \Pi \setminus E = \sigma_{0j}, \sigma_{1j} \cap \Pi \setminus E = \sigma_{1j} \) for all \( j = 1, \ldots, n \).

**Proof.** Let \( j = 1 \) and let, for example, some \( x_0 \in \sigma_{01} \) not to be a boundary point of \( \Pi \setminus E \). Hence there exists a ball \( B(x_0, r) \) such that \( (B(x_0, r) \cap \sigma_{01}) \cap \Pi \setminus E = \emptyset \). Then we will define another coordinate rectangle \( \Pi_1 \subset \Pi \), whose opposite facets \( \sigma_{01}^1, \sigma_{11}^1 \), orthogonal to the \( x_1 \)-axis, lie in \( B(x_0, r) \cap \sigma_{01}, \sigma_{11} \), respectively.

By definition, \( \sigma_{01}^1 \cap \Pi_1 \setminus E = \emptyset \) and therefore
\[
C_{1,\omega}(\sigma_{01}^1, \sigma_{11}^1, \Pi_1 \setminus E) = C_{1,\omega}(\sigma_{01}^1, \sigma_{11}^1, \Pi_1) = 0.
\]
This, by Property 2, contradicts the inequality (12). Thus, \( \sigma_{01} \cap \Pi \setminus E = \sigma_{01}, \sigma_{11} \cap \Pi \setminus E = \sigma_{11} \).

Similarly, we establish the required equalities for \( j = 2, \ldots, n \). This completes the proof of the Property. \( \square \)

**Property 4.** If \( E \) is an \( NC_{1,\omega} \)-set in \( \Omega \) and \( u_j = \frac{x_j - a_j}{b_j - a_j} \) on \( \Pi \subset \Omega \), then
\[
\int_{\Pi \setminus E} |\nabla u_j| \omega \, dx \geq C_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi \setminus E), \quad \int_{\Pi} |\nabla u_j| \omega \, dx \geq C_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi)
\]
for all \( j = 1, \ldots, n \).

**Proof.** As in the proof of Property 3, we derive the inequalities (13) only for \( j = 1 \). Let \( \varepsilon \in \left( 0, \frac{b_1 - a_1}{2} \right) \) and set
\[
 u_{1,\varepsilon}(x) = \begin{cases} x_1 - a_1 - \varepsilon, & x_1 \in [a_1 + \varepsilon, b_1 - \varepsilon], \\ \frac{x_1 - a_1 - \varepsilon}{b_1 - a_1 - 2\varepsilon}, & x_1 < a_1 + \varepsilon, \\ 0, & x_1 = a_1 + \varepsilon, \\ 1, & x_1 > b_1 - \varepsilon \end{cases}
\]
for \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \). Obviously, \( u_{1,\varepsilon} \in \text{Adm}(\sigma_{01}, \sigma_{11}, \Pi) \cap \text{Adm}(\sigma_{01}, \sigma_{11}, \Pi \setminus E) \). This implies
\[
C_{1,\omega}(\sigma_{01}, \sigma_{11}, \Pi) \leq \int_{\Pi} |\nabla u_{1,\varepsilon}| \omega \, dx = \int_{\Pi} |\nabla u_1| \omega \, dx + o(1),
\]
\[
C_{1,\omega}(\sigma_{01}, \sigma_{11}, \Pi \setminus E) \leq \int_{\Pi \setminus E} |\nabla u_{1,\varepsilon}| \omega \, dx = \int_{\Pi \setminus E} |\nabla u_1| \omega \, dx + o(1),
\]
where \( o(1) \to 0 \) as \( \varepsilon \to 0 \). Hence, taking \( \varepsilon \to 0 \), we derive the required inequalities in the Property for \( j = 1 \). \( \square \)

**Corollary 2.** It is clear that for \( u_1 \) from the proof of Property 4 we have \( |\nabla u_1| = \frac{\rho_1}{b_1 - a_1} \), where \( \rho_1 = 1 \) on \( \Pi \setminus E \) and \( \rho_1 = 0 \) on \( \Pi \cap E \). Therefore, if \( E \) is an
NC$_1$,ω-set in Ω then

\[ (14) \quad \int_{\Pi \setminus E} \frac{\rho \omega}{b_1 - a_1} dx \geq C_{1,\omega}(\sigma_{01}, \Pi \setminus E) \]

for all Π ∈ Ω.

Property 5. If E is an NC$_1$,ω-set in Ω then m$_n$(E) = 0.

Proof. Suppose that m$_n$(E) > 0. Consider the coordinate rectangle Π = \{x = (x_1, \ldots, x_n) ∈ R^n : a_i < x_i < b_i, i = 1, \ldots, n\} and m$_n$(E ∩ Π) > 0. Let \( \rho \) be a function from Corollary 2. Given two rationals \( r_1 < r_2 \) in \( (a_1, b_1) \), put \( \Pi(r_1, r_2) = \Pi \cap \{x ∈ R^n : r_1 < x_i < r_2, \sigma_0 = \sigma_0(r_1, r_2) = \Pi \cap \{x : x_1 = r_1\} \text{ and } \sigma_1 = \sigma_1(r_1, r_2) = \Pi \cap \{x : x_1 = r_2\} \). It is clear that m$_n$(Π′) = m$_n$(σ_0) = m$_n$(σ_1), where Π′ = \{x′ = (x_2, \ldots, x_n) : a_i < x_i < b_i, i = 2, \ldots, n\}. Then from Property 2, (12) and (14) we deduce easily

\[ (15) \quad \int_{\Pi'} dx' \int_{r_1}^{r_2} \frac{\rho \omega}{r_2 - r_1} dx_1 \geq C_1 \frac{1}{\sup_{\Pi} (1 + |x|)^n} \cdot C_2 m_{n-1}(\Pi') \]

where \( C_2 = C_1 \frac{1}{\sup_{\Pi} (1 + |x|)^n} \).

Put \( \Phi(x', r_1, r_2) = \int_{r_1}^{r_2} \frac{\rho \omega dx_1}{r_2 - r_1} \). By Fubini's theorem (see [2]), there exists a set \( \tilde{\Pi}'(r_1, r_2) \) in Π′ with m$_n$((Π′)′(r_1, r_2)) = m$_n$((Π′)), all of whose points are Lebesgue points for \( \Phi(x', r_1, r_2) \) (differentiability points of the integral of this function over Π′). Put \( \tilde{\Pi}' = \bigcap_{r_1 < r_2} \tilde{\Pi}'(r_1, r_2) \). It is obvious that m$_n$(Π′) = m$_n$(Π′). Choose \( x' ∈ \tilde{\Pi}' \) so that \( \mathcal{H}^1(\{x'\} ∩ E) > 0 \) and \( \mathcal{H}^1 \)-almost all points on the segment \( l(x') = \{(x_1, x') : a_1 ≤ x_1 ≤ b_1\} \) are Lebesgue points for the function \( \rho \omega \) and \( \omega < ∞ \).

From (15) we obtain

\[ (16) \quad \frac{1}{r_2 - r_1} \int_{r_1}^{r_2} \rho \omega dx_1 \geq C_2 \]

for m$_n$-almost all \( x' ∈ \tilde{\Pi}' \).

Since the choice of \( r_1 \) and \( r_2 \) in \( (a_1, b_1) \) is arbitrary, the absolute continuity of the integral enables us to consider \( r_1 \) and \( r_2 \) in (16) as real from \( (a_1, b_1) \). By construction, there exists \( x_1^0 ∈ (a_1, b_1) \) and \( (x_1^0, x') ∈ E \cap l(x') \) such that \( (x_1^0, x') \) is a Lebesgue point for \( \rho \omega \) with fixed \( x' \).

Putting in (16) \( r_1 = x_1^0 - \delta \) and \( r_2 = x_1^0 + \delta \) with \( \delta > 0 \), and letting \( \delta → 0 \), we get a contradiction to \( 0 = \rho(x_1^0, x') ω(x_1^0, x') ≥ C_2 \). Consequently, m$_n$(E) = 0. Thus the Property is proved. □

Property 6. If E is an NC$_1$,ω-set in Ω then Π \ E is a domain for arbitrary coordinate rectangle Π ∈ Ω.

Proof. Suppose that there exists a coordinate rectangle Π ⊆ Ω, which is partitioned by E into two non-empty open sets D$_0$ and D$_1$ (Π \ E = D$_0$ ∪ D$_1$, D$_0$ ∩ D$_1$ = ∅). Take two points \( d_0 ∈ D_0 \), \( d_1 ∈ D_1 \) and let \( B_0 = B(d_0, r_0), B_1 = B(d_1, r_1) \) be such that \( \overline{B_0} ⊆ D_0, \overline{B_1} ⊆ D_1 \).
Let $L_1 = L_1(a_1, \ldots, a_k)$ be a simple polyline in $\Pi$ composed of straight segments $[a_i, a_{i+1}]$, $i = 1, \ldots, k - 1$, where each segment $[a_i, a_{i+1}]$ is parallel to some coordinate axis and $a_1 = a_0$, $a_k = d_1$. Below, any polyline whose links are parallel to the coordinate axes, will be called a coordinate polyline. Then it follows that there exists some segment $[a, b] \subset \Pi$, parallel to one of the links of $L_1$ and $a \in D_0$, $b \in D_1$. Indeed, in the ball $B(a_2, r_2)$ with $0 < r_2 < \min(r_0, r_1, \text{dist}(L_1, \partial \Pi))$, by virtue of $m_n(E) = 0$, there is a point $a_2 \notin E$. Consider a parallel translation $T_{h_1} : x \mapsto x + h_1$ on $R^n$, where $h_1 = a_2^1 - a_2$. By construction, $T_{h_1}(L_1)$ is a coordinate polyline $L_2 = L_2(a_1^1, \ldots, a_k^1)$ with $a_i^1 = T_{h_1}(a_i) \in D_0$, $a_{i+1}^1 = T_{h_1}(a_{i+1}) \notin E$, ..., $a_k^1 = T_{h_1}(a_k) \in B(a_k, r_1) \subset D_1$. If $a_3^1 \in D_1$ then we choose $[a, b] = [a_1, a_2]$. Suppose that $a_2^1 \in D_0$. Then in $B(a_3^1, r_3)$ with $0 < r_3 < \min(\text{dist}(a_2^1, \partial B(a_2, r_2) \cup E), \text{dist}(L_2, \Pi), \text{dist}(a_1^1, \partial B(a_k, r_1)))$, as above, there is a point $a_2^2 \notin E$. Put $L_3 = L_3(a_2^1, \ldots, a_k^1)$ and let $T_{h_2} : x \mapsto x + h_2$, where $h_2 = a_3^2 - a_2^1$. Hence $T_{h_2}(L_2)$ is a coordinate polyline $L_4 = L_4(a_2^2, \ldots, a_k^2)$ with $a_2^2 \notin D_0, a_2^2 \notin E$, ..., $a_k^2 = T_{h_2}(a_k^1) \in B(d_1, r_1) \subset D_1$.

If $a_3^2 \in D_1$ then we choose $[a, b] = [a_2^2, a_3^2]$. If $a_3^1 \in D_0$ then we continue this process. Taking into account that, by the choice, $T_{h_1}(a_k^1) \in D_1$, $T_{h_2}(a_k^2) \in D_1$ and so far, at most after $k$ steps we get the required segment $[a, b]$. Without loss of generality, we assume that the segment $[a, b]$ is parallel to the $x_1$-axis.

Let’s choose a coordinate rectangle $\Pi_1 \subset \Pi$ so that its opposite facets $\sigma_{011}^1, \sigma_{111}^1$ lie in $D_0$ and $D_1$, respectively, and are orthogonal to the $x_1$-axis. In addition $a \in \sigma_{011}^1$ and $b \in \sigma_{111}^1$.

Set $u = 0$ on $D_0$ and $u = 1$ on $D_1$. Then $u \in \text{Adm}(\sigma_{011}^1, \sigma_{111}^1, \Pi \setminus E)$ and $\int_\Pi |\nabla u| \omega \, dx = 0$. This implies $C_{1,\omega}(\sigma_{011}^1, \sigma_{111}^1, \Pi \setminus E) = 0$. On the other hand, by (4),(12) and Property 2, we see that $C_{1,\omega}(\sigma_{011}^1, \sigma_{111}^1, \Pi \setminus E) > 0$. The resulting contradiction completes the proof of this Property. 

Note that in the proof of the Property 6, only the condition $m_n(E) = 0$ was used to construct the rectangle $\Pi_1$. And $\Pi_1 \in \mathcal{K}_1(e, \Omega)$, where $e = \Pi_1 \cap E$ and $\mathcal{K}_1(e, \Omega)$ is from definition of $\text{NED}_1$-sets in $\Omega$.

It is clear that $\Gamma(\sigma_{011}^1, \sigma_{111}^1, \Pi_1 \setminus e) = \emptyset$ and hence $M_{1,\omega}(\sigma_{011}^1, \sigma_{111}^1, \Pi_1 \setminus e) = 0$. On the other hand, write $\Pi_1 = \{x = (x_1, \ldots, x_n) : a_i^1 < x_i < b_i^1, i = 1, \ldots, n\}$ and set $\Pi_{1,\delta} = \{x = (x_1, \ldots, x_n) : a_i^1 < x_i < b_i^1, a_i^1 + \delta < x_i < b_i^1 - \delta, i = 2, \ldots, n\}$, where $0 < \delta < \min_{1 \leq i \leq n} \frac{b_i^1 - a_i}{2}$. Then, applying (5) and (12), we have

$$C_1 M_{1,\omega}(\sigma_{011}^1, \sigma_{111}^1, \Pi_1 \setminus e) \geq \lim_{\delta \to 0} M_{1,\omega}(\sigma_{011}^1 \cap \Pi_{1,\delta}, \sigma_{111}^1 \cap \Pi_{1,\delta}, \Pi_{1,\delta}) \geq \frac{C_1}{\max(1 + |x|)^n} m_{n-1}(\sigma_{011}^1) > 0,$$

that contradicts the condition $M_{1,\omega}(\sigma_{011}^1, \sigma_{111}^1, \Pi \setminus e) = 0$. Thus, we come to another assertion.

**Property 7.** If $E$ is an $\text{NED}_{1,\omega}$-set in $\Omega$ then $\Pi \setminus E$ is a domain for all $\Pi \in \Omega$.

**Property 8.** If $E$ is an $\text{NED}_{1,\omega}$-set in $\Omega$ then $E$ is an $\text{NED}_{1,\omega}$-set in $\Omega$.

**Proof.** Take some compact $e \subset E$, coordinate rectangle $\Pi = \{x = (x_1, \ldots, x_n) : a_i < x_i < b_i, i = 1, \ldots, n\}$ and make sure that inequalities (5) are true if $j = 1$.
and $\Pi \in \mathcal{K}_j(e, \Omega)$. For the remaining $j$ and other coordinate rectangles $\in \mathcal{K}_j(e, \Omega)$, $j = 2, \ldots, n$ and $e$ the proof of this fact is performed in the same way.

Let’s introduce a few notations. Let $\sigma_0 = \sigma_{01}, \sigma_1 = \sigma_{11}$ be opposite facets of $\Pi$ such as in (4). Let $\Pi_{1,\delta} = \Pi_{1,\delta} = 2\gamma$. Assume that all $\gamma$ such as in (5) and set $\sigma_0 \cap \Pi_{1,\delta} = \sigma_0(\delta), \sigma_1 \cap \Pi_{1,\delta} = \sigma_1(\delta)$. Fix $0 < \delta < \min_i \frac{b_i - a_i}{2}$. Since $\Pi \in \mathcal{K}_j(e, \Omega)$, then $\eta = \text{dist}(\sigma_0 \cup \sigma_1, e) > 0$. Hence there exist two sequences $\{\Omega_0,k\}, \{\Omega_1,k\}$ of coordinate rectangles $\Omega_0,k, \Omega_1,k \in \Omega$ such that $\text{dist}(\Omega_0,k \cup \Omega_1,k, e) > 0$ for all $k = 1, \ldots; \sigma_0 \subset \Omega_0,k+1 \in \Omega_0,k, \bigcap_k \Omega_0,k = \sigma_0; \sigma_1 \subset \Omega_1,k+1 \in \Omega_1,k, \bigcap_k \Omega_1,k = \sigma_1$ and $\Pi_{0,1} = \sigma_1$ and $\Pi_{1,1} = \sigma_1$. 

Set $\sigma^0 = \partial \Omega_0,k \cap \Pi_{1,\delta}, \sigma^1 = \partial \Omega_1,k \cap \Pi_{1,\delta}$. Put in Lemma 2 $\Omega = \Pi \setminus e, F_0 = \sigma_0, F_1 = \sigma_1$. Then for a given $\varepsilon > 0$ there is a metric $\rho \in \text{adm} \Gamma(\sigma_0, \sigma_1, \Pi \setminus e) \cap L_{1,\omega}$ that satisfies the conditions of Lemma 2. In particular, $\rho$ is a continuous function on $\Pi \setminus e$,

$$\inf_K \rho > 0$$

for all compact set $K \subset \mathbb{R}^n$ and

$$M_{1,\omega}(\sigma_0, \sigma_1, \Pi \setminus e) \leq \int_{\mathbb{R}^n} \rho \omega \, dx < C_1 M_{1,\omega}(\sigma_0, \sigma_1, \Pi \setminus e) + \varepsilon.$$ 

In addition, the metric $\rho$ is continuous on $\partial \Pi_{1,\delta} \setminus (e \cup \sigma^0 \cup \sigma^1)$ for all $k = 1, 2, \ldots$ and $\int_{\gamma} \rho \, ds \geq 1$ for any locally rectifiable curve $\gamma$, connecting $\sigma_0(\delta)$ and $\sigma_1(\delta)$ in $\Pi_{1,\delta} \setminus (\sigma_0(\delta) \cup \sigma_1(\delta) \cup e)$. Finally, let $\Pi^0, \Pi^1 = \Pi_{1,\delta} \setminus \Omega_0,k \cup \Omega_1,k, \Gamma_k = \Gamma(\sigma^0, \sigma^1, \Pi_{1,\delta} \setminus e)$. Take $\beta \in (0, 1)$. We shall show that there is $k_0 = k_0(\beta) \in \mathbb{N}$ for which

$$\int_{\gamma} \rho \, ds \geq 1 - \beta$$

for all $\gamma \in \Gamma_k$ if $k \geq k_0$. Indeed, admitting the opposite, we obtain a sequence $\{\gamma_k\}_{k=1}^{\infty}$ such that $\int_{\Gamma_k} \rho \, ds < 1 - \beta$ for all $k = 1, 2, \ldots$.

By (17), it is inferred that $\gamma_k$ is a rectifiable curve and for any $k$ and $j, 1 \leq j < k, \gamma_k$ contains points in $\sigma^0_j$ and $\sigma^1_j$.

Let $x_0(j, k)$ (resp. $x_1(j, k)$) be the first (resp. last) point in $\sigma^0_j$ (resp. $\sigma^1_j$) coming along $\gamma_k$ from $\sigma^0_k$. Choose a subsequence $\{\gamma_{k_j}\}$ of $\{\gamma_k\}$ so that $x_0(j, k_j)$ converges to a point $x_{js} \in \sigma_0, s = 0, 1$. Denote this subsequence by $\{\gamma_{k_j}\}$.

Since $\rho$ is continuous in some neighborhood of $\sigma^0_j$, we can find a closed ball $B_{1s} = \overline{B(x_{1s}, r_{1s})}$ in $\Pi \setminus e$ so that $\int_{\Gamma} \rho \, ds < \frac{\beta}{2}$ for any segment $l \subset B_{1s}$. We may assume that all $\gamma_{k_j}$ meet $B_{1s}, s = 0, 1$.

Similarly we find a subsequence $\{\gamma_{2k}\}$ of $\{\gamma_{k_j}\}$ and a closed balls $B_{20}, B_{21}$ in $\Pi \setminus e$ such that $\int_{\Gamma} \rho \, ds < \frac{\beta}{2}$ for any segment $l \subset B_{20} \cup B_{21}$. We continue this process.

Denote the diagonal subsequence $\{\gamma_{kk}\}$ by $\{\gamma_k\}$.

For each $k \geq 1$ we modify each $\gamma_k$ in every $B_{js}$ so that all $\gamma_k, \gamma_k \cap B_{js} \neq \emptyset$ pass through $x_{js}$, as follows.

We replace $\gamma_k \cap B_{js}$ by two radii, terminating at $\gamma_k \cap \partial B_{js}$ and denote the resulting curve still by $\gamma_k$. Using further standard construction from the proof
of Lemma 4.10 in [13, p. 138–139], we obtain a curve \( \tilde{\gamma} \), satisfying the following conditions:

1. \( \int_{\tilde{\gamma}} \rho \, ds < 1 - \frac{\beta}{\overline{\sigma}} \) and therefore, by (17), \( \tilde{\gamma} \) is a rectifiable curve,

2. \( \tilde{\gamma} \in \Gamma(\sigma_0, \sigma_1, \Pi \setminus e) \) and \( \tilde{\gamma} \subset \Pi_{1,\delta} \setminus (\sigma_0(\delta) \cup \sigma_1(\delta) \cup e) \),

3. \( x_{j0} \) and \( x_{j1} \in \tilde{\gamma} \) for all \( j = 1, 2, \ldots \).

Hence, by condition 2), we have \( \int_{\tilde{\gamma}} \rho \, ds \geq 1 \), that contradicts condition 1). Thus, the inequality (19) is true.

Take \( k_1 > k_0(\beta) \) and let \( \sigma_0^{k_1} \subset \{ x = (x_1, \ldots, x_n) : x_1 = a_{1,k_1} \} \), \( \sigma_1^{k_1} \subset \{ x = (x_1, \ldots, x_n) : x_1 = b_{1,k_1} \} \).

Set

\[
\rho_1 = \begin{cases} 
0, & x_1 \leq a_{1,k_1}, \\
\frac{1}{1 - \beta}, & x_1 > a_{1,k_1}, x \in \Pi_{1,\delta} \setminus e,
\end{cases}
\]

and let \( G_1 = \{ x = (x_1, \ldots, x_n) : x_1 \leq a_{1,k_1} \} \cup (\Pi_{1,\delta} \setminus e) \), \( u_1(x) = 0 \), if \( x_1 \leq a_{1,k_1} \), \( u_1(x) = \inf_{\gamma_2} \int \rho \, ds \), where \( \gamma_2 \) is arbitrary rectifiable curve in \( \Pi_{1,\delta} \setminus (\Omega_{0,k_1+1} \cup e) \), connecting \( \sigma_0^{k_1+1} \) and \( x \in \Pi_{1,\delta} \setminus (\Omega_{0,k_1+1} \cup e) \).

In view of local boundedness \( \rho_1 \) on \( G_1 \) and known properties of functions such as \( u_1 \) [13, Lemma 5.2], it follows that \( u_1 \) satisfies the Lipschitz condition in \( G_1 \), \( |\nabla u_1| \leq \rho_1 \) a.e. on \( G_1 \), \( u_1 = 0 \) on \( \{ x : x_1 \leq a_{1,k_1} \} \).

Taking the cutting \( u_2 = \min(1, u_1) \) on \( G_1 \), we get

\[
\int_{\Pi_{1,\delta} \setminus e} |\nabla u_2| \, \omega \, dx \leq \int_{\Pi_{1,\delta} \setminus e} |\nabla u_1| \, \omega \, dx,
\]

where \( u_2 \), by properties of the cutting, satisfies the Lipschitz condition on \( G_1 \) and \( u_2 = 1 \) on \( \Pi_{1,\delta} \cap \Omega_{1,k_1} \), \( u_2 = 0 \) on \( \Omega_{0,k_1} \). Set

\[
u_3 = \begin{cases} 
u_2, & x \in G_1 \setminus \Omega_{1,k_1+1}, \\
1, & x \in \Omega_{1,k_1+1}.
\end{cases}
\]

It is easily to see that \( u_3 \in \text{Adm}(\sigma_0(\delta), \sigma_1(\delta), \Pi_{1,\delta} \setminus e) \). Then, by (18), (20), Property 2 and the arbitrariness of \( \beta \), we have for \( NC_{1,\omega} \)-set \( E \)

\[
M_{1,\omega}(\sigma_0(\delta), \sigma_1(\delta), \Pi_{1,\delta}) \leq C_{1,\omega}(\sigma_0(\delta), \sigma_1(\delta), \Pi_{1,\delta}) = C_{1,\omega}(\sigma_0(\delta), \sigma_1(\delta), \Pi_{1,\delta} \setminus e) \leq \int_{\Pi_{1,\delta} \setminus e} |\nabla \nu_3| \, \omega \, dx \leq \int_{\Pi_{1,\delta} \setminus e} \rho \, \omega \, dx + o(1) \leq C_1 M_{1,\omega}(\sigma_0, \sigma_1, \Pi \setminus e) + \varepsilon + o(1),
\]

where \( o(1) \to 0 \) as \( \beta \to 0 \). Hence, taking \( \beta \to 0, \delta \to 0, \varepsilon \to 0 \), we deduce (5). \( \square \)

**Remark 1.** In the proof of Property 8 we applied the cutting \( \min(1, u_1) \). Similarly using \( \max(\min(1, u), 0) \) to the function \( u \in \text{Adm}(F_0, F_1, G) \), we come the following definition of the capacity of the condenser.

**Definition.** If \( F_0, F_1 \) are compact disjoint non-empty sets in \( \overline{\Omega} \) then

\[
C_{1,\omega}(F_0, F_1, \Omega) = \inf_u \int_{\Omega} |\nabla u| \omega \, dx,
\]
where the infimum is taken over all real-valued bounded functions \( u \) such that \( u|_{\Omega} \) satisfies locally there Lipschitz condition and \( u = j \) in some neighborhood of \( F_j \), \( j = 0, 1 \).

Remark 2. Let \( F_0, F_1 \) be compact non-empty disjoint sets in \( \Omega \). Let \( \{D_j\} \) be a sequence (maybe finite) of all mutually disjoint connected components \( D_j \) of \( \Omega \).

Since \( F_0, F_1 \) are compact sets, there exists a finite family of components \( D_j \in \{D_j\} \), having common points with \( F_0 \cup F_1 \). Without loss of generality, suppose that only \( D_1, \ldots, D_{j_1} \) intersect \( F_0 \cup F_1 \). Let there is no component \( D_j \), \( 1 \leq j \leq j_1 \), such that simultaneously \( D_j \cap F_0 \neq \emptyset, D_j \cap F_1 \neq \emptyset \). This implies \( \Gamma(F_0, F_1, \Omega) = \emptyset \).

In addition, assuming \( u = 0 \) on \( D_j \) if \( D_j \cap F_0 \neq \emptyset \), \( u = 1 \) on \( D_j \) if \( D_j \cap F_1 \neq \emptyset \), \( u = 1 \) on \( D_j \) if \( j > j_1 \), we get \( u \in \text{Adm}(F_0, F_1, \Omega) \) and \( \int_{\Omega} |\nabla u| \omega \, dx = 0 \). it follows that \( C_{1,\omega}(F_0, F_1, \Omega) = 0 \).

Otherwise, there is a component \( D_j \), \( 1 \leq j \leq j_1 \), for which \( D_j \cap F_0 \neq \emptyset \), \( D_j \cap F_1 \neq \emptyset \) and hence \( \Gamma(F_0, F_1, \Omega) \neq \emptyset \).

Then using the same arguments from the proof of Property 8 (see also [9, Theorem 5.5]), we obtain the next assertion.

Theorem 1. If \( F_0, F_1 \) are compact disjoint non-empty sets in \( \Omega \) then

\[
M_{1,\omega}(F_0, F_1, \Omega) \leq C_{1,\omega}(F_0, F_1, \Omega) \leq C_1 M_{1,\omega}(F_0, F_1, \Omega),
\]

where \( C_1 \) depends only on \( \omega \) and \( C_1 = 1 \) for \( \omega = 1 \).

5. Removable sets

In this section, we will establish three criteria of removable sets \( E \) for \( L^1_{1,\omega}(\Omega) \).

We recall (see Definition 1) that \( E \) is a removable set for \( L^1_{1,\omega}(\Omega) \) if \( E \) is a relatively closed subset of \( \Omega \), \( m_n(E) = 0 \) and for every function \( u \in L^1_{1,\omega}(\Omega \setminus E) \) there exists a function \( v \in L^1_{1,\omega}(\Omega) \) such that \( v|_{\Omega \setminus E} = u \).

We will need the following two lemmas. Further let \( \Omega = \bigcup_j D_j \), where \( \{D_j\} \) is the sequence (maybe finite) of connected components of \( \Omega \) from Remark 2.

Lemma 3. If \( E \) is a removable set for \( L^1_{1,\omega}(\Omega) \) then \( D_j \setminus E \) is a domain for all \( j \).

Proof. Suppose that there exists some component \( D_j \) which is partitioned by \( E \) into non-empty sets \( Q_0 \) and \( Q_1 \), \( D_j \setminus E = Q_0 \cup Q_1 \), \( Q_0 \cap Q_1 = \emptyset \). take two point \( q_0 \in Q_0, q_1 \in Q_1 \) and balls \( B_0 = B(q_0, r_0), B_1 = B(q_1, r_1) \) be such that \( \overline{B_0} \subset Q_0, \overline{B_1} \subset Q_1 \). Let \( L_1 = L_1(a_1, \ldots, a_k) \) be a simple polyline in a domain \( D_j \) composed of straight segments \( [a_i, a_{i+1}], i = 1, \ldots, k - 1 \), where each segment \( [a_i, a_{i+1}] \) is parallel to some coordinate axis and \( a_1 = q_0, a_k = q_1 \). Applying reasoning from the proof of Property 6, we obtain a coordinate rectangle \( \Pi_1 \) with opposite facets \( \sigma_0 \subset Q_0 \) and \( \sigma_1 \subset Q_1 \). Suppose, for example, that \( \sigma_0, \sigma_1 \) are orthogonal to the \( x_1 \)-axis.

Let \( u_0 = 0 \) on \( Q_0 \) and \( u_0 = 1 \) on \( Q_1 \cap \{D_j\} \). Obviously, \( u_0 \in L^1_{1,\omega}(\Omega \setminus E) \) and, by definition of \( E \), there is a function \( v_0 \in L^1_{1,\omega}(\Omega) \), for which \( v_0|_{\Omega \setminus E} = u_0 \).

In view of Proposition 1, \( v_0 \in W^{1,1}(D_j, \text{loc}) \) and therefore \( v_0 \in L^1_{1}(\Pi_1) \). Then (see [11, Sec. 1.1.3, Theorem 1]) function \( v_0 \) is absolutely continuous on \( m_{n-1} \)-almost all segments \( \mathcal{H}^{1}(l \cap E) = 0 \), joining the facets \( \sigma_0, \sigma_1 \) in \( \Pi_1 \) and parallel to the \( x_1 \)-axis. This implies the existence of a limit point \( x_l \in E \cap l \) simultaneously for \( l \cap Q_0 \) and \( l \cap Q_1 \). Hence, \( v_0(x_l) = 0 \) and \( v_0(x_l) = 1 \), that contradicts the definition of function \( v_0 \). Thus, \( D_j \setminus E \) is a domain for all \( j \) and our Lemma is proved. \( \square \)
Lemma 4. Let $B = B(a, r)$ be some ball in an open set $G \subset \Omega$ and $E$ be a removable set for $L^1_{\omega}(\Omega)$. Suppose that $\varphi \in C^\infty_0(B)$ and $u \in L^1_{\omega}(G \setminus E)$. In addition, assume that $u$ is a bounded and locally satisfying the Lipschitz condition on $G \setminus E$. Then there exists a function $g \in W^{1,1}(\mathbb{R}^n)$ for which $g = 0$ on $R^n \setminus B_1$, $g|_{G \setminus E} = u \varphi$.

Proof. Set $E_1 = \overline{B}_1 \cap E$ and, by Lemma 3, note that $B \setminus E_1$ is a domain. In addition, $E_1$ as a closed subset of $E$ is a removable set for $L^1_{\omega}(\Omega)$. Put $v = u \varphi$ on $\overline{B}_1 \setminus E = \overline{B}_1 \setminus E_1$ and $v = 0$ on $R^n \setminus \overline{B}_1$.

We see that $v$ satisfies locally the Lipschitz condition on $(B \setminus E_1) \cap (R^n \setminus \overline{B}_1)$. If $x_0 \in \partial B_1 \setminus E_1$ then there is some ball $B(x_0, r_0)$ such that $B(x_0, r_0) \subset (B \setminus E) \setminus \text{supp} \varphi$. This implies $v = 0$ on $B(x_0, r_0)$ and hence $v$ satisfies the Lipschitz condition on $B(x_0, r_0)$. On account of the arbitrariness of $x_0 \in \partial B_1 \setminus E_1$ we obtain that $v$ satisfies locally the Lipschitz condition on $R^n \setminus E$ and $v|_{G \setminus E} = u \varphi$. Since, $u, \varphi,$ $|\nabla \varphi|$ are bounded functions on $\overline{B}_1 \setminus E_1$, we see that $v \in W^{1,1}(\mathbb{R}^n \setminus E_1)$ and hence $v \in W^{1,1}(\Omega \setminus E_1)$.

It follows that there exists $g \in L^1_{\omega}(\Omega)$ such that $g|_{\Omega \setminus E_1} = v$, $g|_{G \setminus E} = u \varphi$ and $g = 0$ on $\Omega \setminus \overline{B}_1$. Assuming $g = 0$ on $R^n \setminus \Omega$ we get a function that satisfies the condition of Lemma. Thus the Lemma is proved.

Theorem 2. If $E$ is a removable set for $L^1_{\omega}(\Omega)$ then $E$ is an NC$^1_{\omega}$-set in $\Omega$.

Proof. Let $\Pi = \{x = (x_1, \ldots, x_n) : a_i < x_i < b_i, i = 1, \ldots, n\}$ be a coordinate rectangle with its facets $\sigma_{0i}, \sigma_{1i}$ from definition of NC$^1_{\omega}$-set in (4).

In view of $m_\omega(E) = 0$, we see that $\sigma_{0i}, \sigma_{1i} \subset \partial(\Pi \setminus E)$ for all $i = 1, \ldots, n$. Verify that the equality (4) is true if $j = 1$. For the remaining $j = 2, \ldots, n$ the proof of (4) is performed in the same way.

Given $\varepsilon > 0$, by Lemma 1 and Remark 1, we find a bounded function $u \in \text{Adm}(\sigma_{0i}, \sigma_{11}, \Pi \setminus E) \cap L^1_{\omega}(\Pi \setminus E)$ such that

$$
(21) \quad C_{\omega}(\sigma_{0i}, \sigma_{11}, \Pi \setminus E) \leq \int_{\Pi \setminus E} |\nabla u| \omega \, dx < C_{\omega}(\sigma_{0i}, \sigma_{11}, \Pi \setminus E) + \frac{\varepsilon}{2}.
$$

Let $G_s$ be an open neighborhood of the facets $\sigma_{s1}$, in which $u = s$, where $G_s \subset \Omega$, $s = 0, 1$, and dist($G_0, G_1) > 0$. In addition, let $G'_s$ be another open neighborhood of the facets $\sigma_{s1}$, where $G'_s \subset G_s$, $s = 0, 1$.

Set $G = G_0 \cup G_1 \cup \Pi$ and $E_1 = (\Pi \setminus E) \setminus (G_0 \cup G_1)$. It is easily to see that $u \in W^{1,1}(G \setminus E_1)$ and $E_1$ is a relatively closed subset of $G$.

Let the sequence $\{B_k\}$ be a locally finite covering of $G$ by balls $B_k = B(a_k, r_k) \subset G$, $k \geq 1$. Additionally we required that all balls $B_k$ having common points with $G'_s$ be contained in $G_s$, $s = 0, 1$. Let $\{\varphi_k\}$ be a $C^\infty$-partition of unity for $G$ subordinating to the covering $\{B_k\}$. We show that there is a function $z \in \text{Adm}(\sigma_{0i}, \sigma_{11}, \Pi)$ with $\|z\|_{L^1_{\omega}(\Pi)}$ arbitrarily close to $\|u\|_{L^1_{\omega}(\Pi \setminus E)}$.

Indeed, let $B'_k = B(a_k, r'_k)$ be a ball such that $0 < r'_k < r_k$ and supp $\varphi_k \subset B'_k$. Then, by Lemma 4, there is a function $g_k \in W^{1,1}(\mathbb{R}^n)$ for which $g_k = 0$ on $R^n \setminus \overline{B'_k}$, $g|_{G \setminus E} = u \varphi_k$, $k \geq 1$. Set $u_k = u \varphi_k$ on $G \setminus E_1$, $k \geq 1$, and $u = \sum_k u_k$ on $G \setminus E_1$ and $g = \sum_k g_k$ on $G$.

On any bounded open set $Q \subset G$ we have

$$
\sum_k \varphi_k = 1, \quad g = \sum_k g_k
$$
and on $Q \setminus E_1$

\begin{equation}
\label{eq:23}
\sum_{k} u_k,
\end{equation}

where the sums in (22), (23) contain a finite number of terms. This implies

\begin{equation}
\label{eq:24}
\|g\|_{W^{1,1}_\omega(Q)} \leq \infty, \quad \|u - g\|_{W^{1,1}_\omega(Q \setminus E_1)} \leq \sum_{k} \|u_k - g_k\|_{W^{1,1}_\omega(Q \setminus E_1)} = 0
\end{equation}

On account of the arbitrariness of $Q$ we obtain $g \in W^{1,1}_\omega(G)$ and

\[
\int_{\Pi \setminus E} |\nabla g| \omega \, dx = \int_{\Pi \setminus E} |\nabla u| \omega \, dx = \int_{\Pi} |\nabla g| \omega \, dx.
\]

Moreover, if $Q = G_0$ then $u = 0$ on $G_0$, $\|u - g\|_{W^{1,1}_\omega(G_0)} = 0$ and therefore $g = 0$ on $G_0$. Similarly if $Q = G_1$ then $u = 1 = g$ on $G_1$.

Next we approximate the function $g$ in $W^{1,1}_\omega(G)$ by smooth functions as follows.

If $B_k \cap G_0' \neq \emptyset$, we set $z_k = g_k = 0$ on $R^n$. If $B_k \cap G_1' \neq \emptyset$, we set $z_k = \varphi_k \in C^\infty_0(R^n)$. In other cases let $z_k$ denote the mollification of $g_k \in W^{1,1}_\omega(R^n)$ with a radius $0 < \rho_k < \text{dist}(\supp \varphi_k, \partial B_k')$.

We take $\beta \in \left(0, \frac{1}{2}\right)$ and, by Proposition 6, choose $\rho_k$ to satisfy

\[
\|g_k - z_k\|_{W^{1,1}_\omega(R^n)} < \beta^k.
\]

By construction, $z_k \in W^{1,1}_\omega(R^n) \cap C^\infty_0(R^n)$. Set $z = \sum z_k$ and on any set $Q \subseteq G$

we have

\[
\|g - z\|_{W^{1,1}_\omega(G)} \leq \sum_{k} \|g_k - z_k\|_{W^{1,1}_\omega(R^n)} \leq \frac{\beta}{1 - \beta} \leq 2\beta.
\]

Therefore $z \in L^1_\omega(G) \cap C^\infty(G)$ and $z = 0$ on $G_0'$ and $z = 1$ on $G_1'$. This implies $z \in \text{Adm}(\sigma_0, \sigma_1, \Pi)$ and

\begin{equation}
\label{eq:25}
\int_{\Pi} |\nabla z| \omega \, dx = \int_{\Pi} |\nabla g| \omega \, dx + o(1),
\end{equation}

where $o(1) \to 0$ as $\beta \to 0$.

Connecting (21), (24) and (25), we derive

\[
C_{1,\omega}(\sigma_0, \sigma_{11}, \Pi) \leq \int_{\Pi} |\nabla z| \omega \, dx \leq \int_{\Pi \setminus E} |\nabla g| \omega \, dx + o(1) \leq \int_{\Pi \setminus E} |\nabla u| \omega \, dx + o(1) \leq C_{1,\omega}(\sigma_0, \sigma_{11}, \Pi \setminus E) + o(1) + \varepsilon.
\]

Sequentially taking $\beta \to 0$, $\varepsilon \to 0$, we obtain

\[
C_{1,\omega}(\sigma_0, \sigma_{11}, \Pi) = C_{1,\omega}(\sigma_0, \sigma_{11}, \Pi \setminus E).
\]

This is proved our theorem. \hfill \Box

**Theorem 3.** Let $E$ be a relatively closed subset of an open set $\Omega \subset R^n$. Then the following four conditions are equivalent:

1. $E$ is a removable set for $L^1_{1,\omega}(\Omega)$;
2. $E$ is an $\text{NC}_{1,\omega}$-set in $\Omega$;
(3) \( E \) is an \( NED_{1,\omega} \)-set in \( \Omega \);
(4) \( E \in G_{1,\omega}(\Omega) \).

Proof. The implication 1) \( \rightarrow \) 2) is established in Theorem 2 and the implication 2) \( \rightarrow \) 3) is obtained in Proposition 8. So first we get the implication 3) \( \rightarrow \) 4).

Let \( E \) be an \( NED_{1,\omega} \)-set in \( \Omega \) and hence \( m_n(E) = 0 \), \( \Omega \) does not locally partitioned by \( E \). Take a compact set \( e \subset E \) and verify that \( e \) satisfies the \((1,\omega)\)-girth condition with respect to \( X_i \) (see Definition 1 in Sec. 2.6) if \( i = 1 \). For the remaining \( i = 2, \ldots, n \) the proof of the \((1,\omega)\)-girth condition with respect to \( X_i \) is performed in the same way.

Let \( \rho \) be a function from Definition 1 with respect to \( X_1 \) and put \( \rho_0 = 0 \) on \( e \).

Consider also some coordinate rectangle \( \Pi = \{ x = (x_1, \ldots, x_n) : a_i < x_i < b_i, i = 1, \ldots, n \} \) such that \( e \subset \Pi \). Put \( \Pi' = \{ x' = (x_2, \ldots, x_n) : a_i < x_i < b_i, i = 2, \ldots, n \} \) and consider \( x' \in \Pi' \) such that \( l(x') = \{ x = (x_1, x') \in \mathbb{R}^n : a_1 \leq x_1 \leq b_1 \} \) has common points with \( e \).

Verify that \( \varepsilon \)-girth condition holds for \( \rho \) on the \( m_{n-1} \)-almost all segments \( l(x') \). It suffices to prove this fact for the function \( \rho_1 = \rho + \rho_0 \), where \( \rho_0 = 1 \) on \( \Pi \) and \( \rho_0 = 0 \) on \( \mathbb{R}^n \setminus \Pi \). Then \( \int_{\Pi'} \int_{a_1}^{b_1} \rho_1 \omega \, dx = \int_{\Pi} \rho_1 \omega \, dx < \infty \).

Put \( \Phi(x', r_1, r_2) = \int_{r_1}^{r_2} \rho_1 \omega \, dx \) for all rationals \( r_1, r_2 \in [a_1, b_1], r_1 < r_2 \). As in the proof of Property 5, we verify that there exists a \( m_{n-1} \)-measurable set \( \tilde{\Pi}' \subset \Pi' \) with \( m_{n-1}(\tilde{\Pi}') = m_{n-1}(\Pi') \) such that every point \( x' \in \tilde{\Pi}' \) is a Lebesgue point for the function \( \Phi(x', r_1, r_2) \) (a differentiability point of the integral of these functions over \( \Pi' \)). Choose now \( x' = (x_2, \ldots, x_n) \in \tilde{\Pi}' \) so that \( \mathcal{H}^1(l(x') \cap e) = 0 \) (by definition \( l(x') \cap e \neq \emptyset \)).

Cover \( l(x') \cap e \) by the mutually disjoint intervals \( U_k = (c_k, d_k), k = 1, \ldots, k_1 \), where \( c_k = (r_k, x'), d_k = (\tilde{r}_k, x') \), while \( r_k \) and \( \tilde{r}_k \) are rationals in \((a_1, b_1)\) with \( r_k < \tilde{r}_k \) and

\[
\sum_{k=1}^{k_1} \mathcal{H}^1(U_k) < \varepsilon, \quad \sum_{k=1}^{k_1} \int_{U_k} \rho_1 \omega \, dx < \frac{\varepsilon C}{8},
\]

where \( C = \frac{1}{\sup_{\Pi} (1 + |x|)^n} \). For small \( \delta > 0 \) put

\[
\Pi'(\delta) = \{ y' = (y_2, \ldots, y_n) : |y_i - x_i| < \delta, i = 2, \ldots, n \},
\]

\[
\Pi_k(\delta) = \{ (x, y') : r_k < x_1 < \tilde{r}_k, y' \in \Pi'(\delta) \} \subset \Pi.
\]

Denote by \( \sigma_0^k = \sigma_0^k(\delta), c_k \in \sigma_0^k, \) and \( \sigma_1^k = \sigma_1^k(\delta), d_k \in \sigma_1^k, \) the facets of \( \Pi_k(\delta) \), parallel to \( H_1 = \{ x = (x_1, \ldots, x_n) : x_1 = 0 \} \).

In addition, put \( \Gamma_k(\delta) = \Gamma(\sigma_0^k, \sigma_1^k, \Pi_k(\delta) \setminus e) \). It is easily to see that \( \Pi_k(\delta) \in K_1(\varepsilon, \Pi) \). By (12) and the definition of an \( NED_{1,\omega} \)-set,

\[
M_{1,\omega}(\sigma_0^k, \sigma_1^k, \Pi_k(\delta) \setminus e) \geq \frac{m_{n-1}(\sigma_0^k)}{\sup_{\Pi_k(\delta)} (1 + |x|)^n} \geq C \cdot m_{n-1}(\sigma_0^k),
\]

where \( m_{n-1}(\sigma_0^k) = m_{n-1}(\Pi'(\delta)) \).
Let \( L_k(\delta) = \inf \{ \int_{\gamma} \rho_1 \, ds : \gamma \in \Gamma_k(\delta) \} \). Obviously, \( L_k(\delta) \geq \hat{r}_k - r_k \) for all \( k = 1, \ldots, k_1 \). Show that \( L_k(\delta) \) is sufficiently small when \( \delta > 0 \) close to zero. Indeed, \( \frac{\rho_1}{L_k(\delta)} \in \text{adm } \Gamma_k(\delta) \) and

\[
\int_{\Omega(\delta)} dx' \int_{r_k}^{\hat{r}_k} \rho_1 \omega \, dx_1 \geq L_k(\delta) C m_{n-1}(\sigma_0^k).
\]

For sufficiently small \( \delta > 0 \), using the differentiability of the integrals, under consideration,

\[
L_k(\delta) \leq \frac{1}{C} \int_{r_k}^{\hat{r}_k} \rho_1 \omega \, dx_1 + \frac{\varepsilon}{8k_1}
\]

for all \( k = 1, \ldots, k_1 \). By the definition of infimum, there exists a curve \( \gamma_k^0 \in \Gamma_k(\delta) \) such that

\[
\int_{\gamma_k^0} \rho_1 \, ds \leq L_k(\delta) + \frac{\varepsilon}{8k_1} \leq \frac{1}{C} \int_{r_k}^{\hat{r}_k} \rho_1 \omega \, dx_1 + \frac{\varepsilon}{4k_1}.
\]

Since \( \rho_1 \geq 1 \) on \( \Omega \) we have \( \int ds \leq \int \rho_1 \, ds \). Hence \( \gamma_k^0 \) is a rectifiable curve. By the local boundedness of \( \rho_1 \) on \( \Omega \setminus \varepsilon_1 \), for sufficiently small \( \delta > 0 \) there exist segments \( \tau_k^0 \subset \sigma_k^0(\delta) \) and \( \tau_k^1 \subset \sigma_k^1(\delta) \) joining the points \( c_k \) and \( d_k \), respectively, with the end points of \( \gamma_k^0 \) in \( \Pi_k(\delta) \setminus \varepsilon \) so that

\[
\int_{\tau_k^0} \rho_1 \, ds + \int_{\tau_k^1} \rho_1 \, ds < \frac{\varepsilon}{8k_1}.
\]

Consequently, there exist simple curves \( \gamma_k \subset \tau_k^0 \cup \gamma_k^0 \cup \tau_k^1 \subset \Pi_k(\delta) \setminus \varepsilon \) joining the corresponding points \( c_k \) and \( d_k \), \( k = 1, \ldots, k_1 \), and by (26), (27)

\[
\sum_{k=1}^{k_1} \int_{\gamma_k} \rho_1 \, ds < \varepsilon.
\]

This implies

\[
\sum_{k=1}^{k_1} \int ds < \sum_{k=1}^{k_1} \int_{\gamma_k} \rho_1 \, ds < \varepsilon, \quad \sum_{k=1}^{k_1} \int \rho \, ds < \sum_{k=1}^{k_1} \int \rho_1 \, ds < \varepsilon.
\]

In other words, the \( \varepsilon \)-girth condition holds for \( \rho \) and \( \rho_1 \) on \( m_{n-1} \)-almost all segments \( l(x'), x' \in \Pi' \).

Let \( t = (t_1, x') \in l(x') \setminus \varepsilon \), where (28) holds for \( l(x') \). It is easy to verify that the \( \varepsilon \)-girth condition holds for \( \rho, \rho_1 \) on \( l(t_1, x') = \{(x_1, x') : a_1 \leq x_1 \leq t_1\} \) (see Remark to Definition 1) for all such \( t \) on \( l(x') \). We will call this property the \([t, \varepsilon]-\)condition for the function \( \rho \) on the segment \( l(x') \). So the implication 3) \( \rightarrow \) 4) is true.

Let’s go to the proof of implication 4) \( \rightarrow \) 1) which we will present in two steps.

Step 1. Let \( E \in \mathcal{G}_{1,\omega}(\Omega) \). By the definition 3, \( m_\nu(E) = 0 \) and \( \Omega \) does not partitioned locally by \( E \). Let \( u \in L^1_{1,\omega}(\Omega \setminus E) \cap C^\infty(\Omega \setminus E) \) and \( u \) be bounded
function on $\Omega \setminus E$. Then $u$ can extended to a function $g \in L^1_{1,\omega}(\Omega)$ for which $g|_{\Omega \setminus E} = u$, as follows.

Next we use constructions from the proof of Theorem 2. Let the sequence $\{B_j\}$ be a locally finite covering of $\Omega$ by balls $B_j = B(a_j, r_j) \in \Omega$, $j \geq 1$. Let $\{\varphi_j\}$ be a $C^\infty$-partition of unity, subordinated to the covering $\{B_j\}$. Besides that let $B'_j = B(a_j, r'_j)$ be a ball such that $0 < r'_j < r_j$ and supp $\varphi_j \subset B'_j$. Set $e_j = \overline{B}_j \cap E$ and let $v_j = u\varphi_j$ on $\overline{B}_j \setminus e_j$ and $v_j = 0$ on $R^n \setminus \overline{B}_j$.

It is easily to see (proof of Lemma 4) that $v_j \in W^{1,1}(R^n \setminus e_j) \cap C^\infty(R^n \setminus e_j)$ and so, $v_j|_{\Omega \setminus E} = u\varphi_j$, $|\nabla v_j|$ is a continuous function on $R^n \setminus e_j$.

Take some coordinate rectangle $\Pi_j$ such that $\overline{B}_j \subset \Pi_j$ and verify that $v_j$ can be extended to a function $g_j \in L^1_{1,\omega}(R^n)$, when $g|_{R^n \setminus e_j} = v_j$.

Since $v_j = 0$ on $R^n \setminus \overline{B}_j$, it suffices to prove that $v_j|_{\Pi_j \setminus e_j}$ can be extended to function $g_j \in L^1_{1,\omega}(\Pi_j)$. Fix $j \geq 1$.

In the implication 3) $\rightarrow$ 4) proof given above, we will put $e = e_j$, $\rho = |\nabla v_j|$ on $R^n \setminus e$, $\Pi_j = \Pi$ and $\rho = |\nabla v_j| = 0$ on $e$. In addition, we set $\frac{\partial v_j}{\partial x_i} = 0$ on $e$, $i = 1, \ldots, n$, and we use notation from the proof of implication 3) $\rightarrow$ 4).

Show that $v_j$ is an absolutely continuous function on $m_{n-1}$-almost all segments $l(x') = \{(x_1, x') : a_1 \leq x_1 \leq b_1\}$, $x' \in \Pi' = \{(x_2, \ldots, x_n) : a_i < x_i < b_i, i = 2, \ldots, n\}$, if we define $v_j$ properly on $e$.

By Proposition 1, we also note that $\int_\Pi |\nabla v_j| \, dx < \infty$. This implies

\begin{equation}
\int_{l(x')} |\nabla v_j| \, dx_1 < \infty
\end{equation}

for $m_{n-1}$-almost all $l(x')$, $x' \in \Pi'$. In addition, if $l(x') \cap e = \emptyset$ then $v_j$ is infinitely differentiable, and therefore obviously absolutely continuous on $l(x')$.

Thus, we consider only those segments $l(x')$, $x' \in \Pi'$, that satisfy the inequality (29), $l(x') \cap E \neq \emptyset$, $H^1(l(x')) \cap E = 0$ and on which the $\varepsilon$-girth condition holds for $\rho = |\nabla v_j|$ for all $\varepsilon > 0$ (see Remark to Definition 1). The family of all such segments $l(x')$ is denoted by $\Gamma_1$. Take $l(x') \in \Gamma_1$ and given $\eta > 0$, choose $\beta > 0$ such that $\int_F |\nabla v_j| \, dx_1 < \eta$ for every $F \subset l(x')$ with $H^1(F) < \beta$.

On the segment $l(x')$ consider an absolutely continuous function

\begin{equation}
q(t_1, x') = \int_{t_1}^{t_1} \frac{dv_j}{dx_1} \, dx_1,
\end{equation}

where $t = (t_1, x') \in l(x')$. We state that $q(t) = v_j(t)$ for all $t = (t_1, x') \in l(x') \setminus e$.

By the choice of $l(x')$ and $[t, \varepsilon]$-girth condition for $|\nabla v_j|$ on $l(x')$ (see proof of implication 3) $\rightarrow$ 4) above), there exist intervals $U_k = (c_k, d_k) \subset l(t_1, x') = \{(x_1, x') : a_1 \leq x_1 \leq t_1\}$, $k = 1, \ldots, k_1$, with $\bigcup_{k=1}^{k_1} U_k \supset e \cap l(t_1, x')$ and rectifiable curves $\gamma_k \subset \Pi \setminus e$ joining the endpoints of $U_k$ such that

\begin{equation}
\sum_{k=1}^{k_1} \int_{\gamma_k} ds < \varepsilon, \quad \sum_{k=1}^{k_1} \int_{\gamma_k} |\nabla v_j| \, ds < \varepsilon.
\end{equation}
Without loss of generality, let $c_k = (r_k, x')$, $d_k = (\tilde{r}_k, x')$, $k = 1, \ldots, k_1$, $a_1 < r_1 < \tilde{r}_1 < \cdots < r_k < \tilde{r}_k < t_1$. Set $c = (a_1, x')$, $0 < \varepsilon < \eta$. Consider a rectifiable curve 
$$
\gamma = [c, c_1] \cup [d_1, c_2] \cup \cdots \cup [d_{k-1}, c_k] \cup [d_k, t_1].
$$

Since $v_j \in C^\infty(R^n \setminus c)$ we have $\int_\gamma \frac{dv_j}{ds} ds = v_j(t) - v_j(c)$. On the other hand, due to absolute continuity of the integrals in (30) and (31) we deduce

$$
v_j(t) - v_j(c) = \int_{a_1}^{t_1} \frac{dv_j}{dx_1} dx_1 + o(1),
$$

where $o(1) \to 0$ as $\varepsilon \to 0$. Since $v_j(c) = 0$, taking $\varepsilon \to 0$, we have

$$
v_j(t) = \int_{a_1}^{t_1} \frac{dv_j}{dx_1} dx_1 \text{ for all } t = (t_1, x') \in l(x') \setminus e.
$$

Define $v_j(x_1, x')$ as $\int_{a_1}^{x_1} \frac{dv_j}{dx_1} dx_1$ if $x \in l(x') \cap e$, $l(x') \in \Gamma_1$, and $v_j(x) = 0$ if $x = (x_1, x') \in l(x') \cap e, l(x') \notin \Gamma_1$.

Hence $v_j(x)$ is an absolutely continuous function in $R^n$ on $m_{n-1}$-almost all straight lines $l$, parallel to the $x_1$-axis.

In addition, using integration by parts and Fubini’s theorem, we obtain

$$
\int_{R^n} v_j \frac{\partial \varphi}{\partial x_1} dx_1 = - \int_{R^n} \varphi \frac{\partial v_j}{\partial x_1} dx_1
$$

for all $\varphi \in C^\infty_0(R^n)$. Now note that in (32) it is possible to redefine the values of $v_j, \frac{\partial v_j}{\partial x_1}$ on $e = e_j$. Then we get $g_j = v_j$ on $R^n \setminus e$, $g_j = 0$ on $e$, and $\frac{\partial g_j}{\partial x_1} = \frac{\partial v_j}{\partial x_1}$ on $R^n \setminus e$, $\frac{\partial g_j}{\partial x_1} = 0$ on $e_j, j = 1, \ldots, n$.

Similarly, let’s make sure that the function $g_j$ has a weak derivative $\frac{\partial g_j}{\partial x_i}$ in $R^n$

for $i = 2, \ldots, n$. This implies, by construction, $g_j \in W^{1,1}_\omega(R^n)$, $g_j = 0$ on $R^n \setminus \overline{B}_r$, $g_j = w \varphi_j$ on $\Omega \setminus E$. Set $g = \sum g_j$ on $\Omega$. Using technique from the proof of Theorem 2 (see (22)–(24)), we deduce that $\|u - g\|_{H^{1,1}_\omega(\Omega \setminus E)} = 0$ for all open $\Omega' \subseteq \Omega$, $\|u - g\|_{L^1_\omega(\Omega \setminus E)} = 0$, $\|u\|_{L^1_\omega(\Omega \setminus E)} = \|g\|_{L^1_\omega(\Omega)}$.

In other words, $g \in L^1_\omega(\Omega)$ and $g|_{\Omega \setminus E} = u$.

Step 2. Now let $u$ be an arbitrary function in $L^1_\omega(\Omega \setminus E)$ and $\{D_k\}$ be a sequence (maybe finite) of all mutually disjoint connected components $D_k$ of $\Omega$. Then, by the definition of $E$, $D_k \setminus E$ is the connected component of $\Omega \setminus E, k \geq 1$, and $\Omega \setminus E = \bigcup_k (D_k \setminus E)$. By Proposition 7, there exists a sequence of bounded functions

$$
u_j \in L^1_\omega(\Omega \setminus E) \cap C^\infty(\Omega \setminus E)
$$
such that

$$
\lim_{j \to \infty} \|u_j - u\|_{L^1_\omega(\Omega \setminus E)} = 0,
$$

$$
\lim_{j \to \infty} \|u_j - u\|_{L^1_\omega(\Omega)} = 0 \text{ for all } \Omega' \subseteq \Omega \setminus E.
$$
Accordingly to Step 1, we assume that \( u_j \in L^1_{1,\omega}(\Omega) \) for all \( j \geq 1 \). In view of (33) and \( m_n(E) = 0 \) it follows that \( \{u_j\} \) is the Cauchy sequence \( L^1_{1,\omega}(\Omega) \). Then, by Proposition 2, \( \{u_j\} \) converges in \( L^1_{1,\omega}(D_k) \) to some function \( v_k, k \geq 1, \) as \( j \to \infty \).

Moreover, from (33) \( |\nabla(u - v_k)| = 0 \) a.e. on \( D_k \setminus e \) and therefore \( u = v_k + c_k \) (see [5, Sec. 1.1.5]) on \( D_k \setminus E \). Using (34), it is easily to show that \( c_k = 0, k \geq 1 \).

For all \( x \in \Omega \) we set \( v(x) = v_k(x) \) if \( x \in D_k \). By construction,

\[
|u|_{L^1_{1,\omega}(\Omega, E)} = |v|_{L^1_{1,\omega}(\Omega)}, \quad v(x)|_{\Omega \setminus E} = u(x).
\]

Thus, \( E \) is a removable set for \( L^1_{1,\omega}(\Omega) \), that completes the proof of the Theorem.

\( \square \)

A simple modification of the arguments in the proof of Theorem 3 gives another assertion.

**Theorem 4.** If \( E \) is an NC\( 1_{1,\omega} \)-set in \( \Omega \) then \( L^m_{1,\omega}(\Omega \setminus E) = L^m_{1,\omega}(\Omega), W^m_{1,\omega}(\Omega \setminus E) = W^m_{1,\omega}(\Omega), W^m_{1,\omega}(\Omega \setminus E) = W^m_{1,\omega}(\Omega) \).

Following Vodop’yanov and Gol’dstein [8] domains \( G_1 \) and \( G_2 \) \( (G_1 \supset G_2) \) in \( \mathbb{R}^n \) will be called \((1,1,\omega)\)-equivalent, if the restriction operator \( \theta : L^1_{1,\omega}(G_1) \to L^1_{1,\omega}(G_2) \) \( (\theta u = u|_{G_2}) \) is the isomorphism of the vector spaces \( L^1_{1,\omega}(G_1) \) and \( L^1_{1,\omega}(G_2) \).

**Theorem 5.** Domain \( G_1 \) and \( G_2 \) \( (G_1 \supset G_2) \) are \((1,1,\omega)\)-equivalent iff the set \( E = G_1 \setminus G_2 \) is an NC\( 1_{1,\omega} \)-set in \( G_1 \).

**Proof.** Necessity. Let the spaces \( L^1_{1,\omega}(G_1) \) and \( L^1_{1,\omega}(G_2) \) be isomorphic as linear spaces for the restriction isomorphism \( \theta u = u|_{G_2} \) and \( u \in L^1_{1,\omega}(G_1) \). Passing to the factor-spaces \( L^1_{1,\omega}(G_1) \) and \( L^1_{1,\omega}(G_2) \) (see Sec. 2.3 and Proposition 3, 4) and using the Banach theorem, we obtain the boundedness of the operator \( \theta^{-1} \). Let us prove that \( m_n(G_1 \setminus G_2) = 0 \). Assume the converse. Then the set \( G_1 \setminus G_2 \) has at least one density point \( x_0 \), which is also the Lebesgue point for weight \( \omega \).

Let us consider a sequence of open coordinate cubes \( Q_m = Q \left( x_0, \frac{1}{m} \right) \) with the center \( x_0 \) and the edge of length \( \frac{1}{m} \). Let us consider the function \( u_m(x) = \text{dist}(x, \mathbb{R}^n \setminus Q_m) \) on \( \mathbb{R}^n \). It is known (see [7]) that \( |\nabla u_m| = 1 \) a.e. on \( Q_m \) and \( |\nabla u_m| = 0 \) a.e. on \( \mathbb{R}^n \setminus Q_m \). \( |u_m(x') - u_m(x'')| \leq |x' - x''| \) for any \( x', x'' \in \mathbb{R}^n \). Hence \( u_m \in L^1_{1,\omega}(Q_m) \cap L^1_{1,\omega} \) for all \( m \geq 1 \). From the boundedness of the operator \( \theta^{-1} \) we have

\[
\int_{Q_m} \omega dx \leq \int_{G_1} |\nabla u_m| \omega dx \leq \|\theta^{-1}\| \int_{G_2} |\nabla u_m| \omega dx \leq \|\theta^{-1}\| \int_{G_2 \cap Q_m} \omega dx.
\]

This implies

\[
\frac{1}{|Q_m|} \int_{Q_m} \omega dx \leq \frac{\|\theta^{-1}\|}{|Q_m|} \int_{G_2 \cap Q_m} \omega dx.
\]

For \( m \to \infty \) the inequality is not valid. Consequently, \( m_n(G_1 \setminus G_2) = 0 \) and every function \( v \in L^1_{1,\omega}(G_2) \) may be extended to function \( u \in L^1_{1,\omega}(G_1) \) for which \( u|_{G_2} = v \). Hence \( E = G_1 \setminus G_2 \) is a removable set for \( L^1_{1,\omega}(G_1) \), and, by Theorem 3, \( E \) is an NC\( 1_{1,\omega} \)-set in \( G_1 \).
The sufficiency condition in the Theorem follows from the Theorem 3. Indeed, let \( E = G_1 \setminus G_2 \) be an NC\(_{1,\omega}\)-set in \( G_1 \). By Theorem 3, every function \( v \in L_{1,\omega}^1(G_2) \) is extended to the function \( u \in L_{1,\omega}^1(G_1) \) such that \( u|_{G_2} = v \). In view of \( m_{n}(E) = 0 \), the extension is only one (the functions \( u_1, u_2 \in L_{1,\omega}^1(G_1) \) with \( m_n (\{x \in G_1 : u_1(x) \neq u_2(x)\}) = 0 \) will be identified).

Consequently, \( \theta : L_{1,\omega}^1(G_1) \to L_{1,\omega}^1(G_2) \) is the isomorphism of \( L_{1,\omega}^1(G_1) \) and \( L_{1,\omega}^1(G_2) \). The Theorem is proved. □

**Remark 3.** In the case \( 1 < p < \infty \) and \( \omega = 1 \) the criterion of \((1,p)\)-equivalence of domains \( G_1 \) and \( G_2 \) \( (G_1 \supset G_2) \) in terms of NC\(_p\)-sets was established in [18].

### References


**Vladimir Alekseevich Shlyk**

**Vladivostok Branch of Russian Customs Academy,**

16 V, Strelkovaya str.,

Vladivostok, 690041, Russia

**Institute of Applied Mathematics, Vladivostok Branch of the RAS,**

7, Radio str.,

Vladivostok, 690041, Russia

**Email address:** shlykva@yandex.ru