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REMOVABLE SETS FOR SOBOLEV SPACES WITH MUCKENHOUPT A_1-WEIGHT

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Abstract. Let Ω be an open set in R^n, $n \geq 2$, and E be a relatively closed subset of Ω. In this paper we obtain a criterion of equality $L^1_{1,\omega}(\Omega\setminus E) = L^1_{1,\omega}(\Omega)$ in terms of E as an $NC_{1,\omega}$-set in Ω with A_1-weight ω. In addition, we establish exact characterizations of $NC_{1,\omega}$-sets in terms of $NED_{1,\omega}$-sets and of the $(1,\omega)$-girth condition. In the case $\omega \equiv 1$, these results complete the studies of Vodop'yanov and Gol'dstein on removable sets for $L^1_p(\Omega)$, $p \in (1,+\infty)$.

Keywords: Sobolev space, capacity and modulus of condenser, Muckenhoupt weight, removable set.

1. Introduction

In [18] Vodop'yanov and Gol'dstein gave a criterion of removable singularities for $L^1_p(\Omega)$, $W^1_p(\Omega)$ in terms of NC_p-sets, $1 < p < \infty$. An NC_p-set can be considered as a p-analog of an NED-set, earlier introduced by Väisälä [16] as result of generalizing the concept of NED-set in R^2 [1] to R^n, $n \geq 2$. Also note the definitions of an NC_p-set in Ω or an NED-set in R^n are based on condensers whose plates are the pair of arbitrary disjoint continuums located outside this set in Ω or in R^n, respectively.

Latter [6] Dymchenko and Shlyk obtained similar assertions about removable singularities for the space $L^1_{p,\omega}(\Omega)$ in terms of $NC_{p,\omega}$-sets in Ω, where ω is a Muckenhoupt A_p-weight, $1 < p < \infty$. Their definition of $NC_{p,\omega}$-set in Ω (as well as the initial definition of NED-set by Ahlfors–Beurling [1] in R^2) is based on condensers formed by an arbitrary coordinate rectangles Π, $\overline{\Pi} \subset \Omega$, and by any pair of its opposite facets.

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The main aim of this paper is to define $NC_{1,\omega}$-sets, $NED_{1,\omega}$-sets in Ω with Muckenhoupt A_1-weight, using proper coordinate rectangular condensers, and to give criteria of equality $L^1_{1,\omega}(\Omega \setminus E) = L^1_{1,\omega}(\Omega)$ in terms of E as an $NC_{1,\omega}$-set, $NED_{1,\omega}$-set or with the $(1,\omega)$-girth condition (see Theorem 3).

Here we note the difficulties in proving the main results of the paper, which are not present at $p > 1$. Namely, the equality of $(1,\omega)$-modulus and $(1,\omega)$-capacity of the condenser is not known in general, even for $\omega \equiv 1$. Similarly, the equality $M_{1,\omega}(\bigcup_j \Gamma_j) = \lim_{j\to\infty} M_{1,\omega}(\Gamma_j)$ is unknown if $\Gamma_j \subset \Gamma_{j+1}$, $j \geq 1$, and $M_{1,\omega}(\cdot)$ is a $(1,\omega)$-modulus of a curve family in R^n.

2. Preliminaries

2.1. **Some definitions and notations.** Throughout the text the symbol Ω denotes a non-empty open set in Euclidean space $R^n = \{x = (x_1, \dots, x_n)\}$, where $n \geq 2$. Respectively, E denotes a relatively closed subset of Ω, the norm of a point $x = (x_1, \dots x_n)$ is given by $|x| = \left(\sum_{i=1}^{n} x_i^2\right)$. We put $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{R} = (-\infty, +\infty)$. If $F \subset R^n$ then ∂F, $\bar{F}$ stand for the boundary and the closure of F in R^n, respectively. The distance between two sets $A, B \subset R^n$ is denoted by $\operatorname{dist}(A, B)$. For an open set $U \subset R^n$, we use the notation $U \Subset \Omega$ in order to indicate that U is bounded and $\bar{U} \subset \Omega$. The restriction of a function f to a set F is denoted by $f|_F$. Given $x \in R^n$ and $r > 0$, let $B(x, r) = B_r(x) = \{y \in R^n : |y - x| < r\}$. If $a > 0$ then $aB_r(x) = B_{ar}(x)$. The symbol $\mathcal{H}^s$ stands for the usual s-dimensional Hausdorff measure in R^n; m_n is a Lebesgue measure in R^n and put $|F| = m_n(F)$ for m_n-measurable set $F \subset R^n$.

Let $C^\infty(\Omega)$ be the space of infinitely differentiable functions in Ω; the space of functions in $C^\infty(R^n)$ with a compact support in Ω is denoted by $C_0^\infty(\Omega)$.

We will use the abbreviation "a.e." for "almost everywhere" with respect to m_n-measure. Similarly, "measurable" and "locally integrable" always mean Lebesgue measurable and locally integrable with respect to m_n-measure.

Let F be a measurable subset of R^n, and u is a measurable real-valued function on F. For $1 \leq p < \infty$ let

$$\|u\|_{L_p(F)} = \left(\int_F |u(x)|^p \, dx\right)^{1/p}.$$

Assume that $u(x)$ is a measurable function defined on Ω. We say that u is locally integrable to the power $p \in [1, +\infty)$ on Ω (and write $u \in L_p(\Omega, loc)$) if $\|u\|_{L_p(F)} < \infty$ for every compact set $F \subset \Omega$. The class of all functions u such that $\|u\|_{L_p(\Omega)} < \infty$ is denoted by $L_p(\Omega)$.

If $\Omega = R^n$ we shall often omit Ω in notations of spaces and norm. Integration without indication of limits extends over R^n.

Let $C, C_1, C_2, \dots$ denote positive constants that depend on "dimensionless" parameters n, p, m and the like.

We call the quantities a and b equivalent and write $a \sim b$ if $C_1 a \leq b \leq C_2 a$.

If $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n-tuple of nonnegative integers α_i, we call α a multi-index and denote by x^α the monomial $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ which has degree $|\alpha| = \sum_{i=1}^{n} \alpha_i$.

Similarly, if $D_j = \dfrac{\partial}{\partial x_j}$ then $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$ denotes a differential operator of order $|\alpha|$. Note that $D^{(0,\dots,0)}u = u$ for real-valued functions u. We use the notations $\nabla_m = \{D^\alpha : |\alpha| = m\}$, $\nabla = \nabla_1$.

By a weight we shall mean a locally integrable function ω on R^n such that $\omega > 0$ for a.e. $x \in R^n$.

Then for $1 \le p < \infty$ we define $L_{p,\omega}(\Omega)$ as the set of measurable functions f on Ω such that

$$\|f\|_{L_{p,\omega}(\Omega)} = \left(\int\limits_\Omega |f|^p \, \omega dx \right)^{1/p} < \infty.$$

As usual, any two functions f and g in $L_{p,\omega}(\Omega)$ that are equal a.e. on Ω will be identified.

Let $\mathcal{F}_1$ be a class of functions given on Ω, and $\mathcal{F}_2$ be another class of functions given on Ω', where $\Omega' \subset \Omega$. Below if $f \in \mathcal{F}_1$ then $f \in \mathcal{F}_2$ means $f|_{\Omega'} \in \mathcal{F}_2$.

Denote by $L_{p,\omega}(\Omega, loc)$ the class of all measurable functions f on Ω such that $f \in L_{p,\omega}(\Omega')$ for all open sets $\Omega' \Subset \Omega$.

2.2. A_1**-weights.** Following B. Muckenhoupt [12] a weight ω is called an A_1-weight, if there exists a positive constant A such that for every ball $B \subset R^n$,

$$\left(\frac{1}{|B|} \int\limits_B \omega \, dx \right) \operatorname*{ess\,sup}_{x \in B} \frac{1}{\omega(x)} \le A, \tag{1}$$

The infimum over all such constants A is called the A_1-constant of ω. Denote by A_1 the class of A_1-weights. Throughout the text let $m \in \mathbb{N}$, $\omega \in A_1$ unless otherwise stated.

Set

$$M\omega(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} \omega(y) \, dy.$$

We mention two assertions concerning A_1-weight.

Proposition 1 ([15, Remark 1.2.4],[10, Theorem 2.7])**.** *If $\omega \in A_1$ then $L_{1,\omega}(\Omega)$ is complete space in norm $\| \cdot \|_{L_{1,\omega}(\Omega)}$ and $L_{1,\omega}(\Omega) \subset L_1(\Omega, loc)$. In addition, $L_{1,\omega}(\Omega) \subset L_1(\Omega)$ in the case of a bounded $\Omega \subset R^n$.*

Proposition 2 ([15, Remark 1.2.4, Properties 7 and 8])**.** *If $\omega \in A_1$ then there exist constants C, C_1 such that $\omega(x) \ge \dfrac{C}{(1+|x|)^n}$ and $M\omega(x) \le C_1 \omega(x)$ for a.e. $x \in R^n$.*

2.3. **Weighted Sobolev spaces.** Suppose that $u : \Omega \to R$ is a function in $L_1(\Omega, loc)$. This function u on Ω has a weak derivative of order $|\alpha|$ if there is a locally integrable function (denoted by $D^\alpha u$) such that

$$\int\limits_\Omega u \cdot D^\alpha \varphi dx = (-1)^{|\alpha|} \int\limits_\Omega D^\alpha u \cdot \varphi dx$$

for all $\varphi \in C_0^\infty(\Omega)$. For $m \in \mathbb{N}$ and any $\omega \in A_1$, $L^m_{1+,\omega}(\Omega)$ is the space of functions u having on Ω weak derivatives $D^\alpha u$ for all orders $|\alpha|$, $|\alpha| \le m$, and satisfying

$$\|u\|_{L^m_{1,\omega}(\Omega)} = \int\limits_\Omega |\nabla_m u| \omega \, dx < \infty,$$

where $|\nabla_m u| = \left(\sum_{|\alpha|=m} (D^\alpha u)^2 \right)^{1/2}$. For $m = 0$ set $L^0_{1,\omega}(\Omega) = L_{1,\omega}(\Omega)$, $\nabla_0 u = u$.

Introduce the spaces

$$W^m_{1,\omega}(\Omega) = L^m_{1,\omega}(\Omega) \cap L_{1,\omega}(\Omega), \qquad W^{m,1}_\omega(\Omega) = \bigcap_{k=0}^m L^k_{1,\omega}(\Omega),$$

equipped with the norms

$$\|u\|_{W^m_{1,\omega}(\Omega)} = \|u\|_{L^m_{1,\omega}(\Omega)} + \|u\|_{L_{1,\omega}(\Omega)}, \qquad \|u\|_{W^{m,1}_\omega(\Omega)} = \sum_{k=0}^m \|\nabla_k u\|_{L_{1,\omega}(\Omega)}.$$

Next we let $\mathcal{P}_{m-1}$ be the collection of all polynomials of degree $\leq m-1$. Let us consider the factor-space $\check{L}^m_{1,\omega}(\Omega) = L^m_{1,\omega}(\Omega)/\mathcal{P}_{m-1}$ (with norm $\|\cdot\|_{L^m_{1,\omega}(\Omega)}$). Elements of the space $\check{L}^m_{1,\omega}(\Omega)$ on each connected component D of the set Ω are classes $\{u + P_D\}$ where $u \in L^m_{1,\omega}(\Omega)$ and $P_D \in \mathcal{P}_{m-1}$.

In the case $\omega \equiv 1$, the weight spaces considered above with the weight ω will be written below without the symbol ω.

Note that a number of important properties of spaces $W^{m,1}_\omega(\Omega)$, $L^m_{1,\omega}(\Omega)$ (in other notations and with equivalent norms) were obtained in [5, 15]. We use the following ones below.

Proposition 3 ([5, Theorem 4.9]). *If Ω is an open connected set and $\omega \in A_1$ then $\check{L}^m_{1,\omega}(\Omega)$ is a Banach space. In particular, if $\{u_j\}$ is a Cauchy sequence in $L^m_{1,\omega}(\Omega)$ then there exists $u_0 \in L^m_{1,\omega}(\Omega)$ such that $\nabla_m u_j \to \nabla_m u_0$ in $L_{1,\omega}(\Omega)$ as $j \to \infty$.*

Proposition 4 ([5, Corollary 4.10]). *Let Ω be an open connected set, let $\{u_j\}$ be a Cauchy sequence in $L^m_{1,\omega}(\Omega)$, and let u be a function in $L^m_{1,\omega}(\Omega)$ such that $\|\nabla_m(u_j - u)\|_{L_{1,\omega}(\Omega)} \to 0$. Then there exists a sequence of polynomials $\{P_j\} \subset \mathcal{P}_{m-1}$ with $u_i - P_j \to u$ in $L_{1,\omega}(K)$ for all compact sets $K \subset \Omega$.*

Proposition 5 ([5, Theorem 4.2]). *Let $\omega \in A_1$. If $u \in L^m_{1,\omega}(\Omega)$ then*

$$\int_K |D^\alpha u| \omega \, dx < \infty \tag{2}$$

for all compact $K \subset \Omega$, $0 \leq |\alpha| \leq m$.

Definition 1. *If the restriction operator $\theta : L^m_{1,\omega}(\Omega) \to L^m_{1,\omega}(\Omega \setminus E)$ ($\theta u = u|_{\Omega \setminus E}$) induces the isometric isomorphism of the normed spaces $\check{L}^m_{1,\omega}(\Omega)$ and $\check{L}^m_{1,\omega}(\Omega \setminus E)$, then we write $L^m_{1,\omega}(\Omega) = L^m_{1,\omega}(\Omega \setminus E)$. In other words, this means that $|E| = 0$ and for every function $u \in L^m_{1,\omega}(\Omega \setminus E)$ there is a function $v \in L^m_{1,\omega}(\Omega)$ for which $v|_{\Omega \setminus E} = u$. In this case, the function v is called an extension of the function u in $L^m_{1,\omega}(\Omega)$ and E is called a removable set for $L^m_{1,\omega}(\Omega)$.*

Similarly we define removable sets for $W^{m,1}_\omega(\Omega)$ and $W^m_{1,\omega}(\Omega)$.

2.4. **Mollifications.** Let $\psi \in C^\infty_0(R^n)$ be nonnegative function such that $\operatorname{supp} \psi \subset B_1(0)$ and $\int \psi(x) dx = 1$. For any function $u \in L_1(\Omega)$, extended by zero on $R^n \setminus \Omega$, we define the family of its mollifications

$$(M_\varepsilon u)(x) = \varepsilon^{-n} \int u(y) \psi\left(\frac{y-x}{\varepsilon}\right) dy = \int_{|\xi|<1} u(x + \varepsilon\xi) \psi(\xi) \, d\xi, \quad 0 < \varepsilon \leq 1.$$

The number ε shall be called a radius of mollification.

The following result is known.

Proposition 6 ([15, Theorem 2.1.4, Corollary 2.1.5]). *Suppose that $u \in W_\omega^{m,1}(\Omega)$ and let Ω' be an open set, $\Omega' \Subset \Omega$. Then $(M_\varepsilon u)(x) \in C^\infty(\Omega) \cap L_{1,\omega}(\Omega)$ and for $0 < \varepsilon < \min(\mathrm{dist}(\Omega', \partial\Omega), 1)$ the equality $D^\alpha M_\varepsilon u = M_\varepsilon D^\alpha u$ is true on Ω', $1 \le |\alpha| \le m$; $M_\varepsilon u \to u$ in $W_\omega^{m,1}(\Omega')$ as $\varepsilon \to 0$. In the case $\Omega = R^n$ we have convergence $M_\varepsilon u \to u$ in $W_\omega^{m,1}(R^n)$ as $\varepsilon \to 0$.*

Using Proposition 6 and an approach due to Maz'ya [11, Sec. 1.1.5, Theorem 1] (see proof of Theorem 2, Sec. 4), we obtain another assertion.

Proposition 7. *Let $u \in L^1_{1,\omega}(\Omega)$ and $\{\Omega_j\}$ be some sequence of open sets Ω_j such that $\Omega_j \Subset \Omega_{j+1} \subset \Omega$ and $\bigcup\limits_j \Omega_j = \Omega$. Then there exists a sequence of bounded functions $u_j \in L^1_{1,\omega}(\Omega) \cap C^\infty(\Omega)$, $j \ge 1$, such that*

$$\int\limits_{\Omega_j} |u - u_j|\omega\, dx < \frac{1}{j}, \quad \lim_{j\to\infty} \|u - u_j\|_{L^1_{1,\omega}(\Omega)} = 0. \tag{3}$$

2.5. $(1,\omega)$**-modulus and** $(1,\omega)$**-capacity.** Let Γ be a family of locally rectifiable curves in R^n. We denote by $\mathrm{adm}\,\Gamma$ the set of Borel functions $\rho : R^n \to [0; +\infty]$ satisfying the condition: for every $\gamma \in \Gamma$ we have $\int\limits_\gamma \rho\, ds \ge 1$. In the case $\Gamma = \emptyset$ we assume that $\mathrm{adm}\,\Gamma$ contains the function $\rho \equiv 0$. The $(1,\omega)$-modulus of Γ, denoted by $M_{1,\omega}(\Gamma)$, is defined as

$$M_{1,\omega}(\Gamma) = \inf \int\limits_{R^n} \rho\, \omega dx,$$

where the infimum is taken over all $\rho \in \mathrm{adm}\,\Gamma$. For the basic facts about the (p,ω)-modulus, $1 \le p < \infty$, see [13]. Now let F_0, F_1 be compact disjoint sets in R^n. Then a triple of sets (F_0, F_1, Ω) is called a condenser in Ω. Let $\Gamma(F_0, F_1, \Omega)$ be the family of all locally rectifiable curves connecting $F_0 \cap \bar\Omega$ and $F_1 \cap \bar\Omega$ in Ω. More precisely, if $\gamma \in \Gamma(F_0, F_1, \Omega)$ then there exists a representation $x(s) : I \to \Omega$ of curve γ in terms of arc length (see [13, Sec. 2.1]), where I is an open interval, $\overline{x(I)} \cap F_0$ and $\overline{x(I)} \cap F_1$ are both non-empty.

We write $M_{1,\omega}(F_0, F_1, \Omega)$ for the $(1,\omega)$-modulus of $\Gamma(F_0, F_1, \Omega)$. By definition, $M_{1,\omega}(F_0, F_1, \Omega) = M_{1,\omega}(F_0 \cap \bar\Omega, F_1 \cap \bar\Omega, \Omega)$ and $M_{1,\omega}(F_0, F_1, \Omega) = 0$ if at least $F_0 \cap \bar\Omega = \emptyset$ or $F_1 \cap \bar\Omega = \emptyset$. The number $M_{1,\omega}(F_0, F_1, \Omega)$ will also be called the $(1,\omega)$-modulus of condenser (F_0, F_1, Ω).

Now let's define $(1,\omega)$-capacity $C_{1,\omega}(F_0, F_1, \Omega)$ of the condenser (F_0, F_1, Ω). Suppose that $F_0 \cup F_1 \subset \bar\Omega$. Then we set $C_{1,\omega}(F_0, F_1, \Omega) = 0$ if, at least, $F_0 = \emptyset$ or $F_1 = \emptyset$. If F_0 and F_1 are non-empty sets then

$$C_{1,\omega}(F_0, F_1, \Omega) = \inf \int\limits_\Omega |\nabla u|\omega\, dx,$$

where the infimum is taken over all real-valued functions u such that $u|_\Omega$ satisfies locally the Lipschitz condition and $u = j$ in some neighborhood of F_j, $j = 0, 1$.

Denote the set of all admissible functions of this kind by $\mathrm{Adm}(F_0, F_1, \Omega)$. In general, we define $(1,\omega)$-capacity of a condenser (F_0, F_1, Ω) as $C_{1,\omega}(F_0, F_1, \Omega) = C_{1,\omega}(F_0 \cap \bar\Omega, F_1 \cap \bar\Omega, \Omega)$.

By Rademacher's theorem, the function $u \in \mathrm{Adm}(F_0, F_1, \Omega)$ is differentiable a.e. on Ω. Set for $x \in \Omega$

$$L(x,u) = \limsup_{h\to 0} \frac{|u(x+h)-u(x)|}{|h|}.$$

Then $L(x,u)$ is a Borel function on Ω and $|\nabla u(x)| = L(x,u)$ at differentiability points of u. If u is not differentiable at $x \in \Omega$, we set $|\nabla u(x)| = L(x,u)$ (see [17, Theorem 5.1]).

It follows from the Vitali-Carathéodory theorem (see [14, p.37, Theorem 2.24]) that given $f : R^n \to [0,+\infty]$, $f \in L_{1,\omega}$, there exists a lower semi-continuous function $g \geq f$ with $\|g\|_{L_{1,\omega}}$ arbitrarily closed to $\|f\|_{L_{1,\omega}}$. We shall apply below the following assertion.

Proposition 8 ([13, Sec. 2.2, p.19])**.**

$$M_{1,\omega}(\Gamma) = \inf_\rho \left\{ \int \rho\,\omega dx : \rho \textit{ is lower semi-continuous and } \rho \in \mathrm{adm}\,\Gamma \right\}.$$

2.6. **Removable sets.** Here we define three types of sets E in Ω (recall that E is a relatively closed subset of Ω) which will be removable singularities for $L^1_{1,\omega}(\Omega)$. Let here and further Π be any coordinate rectangle

$$\{x = (x_1, \dots, x_n) : a_i < x_i < b_i, i = 1, \dots, n\},$$

where $a_i, b_i \in R$. Denote the facets of this rectangle, parallel to the hyperplane $x_i = 0$, by $\sigma_{0i} \subset \{x : x_i = a_i\}$ and $\sigma_{1i} \subset \{x : x_i = b_i\}$, $i = 1, \dots, n$. If

$$C_{1,\omega}(\sigma_{0i}, \sigma_{1i}, \Pi \setminus E) = C_{1,\omega}(\sigma_{0i}, \sigma_{1i}, \Pi), \quad i = 1, \dots, n, \tag{4}$$

for every coordinate rectangle Π with $\bar{\Pi} \subset \Omega$, then E is called $NC_{1,\omega}$-set in Ω.

Now let $m_n(E) = 0$, and let $e \subset E$ be an arbitrary compact. Set $\mathcal{K}_j(e,\Omega) = \{\Pi : \mathrm{dist}(\sigma_{0j} \cup \sigma_{1j}, e) > 0, \bar{\Pi} \subset \Omega\}$, $j = 1, \dots, n$. In addition, for $\Pi \in \mathcal{K}_j(e,\Omega)$ put $\Pi_{j,\delta} = \{x = (x_1, \dots, x_n) \in R^n : a_j < x_j < b_j, a_i + \delta < x_i < b_i - \delta, i \neq j\}$, where $0 < \delta < \min\limits_{1\leq i\leq n} \dfrac{b_i - a_i}{2}$. If, regardless of choice of compact set $e \subset E$ the estimate

$$C_1 M_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi \setminus e) \geq \lim_{\delta\to 0} M_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi_{j\delta}) \tag{5}$$

is valid for every coordinate rectangle $\Pi \in \mathcal{K}_j(e,\Omega)$ and all $j = 1, \dots, n$, then E is called a $NED_{1,\omega}$-set in Ω. In the definition C_1 is a constant from Proposition 2. Observe that in the case $\omega \equiv 1$, inequality (5) is equivalent to equality (see Corollary 1 from Sec. 4)

$$M_1(\sigma_{0j}, \sigma_{1j}, \Pi \setminus e) = M_1(\sigma_{0j}, \sigma_{1j}, \Pi) \tag{6}$$

for every compact $e \subset E$ and coordinate rectangle $\Pi \in \mathcal{K}_j(e,\Omega)$, $j = 1, \dots, n$.

In order to define another type of removable sets for $L^1_{1,\omega}(\Omega)$, we introduce the following concepts.

Let X_i be the family of straight lines in R^n, parallel to the coordinate x_i-axis, $i = 1, \dots, n$. Index every line $l \in X_i$ by the point $a \in l \cap H_i$, where $H_i = \{x = (x_1, \dots, x_n) \in R^n : x_i = 0\}$. Then say that some property holds for m_{n-1}-almost all lines in X_i (or segments on these lines), whenever the corresponding set of points a on H_i for lines (or their segments) in X_i violating this property, is of m_{n-1}-measure zero.

We shall say that Ω does not partitioned locally by E if $B \setminus E$ is a domain for all balls $B \Subset \Omega$. It is easily to see that in this definition the balls $B \Subset \Omega$ can be replaced with coordinate rectangles $\Pi \Subset \Omega$.

Definition 2. *Take a compact set $e \subset R^n$ with $m_n(e) = 0$ so that R^n does not partitioned locally by e. We say that e satisfies the $(1,\omega)$-girth condition respect to X_i if every Borel function $\rho : R^n \setminus e \to [0, +\infty)$, $\rho \in L_{1,\omega}(R^n \setminus e)$, locally bounded on $R^n \setminus e$, satisfies the following ε-girth condition for given $\varepsilon > 0$:*

Let Π be some coordinate rectangle in R^n and $e \subset \Pi$. Let $\Gamma_i(\Pi) = \{l \cap \bar{\Pi} : l \in X_i, l \cap e \neq \emptyset\}$. Then for m_{n-1}-almost all segments $\tau \in \Gamma_i(\Pi)$ we can indicate a finite sequence of mutually disjoint intervals $(c_k, d_k) \subset \tau$ and rectifiable curves $\gamma_k \subset \Pi \setminus e$, joining c_k with d_k, $k = 1, \dots, k_1$, such that

$$\sum_{k=1}^{k_1} \int_{\gamma_k} \rho\, ds < \varepsilon, \quad \sum_{k=1}^{k_1} \int_{\gamma_k} ds < \varepsilon, \quad \bigcup_{k=1}^{k_1} (c_k, d_k) \supset \tau \cap e.$$

Refer to the last requirement as the ε-girth condition on τ for the ρ.

Remark. *Let's denote the family of all such segments τ for the function ρ by $\Gamma_i(e, \Pi, \varepsilon)$. Take $\varepsilon = \frac{1}{k}$, $k \in \mathbb{N}$ and set $\Gamma_{0i}(e, \Pi) = \bigcap_k \Gamma_i(e, \Pi, \frac{1}{k})$, $F_k = \{l \cap H_i : l \in X_i$ and there exists a segment $\tau \in \Gamma_i(e, \Pi, \frac{1}{k}), \tau \subset l\}$, $F_0 = \{l \cap H_i : l \in X_i$ and there exists a segment $\tau \in \Gamma_{0i}(e, \Pi), \tau \subset l\}$. It is easily to see that $F_0 \subset F_{k+1} \subset F_k$ and $m_{n-1}(F_k) = m_{n-1}(F_0)$ for all $k \geq 1$ and ρ satisfies the ε-girth condition on all $\tau \in \Gamma_{0i}(e, \Pi)$ for any $\varepsilon > 0$.*

Moreover, the function ρ satisfies the ε-girth condition on any segment $[a, b] \subset \tau$, $\tau \in \Gamma_{0i}(e, \Pi)$, if a and $b \notin e$, for arbitrary $\varepsilon > 0$.

Definition 3. *If e satisfies the $(1,\omega)$-girth condition with respect to X_i for all $i = 1, \dots, n$, then we say that e satisfies the $(1,\omega)$-girth condition in R^n.*

Definition 4. *We say that E, $m_n(E) = 0$ and Ω does not partitioned locally by E, satisfies the $(1,\omega)$-girth condition in Ω, if any compact set $e \subset E$ satisfies the $(1,\omega)$-girth condition in R^n.*

The family of all such sets $E \subset \Omega$ is denoted by $\mathcal{G}_{1,\omega}(\Omega)$. Further $\omega \equiv 1$ we shall often omit in notations of spaces, norms, families. For example, in the case $\omega \equiv 1$ we write $\mathcal{G}_{1,\omega}$ as $\mathcal{G}_1$.

3. About one class of admissible metrics

In this section F_0, F_1 will be compact non-empty sets in $\bar{\Omega}$. Any function $\rho \in \operatorname{adm}\Gamma(F_0, F_1, \Omega)$ will also be called an admissible metric for $\Gamma(F_0, F_1, \Omega)$. If $\Gamma(F_0, F_1, \Omega) = \emptyset$ then, by definition, the admissible metric for $\Gamma(F_0, F_1, \Omega)$ is an arbitrary Borel function $\rho : R^n \to [0, +\infty]$. We let $d : R^n \to [0, +\infty]$ be the function defined by $d(x) = \operatorname{dist}(x, (R^n \setminus \Omega) \cup F_0 \cup F_1)$. It is well-known that $d(x)$ satisfies the Lipschitz condition with the Lipschitz constant $Lip(d) \leq 1$. In the case where $d(x)$ is differentiable at $x \in \Omega \setminus (F_0 \cup F_1)$, it follows that $|\nabla d| = 1$ (see [7, Sec. 3.2.34]). Further we use the following technical results.

Lemma 1. *$M_{1,\omega}(F_0, F_1, \Omega)$ and $C_{1,\omega}(F_0, F_1, \Omega)$ are finite.*

Proof. Let U_0 and U_1 be open bounded sets in R^n such that $F_0 \subset U_0$, $F_1 \subset U_1$, $\overline{U_0} \cap \overline{U_1} = \emptyset$. In addition, take the ball $B_0 = B(0, r)$ so that $\overline{U_0} \cup \overline{U_1} \subset B_0$.

Set $\varphi \in C_0^\infty(B_0)$, $0 \le \varphi \le 1$, $\varphi = 0$ on U_0 and $\varphi = 1$ on $\overline{U_1}$ (see [8, Chapter 1, Theorem 2.6]). Because of choice of φ and Proposition 1 it follows that $\varphi \in \mathrm{Adm}(F_0, F_1, R^n) \cap \mathrm{Adm}(F_0, F_1, \Omega)$, $|\nabla\varphi| \in \mathrm{adm}(F_0, F_1, R^n) \cap \mathrm{adm}(F_0, F_1, \Omega)$ and $\int\limits_{R^n} |\nabla\varphi|\omega\, dx < \infty$. This implies $M_{1,\omega}(F_0, F_1, \Omega) < \infty$, $C_{1,\omega}(F_0, F_1, \Omega) < \infty$. The Lemma is proved. □

Lemma 2. *For every $\varepsilon > 0$, there is a function $\rho \in \mathrm{adm}(F_0, F_1, \Omega) \cap L_{1,\omega}(\Omega)$, for which the following conditions are realized:*

(1) *ρ is lower semi-continuous on R^n;*
(2) *ρ is continuous on $\Omega \setminus (F_0 \cup F_1)$;*
(3) *ρ is a positive function on R^n such that for any compact set $K \subset R^n$ $\inf\limits_K \rho > 0$ and*

$$M_{1,\omega}(F_0, F_1, \Omega) \le \int_{R^n} \rho\omega\, dx \le C_1 M_{1,\omega}(F_0, F_1, \Omega) + \varepsilon, \tag{7}$$

where the constant C_1 does not depend on ε. In the case $\omega \equiv 1$, put in (7) *$C_1 = 1$.*

Proof. For $\varepsilon > 0$, by Proposition 8 and Lemma 1, let ρ_1 be some admissible metric for $\Gamma(F_0, F_1, \Omega)$, $\rho_1 \in L_{1,\omega}(R^n)$ and ρ_1 be a lower semi-continuous on R^n with

$$M_{1,\omega}(F_0, F_1, \Omega) \le \int\limits_{R^n} \rho_1\omega\, dx < M_{1,\omega}(F_0, F_1, \Omega) + \frac{\varepsilon}{3C_1}.$$

Here C_1 is the constant from Proposition 2.

Define ρ_2 as

$$\rho_2(x) = T_k\rho_1(x) = \frac{1}{|B(0,1)|} \int\limits_{B(0,1)} \rho_1\left(x + \frac{d(x)}{2k}y\right) dy, \tag{8}$$

where T_k is the averaging operator used in [3] and studied in details in [9, Lemma 4.3]. Due to the known properties of the operator the function ρ_2 is lower semi-continuous on R^n and continuous on $\Omega \setminus (F_0 \cup F_1)$. Here note that to prove these properties, only local integrability of ρ_1 is required in addition.

By integration (8), we get

$$\int\limits_{R^n} \rho_2(x)\omega dx = \int\limits_{R^n} \left[\frac{1}{|B(0,1)|} \int\limits_{B(0,1)} \rho_1\left(x + \frac{d(x)}{2k}y\right) dy\right] \omega(x)\, dx.$$

Interchanging the order of integration gives

$$\|\rho_2\|_{L_{1,\omega}} = \frac{1}{|B(0,1)} \int\limits_{B(0,1)} \int\limits_{R^n} \rho_1\left(x + \frac{d(x)}{2k}y\right) \omega(x)\, dy. \tag{9}$$

Define for $y \in B(0,1)$, $\theta_{y,k} : R^n \to R^n$ by $z = \theta_{y,k}(x) = x + \dfrac{d(x)}{2k}y$. By construction, for all $x, x' \in R^n$ (see [9, Theorem 2.1])

$$\left(1 - \frac{1}{2k}\right)|x - x'| \le |\theta_{y,k}(x) - \theta_{y,k}(x')| \le \left(1 + \frac{1}{2k}\right)|x - x'|.$$

Thus, the mapping $\theta_{y,k}$ is a Lipschitz homeomorphism and $\theta_{y,k}(\Omega \setminus (F_0 \cup F_1)) = \Omega \setminus (F_0 \cup F_1)$. In addition, Jacobian of the mapping $\theta_{y,k}$ is equal $1 + \dfrac{y}{2k}\nabla d(x)$ a.e. on R^n. Hence [9, Lemma 4.3], by the changing of variables formula with $z = \theta_{y,k}$ as the mapping function, we obtain in (9)

$$\|\rho_2\|_{L_{1,\omega}} \le \frac{1}{|B(0,1)|(1-\frac{1}{2k})} \int\limits_{B(0,1)} \int\limits_{R^n} \rho_1(z)\omega(x(z))\, dzdy.$$

Repeated interchanging of order of integration gives

$$\|\rho_2\|_{L_{1,\omega}} \le \frac{1}{(1-\frac{1}{2k})} \int\limits_{R^n} \left(\frac{1}{|B(0,1)|} \int\limits_{B(0,1)} w(x(z))dy \right) \rho_1(z)\, dz.$$

By Proposition 2 and $x = z - \frac{1}{2k} d(x(z))y$, we deduce

$$\frac{1}{|B(0,1)|} \int\limits_{B(0,1)} \omega\left(z - \frac{1}{2k} d(x(z))y\right) dy = \frac{1}{\left|B\left(z, \frac{d(x(z))}{2k}\right)\right|} \int\limits_{B\left(z,\frac{d(x(z))}{2k}\right)} \omega(y)\, dy \le$$

$$M\omega(z) \le C_1\omega(z) \tag{10}$$

a.e. on R^n. Hence

$$\|\rho_2\|_{L_{1,\omega}} \le \frac{C_1}{\left(1-\frac{1}{2k}\right)} \int\limits_{R^n} \rho_1(z)\omega(z)\, dz = \frac{C_1}{\left(1-\frac{1}{2k}\right)} \|\rho_1\|_{L_{1,\omega}}.$$

Here note that for $\omega \equiv 1$ we have the equality $M\omega(x) = 1$ on R^n. In other words, we can assume $C_1 = 1$ for $\omega \equiv 1$ in (10).

Moreover, using standard arguments (see proofs [9, Lemma 4.3], [4, Theorem 2.1]), we have $g_k = \left(1 + \frac{1}{2k}\right) \rho_2 \in \operatorname{adm}(F_0, F_1, \Omega)$. This implies

$$M_{1,\omega}(F_0, F_1, \Omega) \le \int\limits_{R^n} g_k \omega\, dx \le$$

$$\frac{C_1\left(1+\frac{1}{2k}\right)}{\left(1-\frac{1}{2k}\right)} \int\limits_{R^n} \rho_1\omega\, dx \le C_1 \frac{\left(1+\frac{1}{2k}\right)}{\left(1-\frac{1}{2k}\right)} M_{1,\omega}(F_0, F_1, \Omega) + \frac{\varepsilon}{3}.$$

Let $g(x)$ be a positive continuous function on R^n with $\int\limits_{R^n} g(x)\omega(x)\, dx < \dfrac{\varepsilon}{3}$. Its construction is similar to the construction of a positive function $\alpha(x) \in C^\infty$ in [13, Lemma 2.4.1]. Then it is clear that for large $k \in \mathbb{N}$ the function $\rho = g_k + g$ satisfies the conditions of the Lemma. Thus, the Lemma is proved. □

4. Some properties of $NED_{1,\omega}$-, $NC_{1,\omega}$-sets

In this section, we will establish a number of properties of sets that will be removable for $L^1_{1,\omega}(\Omega)$ and use these properties in proving the main results of our paper.

Property 1. *For any coordinate rectangle* $\Pi = \{x \in R^n : a_i < x_i < b_i, i = 1, \dots, n\}$ *we have (see* (4)*)*

$$M_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi) \operatorname*{ess\,sup}_{\Pi} \frac{1}{\omega} \ge m_{n-1}(\sigma_{0j}), j = 1, \dots, n. \tag{11}$$

Proof. By Lemma 1, it follows that $M_{1,\omega}(\sigma_{0j},\sigma_{1j},\Pi)<\infty$ for all $j=1,\dots,n$. Fix $j\in\{1,\dots,n\}$. Then for each $\varepsilon>0$ we can find $\rho\in \operatorname{adm}(\sigma_{0j},\sigma_{1j},\Pi)$ such that

$$M_{1,\omega}(\sigma_{0j},\sigma_{1j},\Pi)\le \int\limits_{R^n}\rho\omega\,dx<M_{1,\omega}(\sigma_{0j},\sigma_{1j},\Pi)+\varepsilon.$$

In addition, $\int\limits_{a_j}^{b_j}\rho(x)\,dx_j\ge 1$ for all x', where

$x'\in\Pi'=\{x'=(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)\in R^{n-1}: a_i<x_i<b_i, i\ne j\}$. This implies

$$\int\limits_{\Pi'}\left(\int\limits_{a_j}^{b_j}\rho\omega\frac{1}{\omega}\,dx_j\right)dx'\ge m_{n-1}(\sigma_{0j}).$$

Hence, $\int\limits_{\Pi}\rho\omega\,dx\cdot \operatorname*{ess\,sup}\limits_{\Pi}\frac{1}{\omega}\ge m_{n-1}(\sigma_{0j})$. Thus, the Property is proved. □

Set $\omega\equiv 1$ in (11) and let $\rho_0=\dfrac{1}{b_j-a_j}$ on Π and $\rho_0=0$ on $R^n\setminus\Pi$. By choice, $\operatorname*{ess\,sup}\limits_{\Pi}\dfrac{1}{\omega}=1$, $\rho_0\in\operatorname{adm}(\sigma_{0j},\sigma_{1j},\Pi)$ and $\int\limits_{R^n}\rho_0\,dx=m_{n-1}(\sigma_{0j})$. It follows from (11) another assertion.

Corollary 1. *In the case of* $\omega\equiv 1$ *we see*

$$M_1(\sigma_{0j},\sigma_{1j},\Pi)=m_{n-1}(\sigma_{0j}),\quad j=1,\dots,n,$$

and, by (5)*,*

$$\lim_{\delta\to+0}M_1(\sigma_{0j},\sigma_{1j},\Pi_{j,\delta})=\lim_{\delta\to+0}M_1(\sigma_{0j}\cap\overline{\Pi_{j,\delta}},\sigma_{1j}\cap\overline{\Pi_{j,\delta}},\Pi_{j,\delta})=$$

$$\lim_{\delta\to+0}m_{n-1}(\sigma_{0j}\cap\overline{\Pi_{j,\delta}})=m_{n-1}(\sigma_{0j})=M_1(\sigma_{0j},\sigma_{1j},\Pi).$$

Moreover, in view of Proposition 2, we have also the estimate

$$M_{1,\omega}(\sigma_{0j},\sigma_{1j},\Pi)\ge C_1\cdot\frac{1}{\sup\limits_{\Pi}(1+|x|)^n}m_{n-1}(\sigma_{0j})\tag{12}$$

for all $j=1,\dots,n$.

Property 2.

$$C_{1,\omega}(\sigma_{0j},\sigma_{1j},\Pi)\ge M_{1,\omega}(\sigma_{0j},\sigma_{1j},\Pi).$$

Proof. For given $\varepsilon>0$ we can find $u_\varepsilon\in\operatorname{Adm}(\sigma_{0j},\sigma_{1j},\Pi)$ such that

$$C_{1,\omega}(\sigma_{0j},\sigma_{1j},\Pi)\le\int\limits_{\Pi}|\nabla u_\varepsilon|\omega\,dx<C_{1,\omega}(\sigma_{0j},\sigma_{1j},\Pi)+\varepsilon.$$

In addition, we see

$$1\le\left|\int\limits_{\gamma}\frac{\partial u_\varepsilon}{\partial s}ds\right|\le\int\limits_{\gamma}|\nabla u_\varepsilon|\,ds$$

for all $\gamma \in \Gamma(\sigma_{0j}, \sigma_{1j}, \Pi)$. Hence $\rho \in \mathrm{adm}(\sigma_{0j}, \sigma_{1j}, \Pi)$ if $\rho = |\nabla u_\varepsilon|$ on Π and $\rho = 0$ on $R^n \setminus \Pi$. This implies

$$M_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi) \leq \int\limits_{\Pi} \rho\omega \, dx \leq C_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi) + \varepsilon.$$

The arbitrariness of ε yields the inequality required in the Property. □

Property 3. *If E is an $NC_{1,\omega}$-set in Ω then $\sigma_{0j} \cap \overline{\Pi \setminus E} = \sigma_{0j}$, $\sigma_{1j} \cap \overline{\Pi \setminus E} = \sigma_{1j}$ for all $j = 1, \dots, n$.*

Proof. Let $j = 1$ and let, for example, some $x_0 \in \sigma_{01}$ not to be a boundary point of $\Pi \setminus E$. Hence there exists a ball $B(x_0, r)$ such that $(B(x_0, r) \cap \sigma_{01}) \cap \overline{\Pi \setminus E} = \emptyset$. Then we will define another coordinate rectangle $\Pi_1 \subset \Pi$, whose opposite facets σ_{01}^1, σ_{11}^1, orthogonal to the x_1-axis, lie in $B(x_0, r) \cap \sigma_{01}$, σ_{11}, respectively.

By definition, $\sigma_{01}^1 \cap \overline{\Pi_1 \setminus E} = \emptyset$ and therefore
$C_{1,\omega}(\sigma_{01}^1, \sigma_{11}^1, \Pi_1 \setminus E) = C_{1,\omega}(\sigma_{01}^1, \sigma_{11}^1, \Pi_1) = 0$. This, by Property 2, contradicts the inequality (12). Thus, $\sigma_{01} \cap \overline{\Pi \setminus E} = \sigma_{01}$, $\sigma_{11} \cap \overline{\Pi \setminus E} = \sigma_{11}$.

Similarly, we establish the required equalities for $j = 2, \dots, n$. This completes the proof of the Property. □

Property 4. *If E is an $NC_{1,\omega}$-set in Ω and $u_j = \dfrac{x_j - a_j}{b_j - a_j}$ on $\Pi \Subset \Omega$, then*

$$(13) \qquad \int\limits_{\Pi \setminus E} |\nabla u_j| \omega \, dx \geq C_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi \setminus E), \quad \int\limits_{\Pi} |\nabla u_j| \omega \, dx \geq C_{1,\omega}(\sigma_{0j}, \sigma_{1j}, \Pi)$$

for all $j = 1, \dots, n$.

Proof. As in the proof of Property 3, we derive the inequalities (13) only for $j = 1$. Let $\varepsilon \in \left(0, \dfrac{b_1 - a_1}{2}\right)$ and set

$$u_{1,\varepsilon}(x) = \begin{cases} \dfrac{x_1 - a_1 - \varepsilon}{b_1 - a_1 - 2\varepsilon}, & x_1 \in [a_1 + \varepsilon, b_1 - \varepsilon], \\ 0, & x_1 < a_1 + \varepsilon, \\ 1, & x_1 > b_1 - \varepsilon \end{cases}$$

for $x = (x_1, \dots, x_n) \in R^n$. Obviously, $u_{1,\varepsilon} \in \mathrm{Adm}(\sigma_{01}, \sigma_{11}, \Pi) \cap \mathrm{Adm}(\sigma_{01}, \sigma_{11}, \Pi \setminus E)$. This implies

$$C_{1,\omega}(\sigma_{01}, \sigma_{11}, \Pi) \leq \int\limits_{\Pi} |\nabla u_{1,\varepsilon}| \omega \, dx = \int\limits_{\Pi} |\nabla u_1| \omega \, dx + o(1),$$

$$C_{1,\omega}(\sigma_{01}, \sigma_{11}, \Pi \setminus E) \leq \int\limits_{\Pi \setminus E} |\nabla u_{1,\varepsilon}| \omega \, dx = \int\limits_{\Pi \setminus E} |\nabla u_1| \omega \, dx + o(1),$$

where $o(1) \to 0$ as $\varepsilon \to 0$. Hence, taking $\varepsilon \to 0$, we derive the required inequalities in the Property for $j = 1$. □

Corollary 2. *It is clear that for u_1 from the proof of Property 4 we have $|\nabla u_1| = \dfrac{\rho_1}{b_1 - a_1}$, where $\rho_1 = 1$ on $\Pi \setminus E$ and $\rho_1 = 0$ on $\Pi \cap E$. Therefore, if E is an*

$NC_{1,\omega}$-set in Ω then

$$\int\limits_{\Pi\setminus E} \frac{\rho_1}{b_1-a_1}\omega\,dx \geq C_{1,\omega}(\sigma_{01},\sigma_{11},\Pi\setminus E) \tag{14}$$

for all $\Pi \Subset \Omega$.

Property 5. *If E is an $NC_{1,\omega}$-set in Ω then $m_n(E)=0$.*

Proof. Suppose that $m_n(E)>0$. Consider the coordinate rectangle $\Pi=\{x=(x_1,\dots,x_n)\in R^n : a_i<x_i<b_i, i=1,\dots,n\}$ and $m_n(E\cap\Pi)>0$. Let ρ_1 be a function from Corollary 2. Given two rationals $r_1<r_2$ in (a_1,b_1), put $\Pi(r_1,r_2)=\Pi\cap\{x\in R^n : r_1<x_1<r_2\}$, $\sigma_0=\sigma_0(r_1,r_2)=\bar\Pi\cap\{x : x_1=r_1\}$ and $\sigma_1=\sigma_1(r_1,r_2)=\bar\Pi\cap\{x : x_1=r_2\}$. It is clear that $m_{n-1}(\Pi')=m_{n-1}(\sigma_0)=m_{n-1}(\sigma_1)$, where $\Pi'=\{x'=(x_2,\dots,x_n) : a_i<x_i<b_i, i=2,\dots,n\}$. Then from Property 2, (12) and (14) we deduce easily

$$\int\limits_{\Pi'} dx' \int\limits_{r_1}^{r_2} \frac{\rho_1\omega}{r_2-r_1}dx_1 \geq C_1\frac{1}{\sup\limits_{\Pi}(1+|x|)^n}\cdot m_{n-1}(\Pi') \geq C_2 m_{n-1}(\Pi'), \tag{15}$$

where $C_2=C_1\dfrac{1}{\sup\limits_{\Pi}(1+|x|)^n}$.

Put $\Phi(x',r_1,r_2)=\int_{r_1}^{r_2}\frac{\rho_1\omega\,dx_1}{r_2-r_1}$. By Fubini's theorem (see [2]), there exists a set $\tilde\Pi'(r_1,r_2)$ in Π' with $m_{n-1}(\tilde\Pi'(r_1,r_2))=m_{n-1}(\Pi')$, all of whose points are Lebesgue points for $\Phi(x',r_1,r_2)$ (differentiability points of the integral of this function over Π'). Put $\tilde\Pi'=\bigcap\limits_{r_1<r_2}\tilde\Pi'(r_1,r_2)$. It is obvious that $m_{n-1}(\tilde\Pi')=m_{n-1}(\Pi')$. Choose $x'\in\tilde\Pi'$ so that $\mathcal{H}^1(l(x')\cap E)>0$ and $\mathcal{H}^1$-almost all points on the segment $l(x')=\{(x_1,x') : a_1\leq x_1\leq b_1\}$ are Lebesgue points for the function $\rho_1\omega$ and $\omega<\infty$.

From (15) we obtain

$$\frac{1}{r_2-r_1}\int\limits_{r_1}^{r_2}\rho_1\omega\,dx_1 \geq C_2 \tag{16}$$

for m_{n-1}-almost all $x'\in\Pi'$.

Since the choice of r_1 and r_2 in (a_1,b_1) is arbitrary, the absolute continuity of the integral enables us to consider r_1 and r_2 in (16) as real from (a_1,b_1). By construction, there exists $x_1^0\in(a_1,b_1)$ and $(x_1^0,x')\in E\cap l(x')$ such that (x_1^0,x') is a Lebesgue point for $\rho_1\omega$ with fixed x'.

Putting in (16) $r_1=x_1^0-\delta$ and $r_2=x_1^0+\delta$ with $\delta>0$, and letting $\delta\to0$, we get a contradiction to $0=\rho(x_1^0,x')\omega(x_1^0,x')\geq C_2$. Consequently, $m_n(E)=0$. Thus the Property is proved. □

Property 6. *If E is an $NC_{1,\omega}$-set in Ω then $\Pi\setminus E$ is a domain for arbitrary coordinate rectangle $\Pi\Subset\Omega$.*

Proof. Suppose that there exists a coordinate rectangle $\Pi\Subset\Omega$, which is partitioned by E into two non-empty open sets D_0 and D_1 ($\Pi\setminus E=D_0\cup D_1$, $D_0\cap D_1=\emptyset$). Take two points $d_0\in D_0$, $d_1\in D_1$ and let $B_0=B(d_0,r_0)$, $B_1=B(d_1,r_1)$ be such that $\overline{B_0}\subset D_0$, $\overline{B_1}\subset D_1$.

Let $L_1 = L_1(a_1, \dots, a_k)$ be a simple polyline in Π composed of straight segments $[a_i, a_{i+1}]$, $i = 1, \dots, k-1$, where each segment $[a_i, a_{i+1}]$ is parallel to some coordinate axis and $a_1 = d_0$, $a_k = d_1$. Below, any polyline whose links are parallel to the coordinate axes, will be called a coordinate polyline. Then it follows that there exists some segment $[a, b] \subset \Pi$, parallel to one of the links of L_1 and $a \in D_0$, $b \in D_1$. Indeed, in the ball $B(a_2, r_2)$ with $0 < r_2 < \min(r_0, r_1, \operatorname{dist}(L_1, \partial\Pi))$, by virtue of $m_n(E) = 0$, there is a point $a_2^1 \notin E$. Consider a parallel translation $T_{h_1} : x \to x + h_1$ on R^n, where $h_1 = a_2^1 - a_2$. By construction, $T_{h_1}(L_1)$ is a coordinate polyline $L_2 = L_2(a_1^1, \dots, a_k^1)$ with $a_1^1 = T_{h_1}(a_1) \in D_0$, $a_2^1 = T_{h_1}(a_2) \notin E$, ..., $a_k^1 = T_{h_1}(a_k) \in B(a_k, r_1) \subset D_1$. If $a_2^1 \in D_1$ then we choose $[a, b] = [a_1^1, a_2^1]$.

Suppose that $a_2^1 \in D_0$. Then in $B(a_3^1, r_3)$ with $0 < r_3 < \min(\operatorname{dist}(a_2^1, \partial B(a_2, r_2) \cup E), \operatorname{dist}(L_2, \Pi), \operatorname{dist}(a_k^1, \partial B(a_k, r_1)))$, as above, there is a point $a_3^2 \notin E$. Put $L_3 = L_3(a_2^1, \dots, a_k^1)$ and let $T_{h_2} : x \to x + h_2$, where $h_2 = a_3^2 - a_3^1$. Hence $T_{h_2}(L_3)$ is a coordinate polyline $L_4 = L_4(a_2^2, \dots, a_k^2)$ with $a_2^2 \in D_0$, $a_3^2 \notin E$, ..., $a_k^2 = T_{h_2}(a_k^1) \in B(d_1, r_1) \subset D_1$.

If $a_3^2 \in D_1$ then we choose $[a, b] = [a_2^2, a_3^2]$. If $a_3^2 \in D_0$ then we continue this process. Taking into account that, by the choice, $T_{h_1}(a_k^1) \in D_1$, $T_{h_2}(a_k^2) \in D_1$ and so far, at most after k steps we get the required segment $[a, b]$. Without loss of generality, we assume that the segment $[a, b]$ os parallel to the x_1-axis.

Let's choose a coordinate rectangle $\Pi_1 \subset \Pi$ so that its opposite facets σ_{01}^1, σ_{11}^1 lie in D_0 and D_1, respectively, and are orthogonal to the x_1-axis. In addition $a \in \sigma_{01}^1$ and $b \in \sigma_{11}^1$.

Set $u = 0$ on D_0 and $u = 1$ on D_1. Then $u \in \operatorname{Adm}(\sigma_{01}^1, \sigma_{11}^1, \Pi \setminus E)$ and $\int\limits_{\Pi\setminus E} |\nabla u| \omega \, dx = 0$. This implies $C_{1,\omega}(\sigma_{01}^1, \sigma_{11}^1, \Pi \setminus E) = 0$. On the other hand, by (4),(12) and Property 2, we see that $C_{1,\omega}(\sigma_{01}^1, \sigma_{11}^1, \Pi \setminus E) > 0$. The resulting contradiction completes the proof of this Property. □

Note that in the proof of the Property 6, only the condition $m_n(E) = 0$ was used to construct the rectangle Π_1. And $\Pi_1 \in \mathcal{K}_1(e, \Omega)$, where $e = \overline{\Pi_1} \cap E$ and $\mathcal{K}_1(e, \Omega)$ is from definition of $NED_{1,\omega}$-sets in Ω.

It is clear that $\Gamma(\sigma_{01}^1, \sigma_{11}^1, \Pi_1 \setminus e) = \emptyset$ and hence $M_{1,\omega}(\sigma_{01}^1, \sigma_{11}^1, \Pi_1 \setminus e) = 0$. On the other hand, write $\Pi_1 = \{x = (x_1, \dots, x_n) : a_i^1 < x_i < b_i^1, i = 1, \dots, n\}$ and set $\Pi_{1,\delta} = \{x = (x_1, \dots, x_n) : a_1^1 < x_1 < b_1^1, a_i^1 + \delta < x_i < b_i^1 - \delta, i = 2, \dots, n\}$, where $0 < \delta < \min\limits_{1 \le i \le n} \dfrac{b_i - a_i}{2}$. Then, applying (5) and (12), we have

$$C_1 M_{1,\omega}(\sigma_{01}^1, \sigma_{11}^1, \Pi_1 \setminus e) \ge \lim_{\delta \to 0} M_{1,\omega}(\sigma_{01}^1 \cap \overline{\Pi_{1,\delta}}, \sigma_{11}^1 \cap \overline{\Pi_{1,\delta}}, \Pi_{1,\delta}) \ge$$

$$\frac{C_1}{\max\limits_{\Pi_1}(1 + |x|)^n} m_{n-1}(\sigma_{01}^1) > 0,$$

that contradicts the condition $M_{1,\omega}(\sigma_{01}^1, \sigma_{11}^1, \Pi_1 \setminus e) = 0$. Thus, we come to another assertion

Property 7. *If E is an $NED_{1,\omega}$-set in Ω then $\Pi \setminus E$ is a domain for all $\Pi \Subset \Omega$.*

Property 8. *If E is an $NC_{1,\omega}$-set in Ω then E is an $NED_{1,\omega}$-set in Ω.*

Proof. Take some compact $e \subset E$, coordinate rectangle $\Pi = \{x = (x_1, \dots, x_n) : a_i < x_i < b_i, i = 1, \dots, n\}$ and make sure that inequalities (5) are true if $j = 1$

and $\Pi \in \mathcal{K}_1(e,\Omega)$. For the remaining j and other coordinate rectangles $\in \mathcal{K}_j(e,\Omega)$, $j=2,\dots,n$ and e the proof of this fact is performed in the same way.

Let's introduce a few notations. Let $\sigma_0=\sigma_{01}$, $\sigma_1=\sigma_{11}$ be opposite facets of Π such as in (4). Let $\Pi_{1,\delta}$ be such as in (5) and set $\sigma_0\cap\overline{\Pi_{1,\delta}}=\sigma_0(\delta)$, $\sigma_1\cap\overline{\Pi_{1,\delta}}=\sigma_1(\delta)$. Fix $0<\delta<\min\limits_i \dfrac{b_i-a_i}{2}$.

Since $\Pi \in \mathcal{K}_1(e,\Omega)$, then $\eta=\operatorname{dist}(\sigma_0\cup\sigma_1,e)>0$. Hence there exist two sequences $\{\Omega_{0,k}\}$, $\{\Omega_{1,k}\}$ of coordinate rectangles $\Omega_{0,k},\Omega_{1,k}\Subset\Omega$ such that $\operatorname{dist}(\Omega_{0,k}\cup\Omega_{1,k},e)>0$ for all $k=1,\dots$; $\sigma_0\subset\Omega_{0,k+1}\Subset\Omega_{0,k}$, $\bigcap\limits_k\Omega_{0,k}=\sigma_0$; $\sigma_1\subset\Omega_{1,k+1}\Subset\Omega_{1,k}$, $\bigcap\limits_k\Omega_{1,k}=\sigma_1$ and $\overline{\Omega_{0,1}}\cap\overline{\Omega_{1,1}}=\emptyset$.

Set $\sigma_0^k=\partial\Omega_{0,k}\cap\overline{\Pi_{1,\delta}}$, $\sigma_1^k=\partial\Omega_{1,k}\cap\overline{\Pi_{1,\delta}}$.

Put in Lemma 2 $\Omega=\Pi\setminus e$, $F_0=\sigma_0$, $F_1=\sigma_1$. Then for a given $\varepsilon>0$ there is a metric $\rho\in\operatorname{adm}\Gamma(\sigma_0,\sigma_1,\Pi\setminus e)\cap L_{1,\omega}$ that satisfies the conditions of Lemma 2. In particular, ρ is a continuous function on $\Pi\setminus e$,

$$\inf_K \rho>0 \text{ for all compact set } K\subset R^n \tag{17}$$

and

$$M_{1,\omega}(\sigma_0,\sigma_1,\Pi\setminus e)\le\int\limits_{R^n}\rho\omega\,dx<C_1M_{1,\omega}(\sigma_0,\sigma_1,\Pi\setminus e)+\varepsilon. \tag{18}$$

In addition, the metric ρ is continuous on $\partial\Pi_{1,\delta}\setminus(e\cup\sigma_0^k\cup\sigma_1^k)$ for all $k=1,2,\dots$ and $\int\limits_\gamma\rho\,ds\ge1$ for any locally rectifiable curve γ, connecting $\sigma_0(\delta)$ and $\sigma_1(\delta)$ in $\overline{\Pi_{1,\delta}}\setminus(\sigma_0(\delta)\cup\sigma_1(\delta)\cup e)$. Finally, let $\Pi_{1,\delta}^k=\Pi_{1,\delta}\setminus\overline{\Omega_{0,k}\cup\Omega_{1,k}}$, $\Gamma_k=\Gamma(\sigma_0^k,\sigma_1^k,\Pi_{1,\delta}^k\setminus e)$.

Take $\beta\in(0,1)$. We shall show that there is $k_0=k_0(\beta)\in\mathbb{N}$ for which

$$\int\limits_\gamma\rho\,ds\ge1-\beta \tag{19}$$

for all $\gamma\in\Gamma_k$ if $k\ge k_0$. Indeed, admitting the opposite, we obtain a sequence $\{\gamma_k\}_{k=1}^\infty$ such that $\int\limits_{\gamma_k}\rho\,ds<1-\beta$ for all $k=1,2,\dots$.

By (17), it is inferred that γ_k is a rectifiable curve and for any k and j, $1\le j<k$, γ_k contains points in σ_0^j and σ_1^j.

Let $x_0(j,k)$ (resp. $x_1(j,k)$) be the first (resp. last) point in σ_0^j (resp. σ_1^j) coming along γ_k from σ_0^k. Choose a subsequence $\{\gamma_{k_j}\}$ of $\{\gamma_k\}$ so that $x_s(j,k_j)$ converges to a point $x_{js}\in\sigma_s$, $s=0,1$. Denote this subsequence by $\{\gamma_{1k}\}$.

Since ρ is continuous in some neighborhood of σ_s^j, we can find a closed ball $B_{1s}=\overline{B(x_{1s},r_{1s})}$ in $\Pi\setminus e$ so that $\int\limits_l\rho\,ds<\frac{\beta}{2^6}$ for any segment l in B_{1s}. We may assume that all γ_{1k} meet B_{1s}, $s=0,1$.

Similarly we find a subsequence $\{\gamma_{2k}\}$ of $\{\gamma_{1k}\}$ and a closed balls B_{20}, B_{21} in $\Pi\setminus e$ such that $\int\limits_l\rho\,ds<\frac{\beta}{2^7}$ for any segment $l\subset B_{20}\cup B_{21}$. We continue this process.

Denote the diagonal subsequence $\{\gamma_{kk}\}$ by $\{\gamma_k\}$.

For each $k\ge1$ we modify each γ_k in every B_{js} so that all γ_k, $\gamma_k\cap B_{js}\ne\emptyset$, pass through x_{js}, as follows.

We replace $\gamma_k\cap B_{js}$ by two radii, terminating at $\gamma_k\cap\partial B_{js}$ and denote the resulting curve still by γ_k. Using further standard construction from the proof of Lemma 4.10 in [13, p. 138–139], we obtain a curve $\tilde{\gamma}$, satisfying the following conditions:

(1) $\int\limits_{\tilde{\gamma}} \rho\, ds < 1 - \frac{\beta}{8}$ and therefore, by (17), $\tilde{\gamma}$ is a rectifiable curve,
(2) $\tilde{\gamma} \in \Gamma(\sigma_0, \sigma_1, \Pi \setminus e)$ and $\tilde{\gamma} \subset \overline{\Pi_{1,\delta}} \setminus (\sigma_0(\delta) \cup \sigma_1(\delta) \cup e)$,
(3) x_{j0} and $x_{j1} \in \tilde{\gamma}$ for all $j = 1, 2, \dots$

Hence, by condition 2), we have $\int\limits_{\tilde{\gamma}} \rho\, ds \geq 1$, that contradicts condition 1). Thus, the inequality (19) is true.

Take $k_1 > k_0(\beta)$ and let $\sigma_0^{k_1} \subset \{x = (x_1, \dots, x_n) : x_1 = a_{1,k_1}\}$, $\sigma_1^{k_1} \subset \{x = (x_1, \dots, x_n) : x_1 = b_{1,k_1}\}$.

Set

$$\rho_1 = \begin{cases} 0, & x_1 \leq a_{1,k_1}, \\ \frac{\rho}{1-\beta}, & x_1 > a_{1,k_1}, x \in \Pi_{1,\delta} \setminus e, \end{cases}$$

and let $G_1 = \{x = (x_1, \dots, x_n) : x_1 \leq a_{1,k_1}\} \cup (\Pi_{1,\delta} \setminus e)$, $u_1(x) = 0$, if $x_1 \leq a_{1,k_1}$, $u_1(x) = \inf\limits_{\gamma_x} \int\limits_{\gamma_x} \rho\, ds$, where γ_x is arbitrary rectifiable curve in $\Pi_{1,\delta} \setminus (\Omega_{0,k_1+1} \cup e)$, connecting $\sigma_0^{k_1+1}$ and $x \in \Pi_{1,\delta} \setminus (\Omega_{0,k_1+1} \cup e)$.

In view of local boundedness ρ_1 on G_1 and known properties of functions such as u_1 [13, Lemma 5.2], it follows that u_1 satisfies the Lipschitz condition in G_1, $|\nabla u_1| \leq \rho_1$ a.e. on G_1, $u_1 = 0$ on $\{x : x_1 \leq a_{1,k_1}\}$.

Taking the cutting $u_2 = \min(1, u_1)$ on G_1, we get

$$\int\limits_{\Pi_{1,\delta} \setminus e} |\nabla u_2| \omega\, dx \leq \int\limits_{\Pi_{1,\delta} \setminus e} |\nabla u_1| \omega\, dx, \tag{20}$$

where u_2, by properties of the cutting, satisfies the Lipschitz condition on G_1 and $u_2 = 1$ on $\Pi_{1,\delta} \cap \Omega_{1,k_1}$, $u_2 = 0$ on Ω_{0,k_1}. Set

$$u_3 = \begin{cases} u_2, & x \in G_1 \setminus \Omega_{1,k_1+1}, \\ 1, & x \in \Omega_{1,k_1+1}. \end{cases}$$

It is easily to see that $u_3 \in \mathrm{Adm}(\sigma_0(\delta), \sigma_1(\delta), \Pi_{1,\delta} \setminus e)$. Then, by (18), (20), Property 2 and the arbitrariness of β, we have for $NC_{1,\omega}$-set E

$$M_{1,\omega}(\sigma_0(\delta), \sigma_1(\delta), \Pi_{1,\delta}) \leq C_{1,\omega}(\sigma_0(\delta), \sigma_1(\delta), \Pi_{1,\delta}) = C_{1,\omega}(\sigma_0(\delta), \sigma_1(\delta), \Pi_{1,\delta} \setminus e) \leq$$

$$\int\limits_{\Pi_{1,\delta} \setminus e} |\nabla u_3| \omega\, dx \leq \int\limits_{\Pi_{1,\delta} \setminus e} \rho \omega\, dx + o(1) \leq C_1 M_{1,\omega}(\sigma_0, \sigma_1, \Pi \setminus e) + \varepsilon + o(1),$$

where $o(1) \to 0$ as $\beta \to 0$. Hence, taking $\beta \to 0$, $\delta \to 0$, $\varepsilon \to 0$, we deduce (5). □

Remark 1. *In the proof of Property 8 we aplied the cutting* $\min(1, u_1)$*. Similarly using* $\max(\min(1, u), 0)$ *to the function* $u \in \mathrm{Adm}(F_0, F_1, G)$*, we come the following definition of the capacity of the condenser.*

Definition. If F_0*,* F_1 *are compact disjoint non-empty sets in* $\bar{\Omega}$ *then*

$$C_{1,\omega}(F_0, F_1, \Omega) = \inf_u \int\limits_{\Omega} |\nabla u| \omega\, dx,$$

where the infimum is taken over all real-valued bounded functions u such that $u|_\Omega$ satisfies locally ther Lipschitz condition and $u = j$ in some neighborhood of F_j, $j = 0, 1$.

Remark 2. *Let F_0, F_1 be compact non-empty disjoint sets in Ω. Let $\{D_j\}$ be a sequence (maybe finite) of all mutually disjoint connected components D_j of Ω.*

Since F_0, F_1 are compact sets, there exists a finite family of components $D_j \in \{D_j\}$, having common points with $F_0 \cup F_1$. Without loss of generality, suppose that only $D_1, \ldots, D_{j_1}$ intersect $F_0 \cup F_1$. Let there is no component D_j, $1 \le j \le j_1$, such that simultaneously $D_j \cap F_0 \ne \emptyset$, $D_j \cap F_1 \ne \emptyset$. This implies $\Gamma(F_0, F_1, \Omega) = \emptyset$. In addition, assuming $u = 0$ on D_j if $D_j \cap F_0 \ne \emptyset$, $u = 1$ on D_j if $D_j \cap F_1 \ne \emptyset$, $u = 1$ on D_j if $j > j_1$, we get $u \in \mathrm{Adm}(F_0, F_1, \Omega)$ and $\int_\Omega |\nabla u| \omega \, dx = 0$. it follows that $C_{1,\omega}(F_0, F_1, \Omega) = 0$.

Otherwise, there is a component D_j, $1 \le j \le j_1$, for which $D_j \cap F_0 \ne \emptyset$, $D_j \cap F_1 \ne \emptyset$ and hence $\Gamma(F_0, F_1, \Omega) \ne \emptyset$.

Then using the same arguments from the proof of Property 8 (see also [9, Theorem 5.5]), we obtain the next assertion.

Theorem 1. *If F_0, F_1 are compact disjoint non-empty sets in Ω then*

$$M_{1,\omega}(F_0, F_1, \Omega) \le C_{1,\omega}(F_0, F_1, \Omega) \le C_1 M_{1,\omega}(F_0, F_1, \Omega),$$

where C_1 depends only on ω and $C_1 = 1$ for $\omega \equiv 1$.

5. Removable sets

In this section, we will establish three criteria of removable sets E for $L^1_{1,\omega}(\Omega)$. We recall (see Definition 1) that E is a removable set for $L^1_{1,\omega}(\Omega)$ if E is a relatively closed subset of Ω, $m_n(E) = 0$ and for every function $u \in L^1_{1,\omega}(\Omega \setminus E)$ there exists a function $v \in L^1_{1,\omega}(\Omega)$ such that $v|_{\Omega \setminus E} = u$.

We will need the following two lemmas. Further let $\Omega = \bigcup_j D_j$, where $\{D_j\}$ is the sequence (maybe finite) of connected components of Ω from Remark 2.

Lemma 3. *If E is a removable set for $L^1_{1,\omega}(\Omega)$ then $D_j \setminus E$ is a domain for all j.*

Proof. Suppose that there exists some component D_j which is partitioned by E into non-empty sets Q_0 and Q_1, $D_j \setminus E = Q_0 \cup Q_1$, $Q_0 \cap Q_1 = \emptyset$. take two point $q_0 \in Q_0$, $q_1 \in Q_1$ and balls $B_0 = B(q_0, r_0)$, $B_1 = B(q_1, r_1)$ be such that $\overline{B_0} \subset Q_0$, $\overline{B_1} \subset Q_1$. Let $L_1 = L_1(a_1, \ldots, a_k)$ be a simple polyline in a domain D_j composed of straight segments $[a_i, a_{i+1}]$, $i = 1, \ldots, k-1$, where each segment $[a_i, a_{i+1}]$ is parallel to some coordinate axis and $a_1 = q_0$, $a_k = q_1$. Applying reasoning from the proof of Property 6, we obtain a coordinate rectangle Π_1 with opposite facets $\sigma_0 \subset Q_0$ and $\sigma_1 \subset Q_1$. Suppose, for example, that σ_0, σ_1 are orthogonal to the x_1-axis. Let $u_0 = 0$ on Q_0 and $u_0 = 1$ on $Q_1 \cap (\Omega \setminus D_j)$. Obviously, $u_0 \in L^1_{1,\omega}(\Omega \setminus E)$ and, by definition of E, there is a function $v_0 \in L^1_{1,\omega}(\Omega)$, for which $v_0|_{\Omega \setminus E} = u_0$. In view of Proposition 1, $v_0 \in W^{1,1}(D_j, loc)$ and therefore $v_0 \in L^1_1(\Pi_1)$. Then (see [11, Sec. 1.1.3, Theorem 1]) function v_0 is absolutely continuous on m_{n-1}-almost all segments l, $\mathcal{H}^1(l \cap E) = 0$, joining the facets σ_0, σ_1 in $\overline{\Pi_1}$ and parallel to the x_1-axis. This implies the existence of a limit point $x_l \in E \cap l$ simultaneously for $l \cap Q_0$ and $l \cap Q_1$. Hence, $v_0(x_l) = 0$ and $v_0(x_l) = 1$, that contradicts the definition of function v_0. Thus, $D_j \setminus E$ is a domain for all j and our Lemma is proved. □

Lemma 4. *Let $B = B(a, r)$ be some ball in an open set $G \subset \Omega$ and E be a removable set for $L^1_{1,\omega}(\Omega)$. Suppose that $\varphi \in C_0^\infty(B)$ and $u \in L^1_{1,\omega}(G \setminus E)$. In addition, assume that u is a bounded and locally satisfying the Lipschitz condition on $G \setminus E$, $B_1 = B(a, r_1)$ is some ball such that $0 < r_1 < r$ and* $\operatorname{supp}\varphi \in B_1$. *Then there exists a function $g \in W^{1,1}_\omega(R^n)$ for which $g = 0$ on $R^n \setminus \overline{B_1}$, $g|_{G\setminus E} = u\varphi$.*

Proof. Set $E_1 = \overline{B_1} \cap E$ and, by Lemma 3, note that $B \setminus E_1$ is a domain. In addition, E_1 as a closed subset of E is a removable set for $L^1_{1,\omega}(\Omega)$. Put $v = u\varphi$ on $\overline{B_1} \setminus E = \overline{B_1} \setminus E_1$ and $v = 0$ on $R^n \setminus \overline{B_1}$.

We see that v satisfies locally the Lipschitz condition on $(B_1 \setminus E_1) \cap (R^n \setminus \overline{B_1})$. If $x_0 \in \partial B_1 \setminus E_1$ then there is some ball $B(x_0, r_0)$ such that $B(x_0, r_0) \Subset (B \setminus E) \setminus \operatorname{supp}\varphi$. This implies $v = 0$ on $B(x_0, r_0)$ and hence v satisfies the Lipschitz condition on $B(x_0, r_0)$. On account of the arbitrariness of $x_0 \in \partial B_1 \setminus E_1$ we obtain that v satisfies locally the Lipschitz condition on $R^n \setminus E$ and $v|_{G\setminus E} = u\varphi$. Since, u, φ, $|\nabla\varphi|$ are bounded functions on $\overline{B_1} \setminus E_1$, we see that $v \in W^{1,1}_\omega(R^n \setminus E_1)$ and hence $v \in W^{1,1}_\omega(\Omega \setminus E_1)$. It follows that there exists $g \in L^1_{1,\omega}(\Omega)$ such that $g|_{\Omega\setminus E_1} = v$, $g|_{G\setminus E} = u\varphi$ and $g = 0$ on $\Omega \setminus \overline{B_1}$. Assuming $g = 0$ on $R^n \setminus \Omega$ we get a function that satisfies the condition of Lemma. Thus the Lemma is proved. □

Theorem 2. *If E is a removable set for $L^1_{1,\omega}(\Omega)$ then E is an $NC_{1,\omega}$-set in Ω.*

Proof. Let $\Pi = \{x = (x_1, \dots, x_n) : a_i < x_i < b_i, i = 1, \dots, n\}$ be a coordinate rectangle with its facets σ_{0i}, σ_{1i} from definition of $NC_{1,\omega}$-set in (4).

In view of $m_n(E) = 0$, we see that $\sigma_{0i}, \sigma_{1i} \subset \partial(\Pi \setminus E)$ for all $i = 1, \dots, n$. Verify that the equality (4) is true if $j = 1$. For the remaining $j = 2, \dots, n$ the proof of (4) is performed in the same way.

Given $\varepsilon > 0$, by Lemma 1 and Remark 1, we find a bounded function $u \in \operatorname{Adm}(\sigma_{01}, \sigma_{11}, \Pi \setminus E) \cap L^1_{1,\omega}(\Pi \setminus E)$ such that

$$C_{1,\omega}(\sigma_{01}, \sigma_{11}, \Pi \setminus E) \le \int\limits_{\Pi\setminus E} |\nabla u|\omega\, dx < C_{1,\omega}(\sigma_{01}, \sigma_{11}, \Pi \setminus E) + \frac{\varepsilon}{2}. \tag{21}$$

Let G_s be an open neighborhood of the facets σ_{s1}, in which $u = s$, where $G_s \Subset \Omega$, $s = 0, 1$, and $\operatorname{dist}(G_0, G_1) > 0$. In addition, let G'_s be another open neighborhood of the facets σ_{s1}, where $G'_s \Subset G_s$, $s = 0, 1$.

Set $G = G_0 \cup G_1 \cup \Pi$ and $E_1 = (\Pi \cap E) \setminus (G_0 \cup G_1)$. It is easily to see that $u \in W^{1,1}_\omega(G \setminus E_1)$ and E_1 is a relatively closed subset of G.

Let the sequence $\{B_k\}$ be a locally finite covering of G by balls $B_k = B(a_k, r_k) \Subset G$, $k \ge 1$. Additionally we required that all balls B_k having common points with G'_s be contained in G_s, $s = 0, 1$. Let $\{\varphi_k\}$ be a C^∞-partition of unity for G subordinating to the covering $\{B_k\}$. We show that there is a function $z \in \operatorname{Adm}(\sigma_{01}, \sigma_{11}, \Pi)$ with $\|z\|_{L^1_{1,\omega}(\Pi)}$ arbitrarily close to $\|u\|_{L^1_{1,\omega}(\Pi\setminus E)}$.

Indeed, let $B'_k = B(a_k, r'_k)$ be a ball such that $0 < r'_k < r_k$ and $\operatorname{supp}\varphi_k \subset B'_k$. Then, by Lemma 4, there is a function $g_k \in W^{1,1}_\omega(R^n)$ for which $g_k = 0$ on $R^n \setminus \overline{B'_k}$, $g|_{G\setminus E_1} = u\varphi_k$, $k \ge 1$. Set $u_k = u\varphi_k$ on $G \setminus E_1$, $k \ge 1$, and $u = \sum\limits_k u_k$ on $G \setminus E_1$ and $g = \sum\limits_k g_k$ on G.

On any bounded open set $Q \Subset G$ we have

$$1 = \sum_k \varphi_k, \qquad g = \sum_k g_k \tag{22}$$

and on $Q \setminus E_1$

$$u = \sum_k u_k, \tag{23}$$

where the sums in (22), (23) contain a finite number of terms. This implies

$$\|g\|_{W^{1,1}_\omega(Q)} \le \infty, \qquad \|u - g\|_{W^{1,1}_\omega(Q\setminus E_1)} \le \sum_k \|u_k - g_k\|_{W^{1,1}_\omega(Q\setminus E_1)} = 0 \tag{24}$$

On account of the arbitrariness of Q we obtain $g \in W^{1,1}_\omega(G)$ and

$$\int\limits_{\Pi\setminus E} |\nabla g|\omega\,dx = \int\limits_{\Pi\setminus E} |\nabla u|\omega\,dx = \int\limits_{\Pi} |\nabla g|\omega\,dx.$$

Moreover, if $Q = G_0$ then $u = 0$ on G_0, $\|u-g\|_{W^{1,1}_\omega(G_0)} = 0$ and therefore $g = 0$ on G_0. Similarly if $Q = G_1$ then $u = 1 = g$ on G_1.

Next we approximate the function g in $W^{1,1}_\omega(G)$ by smooth functions as follows.

If $B_k \cap G'_0 \ne \emptyset$, we set $z_k = g_k = 0$ on R^n. If $B_k \cap G'_1 \ne \emptyset$, we set $z_k = \varphi_k \in C^\infty_0(R^n)$. In other cases let z_k denote the mollification of $g_k \in W^{1,1}_\omega(R^n)$ with a radius $0 < \rho_k < \operatorname{dist}(\operatorname{supp}\varphi_k, \partial B'_k)$.

We take $\beta \in \left(0, \dfrac{1}{2}\right)$ and, by Proposition 6, choose ρ_k to satisfy

$$\|g_k - z_k\|_{W^{1,1}_\omega(R^n)} < \beta^k.$$

By construction, $z_k \in W^{1,1}_\omega(R^n) \cap C^\infty_0(R^n)$. Set $z = \sum\limits_k z_k$ and on any set $Q \Subset G$ we have

$$\|g - z\|_{W^{1,1}_\omega(\Omega)} \le \sum_k \|g_k - z_k\|_{W^{1,1}_\omega(R^n)} \le \frac{\beta}{1-\beta} \le 2\beta.$$

Therefore $z \in L^1_{1,\omega}(G) \cap C^\infty(G)$ and $z = 0$ on G'_0 and $z = 1$ on G'_1. This implies $z \in \operatorname{Adm}(\sigma_0, \sigma_1, \Pi)$ and

$$\int\limits_{\Pi} |\nabla z|\omega\,dx = \int\limits_{\Pi} |\nabla g|\omega\,dx + o(1), \tag{25}$$

where $o(1) \to 0$ as $\beta \to 0$.

Connecting (21), (24) and (25), we derive

$$C_{1,\omega}(\sigma_{01}, \sigma_{11}, \Pi) \le \int\limits_{\Pi} |\nabla z|\omega\,dx \le \int\limits_{\Pi\setminus E} |\nabla g|\omega\,dx + o(1) \le$$

$$\int\limits_{\Pi\setminus E} |\nabla u|\omega\,dx + o(1) \le C_{1,\omega}(\sigma_{01}, \sigma_{11}, \Pi \setminus E) + o(1) + \varepsilon.$$

Sequentially taking $\beta \to 0$, $\varepsilon \to 0$, we obtain

$$C_{1,\omega}(\sigma_{01}, \sigma_{11}, \Pi) = C_{1,\omega}(\sigma_{01}, \sigma_{11}, \Pi \setminus E).$$

This is proved our theorem. □

Theorem 3. *Let E be a relatively closed subset of an open set $\Omega \subset R^n$. Then the following four conditions are equivalent:*

1. *E is a removable set for $L^1_{1,\omega}(\Omega)$;*
2. *E is an $NC_{1,\omega}$-set in Ω;*

(3) *E is an $NED_{1,\omega}$-set in Ω;*
(4) $E \in \mathcal{G}_{1,\omega}(\Omega)$.

Proof. The implication 1) $\to$ 2) is established in Theorem 2 and the implication 2) $\to$ 3) is obtained in Proposition 8. So first we get the implication 3) $\to$ 4).

Let E be an $NED_{1,\omega}$-set in Ω and hence $m_n(E) = 0$, Ω does not locally partitioned by E. Take a compact set $e \subset E$ and verify that e satisfies the $(1,\omega)$-girth condition with respect to X_i (see Definition 1 in Sec. 2.6) if $i = 1$. For the remaining $i = 2, \dots, n$ the proof of the $(1,\omega)$-girth condition with respect to X_i is performed in the same way.

Let ρ be a function from Definition 1 with respect to X_1 and put $\rho = 0$ on e.

Consider also some coordinate rectangle $\Pi = \{x = (x_1, \dots, x_n) : a_i < x_i < b_i, i = 1, \dots, n\}$ such that $e \subset \Pi$. Put $\Pi' = \{x' = (x_2, \dots, x_n) : a_i < x_i < b_i, i = 2, \dots, n\}$ and consider $x' \in \Pi'$ such that $l(x') = \{x = (x_1, x') \in R^n : a_1 \le x_1 \le b_1\}$ has common points with e.

Verify that ε-girth condition holds for ρ on the m_{n-1}-almost all segments $l(x')$. It suffices to prove this fact for the function $\rho_1 = \rho + \rho_0$, where $\rho_0 = 1$ on $\bar{\Pi}$ and $\rho_0 = 0$ on $R^n \setminus \bar{\Pi}$. Then $\int\limits_{\Pi'} dx' \int\limits_{a_1}^{b_1} \rho_1 \omega \, dx_1 = \int\limits_{\Pi} \rho_1 \omega \, dx < \infty$.

Put $\Phi(x', r_1, r_2) = \int\limits_{r_1}^{r_2} \rho_1 \omega \, dx$ for all rationals $r_1, r_2 \in [a_1, b_1]$, $r_1 < r_2$. As in the proof of Property 5, we verify that there exists a m_{n-1}-measurable set $\tilde{\Pi}' \subset \Pi'$ with $m_{n-1}(\tilde{\Pi}') = m_{n-1}(\Pi')$ such that every point $x' \in \tilde{\Pi}'$ is a Lebesgue point for the function $\Phi(x', r_1, r_2)$ (a differentiability point of the integral of these functions over Π'). Choose now $x' = (x_2, \dots, x_n) \in \tilde{\Pi}'$ so that $\mathcal{H}^1(l(x) \cap e) = 0$ (by definition $l(x') \cap e \ne \emptyset$).

Cover $l(x') \cap e$ by the mutually disjoint intervals $U_k = (c_k, d_k)$, $k = 1, \dots, k_1$, where $c_k = (r_k, x')$, $d_k = (\tilde{r}_k, x')$, while r_k and $\tilde{r}_k$ are rationals in (a_1, b_1) with $r_k < \tilde{r}_k$ and

$$\sum_{k=1}^{k_1} \mathcal{H}^1(U_k) < \varepsilon, \quad \sum_{k=1}^{k_1} \int\limits_{U_k} \rho_1 \omega \, dx < \frac{\varepsilon C}{8}, \tag{26}$$

where $C = \dfrac{1}{\sup\limits_{\Pi} (1 + |x|)^n}$. For small $\delta > 0$ put

$$\Pi'(\delta) = \{y' = (y_2, \dots, y_n) : |y_i - x_i| < \delta, i = 2, \dots, n\},$$

$$\Pi_k(\delta) = \{(x, y') : r_k < x_1 < \tilde{r}_k, y' \in \Pi'(\delta)\} \subset \Pi.$$

Denote by $\sigma_0^k = \sigma_0^k(\delta)$, $c_k \in \sigma_0^k$, and $\sigma_1^k = \sigma_1^k(\delta)$, $d_k \in \sigma_1^k$, the facets of $\Pi_k(\delta)$, parallel to $H_1 = \{x = (x_1, \dots, x_n) : x_1 = 0\}$.

In addition, put $\Gamma_k(\delta) = \Gamma(\sigma_0^k, \sigma_1^k, \Pi_k(\delta) \setminus e)$. It is easily to see that $\Pi_k(\delta) \in \mathcal{K}_1(e, \Pi)$. By (12) and the definition of an $NED_{1,\omega}$-set,

$$M_{1,\omega}(\sigma_0^k, \sigma_1^k, \Pi_k(\delta) \setminus e) \ge \frac{m_{n-1}(\sigma_0^k)}{\sup\limits_{\Pi_k(\delta)} (1 + |x|)^n} \ge C \cdot m_{n-1}(\sigma_0^k),$$

where $m_{n-1}(\sigma_0^k) = m_{n-1}(\Pi'(\delta))$.

Let $L_k(\delta) = \inf\{\int\limits_{\gamma} \rho_1\, ds : \gamma \in \Gamma_k(\delta)\}$. Obviously, $L_k(\delta) \geq \tilde{r}_k - r_k$ for all $k = 1, \dots, k_1$. Show that $L_k(\delta)$ is sufficiently small when $\delta > 0$ close to zero. Indeed, $\dfrac{\rho_1}{L_k(\delta)} \in \operatorname{adm} \Gamma_k(\delta)$ and

$$\int\limits_{\Pi'(\delta)} dx' \int\limits_{r_k}^{\tilde{r}_k} \rho_1 \omega\, dx_1 \geq L_k(\delta) C m_{n-1}(\sigma_0^k).$$

For sufficiently small $\delta > 0$, using the differentiability of the integrals, under consideration,

$$L_k(\delta) \leq \frac{1}{C} \int\limits_{r_k}^{\tilde{r}_k} \rho_1 \omega\, dx_1 + \frac{\varepsilon}{8k_1}$$

for all $k = 1, \dots, k_1$. By the definition of infimum, there exists a curve $\gamma_k^0 \in \Gamma_k(\delta)$ such that

$$\int\limits_{\gamma_k^0} \rho_1\, ds \leq L_k(\delta) + \frac{\varepsilon}{8k_1} \leq \frac{1}{C} \int\limits_{r_k}^{\tilde{r}_k} \rho_1 \omega\, dx_1 + \frac{\varepsilon}{4k_1}. \tag{27}$$

Since $\rho_1 \geq 1$ on Π we have $\int\limits_{\gamma_k^0} ds \leq \int\limits_{\gamma_k^0} \rho_1\, ds$. Hence γ_k^0 is a rectifiable curve. By the local boundedness of ρ_1 on $\Pi \setminus e$, for sufficiently small $\delta > 0$ there exist segments $\tau_k^0 \subset \sigma_k^0(\delta)$ and $\tau_k^1 \subset \sigma_k^1(\delta)$ joining the points c_k and d_k, respectively, with the end points of γ_k^0 in $\overline{\Pi_k(\delta)} \setminus e$ so that

$$\int\limits_{\tau_k^0} \rho_1\, ds + \int\limits_{\tau_k^1} \rho_1\, ds < \frac{\varepsilon}{8k_1}.$$

Consequently, there exist simple curves $\gamma_k \subset \tau_k^0 \cup \gamma_k^0 \cup \tau_k^1 \subset \overline{\Pi_k(\delta)} \setminus e$ joining the corresponding points c_k and d_k, $k = 1, \dots, k_1$, and by (26), (27)

$$\sum_{k=1}^{k_1} \int\limits_{\gamma_k} \rho_1\, ds < \varepsilon.$$

This implies

$$\sum_{k=1}^{k_1} \int\limits_{\gamma_k} ds < \sum_{k=1}^{k_1} \int\limits_{\gamma_k} \rho_1\, ds < \varepsilon, \quad \sum_{k=1}^{k_1} \int\limits_{\gamma_k} \rho\, ds < \sum_{k=1}^{k_1} \int\limits_{\gamma_k} \rho_1\, ds < \varepsilon. \tag{28}$$

In other words, the ε-girth condition holds for ρ and ρ_1 on m_{n-1}-almost all segments $l(x')$, $x' \in \tilde{\Pi}'$.

Let $t = (t_1, x') \in l(x') \setminus e$, where (28) holds for $l(x')$. It is easy to verify that the ε-girth condition holds for ρ, ρ_1 on $l(t_1, x') = \{(x_1, x') : a_1 \leq x_1 \leq t_1\}$ (see Remark to Definition 1) for all such t on $l(x')$. We will call this property the $[t, \varepsilon]$-condition for the function ρ on the segment $l(x')$. So the implication 3) → 4) is true.

Let's go to the proof of implication 4) → 1) which we will present in two steps.

Step 1. Let $E \in \mathcal{G}_{1,\omega}(\Omega)$. By the definition 3, $m_n(E) = 0$ and Ω does not partitioned locally by E. Let $u \in L^1_{1,\omega}(\Omega \setminus E) \cap C^\infty(\Omega \setminus E)$ and u be bounded function on $\Omega \setminus E$. Then u can extended to a function $g \in L^1_{1,\omega}(\Omega)$ for which $g|_{\Omega\setminus E} = u$, as follows.

Next we use constructions from the proof of Theorem 2. Let the sequence $\{B_j\}$ be a locally finite covering of Ω by balls $B_j = B(a_j, r_j) \Subset \Omega$, $j \geq 1$. Let $\{\varphi_j\}$ be a C^∞-partition of unity, subordinated to the covering $\{B_j\}$. Besides that let $B'_j = B(a_j, r'_j)$ be a ball such that $0 < r'_j < r_j$ and $\operatorname{supp}\varphi_j \subset B'_j$. Set $e_j = \overline{B'_j} \cap E$ and let $v_j = u\varphi_j$ on $\overline{B'_j} \setminus e_j$ and $v_j = 0$ on $R^n \setminus \overline{B'_j}$.

It is easily to see (proof of Lemma 4) that $v_j \in W^{1,1}_\omega(R^n \setminus e_j) \cap C^\infty(R^n \setminus e_j)$ and so, $v_j|_{\Omega\setminus E} = u\varphi_j$, $|\nabla v_j|$ is a continuous function on $R^n \setminus e_j$.

Take some coordinate rectangle Π_j such that $\overline{B_j} \subset \Pi_j$ and verify that v_j can be extended to a function $g_j \in L^1_{1,\omega}(R^n)$, when $g|_{R^n\setminus e_j} = v_j$.

Since $v_j = 0$ on $R^n \setminus \overline{B'_j}$, it suffices to prove that $v_j|_{\Pi_j\setminus e_j}$ can be extended to function $g_j \in L^1_{1,\omega}(\Pi_j)$. Fix $j \geq 1$.

In the implication 3) $\to$ 4) proof given above, we will put $e = e_j$, $\rho = |\nabla v_j|$ on $R^n \setminus e$, $\Pi_j = \Pi$ and $\rho = |\nabla v_j| = 0$ on e. In addition, we set $\dfrac{\partial v_j}{\partial x_i} = 0$ on e, $i = 1, \dots, n$, and we use notation from the proof of implication 3) $\to$ 4).

Show that v_j is an absolutely continuous function on m_{n-1}-almost all segments $l(x') = \{(x_1, x') : a_1 \leq x_1 \leq b_1\}$, $x' \in \Pi' = \{(x_2, \dots, x_n) : a_i < x_i < b_i, i = 2, \dots, n\}$, if we define v_j properly on e.

By Proposition 1, we also note that $\int\limits_\Pi |\nabla v_j|\, dx < \infty$. This implies

$$\int\limits_{l(x')} |\nabla v_j|\, dx_1 < \infty \tag{29}$$

for m_{n-1}-almost all $l(x')$, $x' \in \Pi'$. In addition, if $l(x') \cap e = \emptyset$ then v_j is infinitely differentiable, and therefore obviously absolutely continuous on $l(x')$.

Thus, we consider only those segments $l(x')$, $x' \in \Pi'$, that satisfy the inequality (29), $l(x') \cap E \neq \emptyset$, $\mathcal{H}^1(l(x') \cap E) = 0$ and on which the ε-girth condition holds for $\rho = |\nabla v_j|$ for all $\varepsilon > 0$ (see Remark to Definition 1). The family of all such segments $l(x')$ is denoted by Γ_1. Take $l(x') \in \Gamma_1$ and given $\eta > 0$, choose $\beta > 0$ such that $\int\limits_F |\nabla v_j|\, dx_1 < \eta$ for every $F \subset l(x')$ with $\mathcal{H}^1(F) < \beta$.

On the segment $l(x')$ consider an absolutely continuous function

$$q(t_1, x') = \int_{a_1}^{t_1} \frac{dv_j}{dx_1}\, dx_1, \tag{30}$$

where $t = (t_1, x') \in l(x')$. We state that $q(t) = v_j(t)$ for all $t = (t_1, x') \in l(x') \setminus e$.

By the choice of $l(x')$ and $[t, \varepsilon]$-girth condition for $|\nabla v_j|$ on $l(x')$ (see proof of implication 3) $\to$ 4) above), there exist intervals $U_k = (c_k, d_k) \subset l(t_1, x') = \{(x_1, x') : a_1 \leq x_1 \leq t_1\}$, $k = 1, \dots, k_1$, with $\bigcup\limits_{k=1}^{k_1} U_k \supset e \cap l(t_1, x')$ and rectifiable curves $\gamma_k \subset \Pi \setminus e$ joining the endpoints of U_k such that

$$\sum_{k=1}^{k_1} \int\limits_{\gamma_k} ds < \varepsilon, \quad \sum_{k=1}^{k_1} \int\limits_{\gamma_k} |\nabla v_j|\, ds < \varepsilon. \tag{31}$$

Without loss of generality, let $c_k = (r_k, x')$, $d_k = (\tilde{r}_k, x')$, $k = 1, \dots, k_1$, $a_1 < r_1 < \tilde{r}_1 < \dots < r_k < \tilde{r}_k < t_1$. Set $c = (a_1, x')$, $0 < \varepsilon < \eta$. Consider a rectifiable curve $\gamma = [c, c_1] \cup \gamma_1 \cup [d_1, c_2] \cup \gamma_2 \dots \cup [d_{k-1}, c_k] \cup \gamma_k \cup [d_k, t_1]$.

Since $v_j \in C^\infty(R^n \setminus e)$ we have $\int\limits_\gamma \frac{dv_j}{ds}\, ds = v_j(t) - v_j(c)$. On the other hand, due to absolute continuity of the integrals in (30) and (31) we deduce

$$v_j(t) - v_j(c) = \int\limits_{a_1}^{t_1} \frac{dv_j}{dx_1}\, dx_1 + o(1),$$

where $o(1) \to 0$ as $\varepsilon \to 0$. Since $v_j(c) = 0$, taking $\varepsilon \to 0$, we have

$$v_j(t) = \int\limits_{a_1}^{t_1} \frac{dv_j}{dx_1}\, dx_1 \text{ for all } t = (t_1, x') \in l(x') \setminus e.$$

Define $v_j(x_1, x')$ as $\int\limits_{a_1}^{x_1} \frac{dv_j}{dx_1}\, dx_1$ if $x \in l(x') \cap e$, $l(x') \in \Gamma_1$, and $v_j(x) = 0$ if $x = (x_1, x') \in l(x') \cap e$, $l(x') \notin \Gamma_1$.

Hence $v_j(x)$ is an absolutely continuous function in R^n on m_{n-1}-almost all straight lines l, parallel to the x_1-axis.

In addition,using integration by parts and Fubini's theorem, we obtain

$$\int\limits_{R^n} v_j \frac{\partial \varphi}{\partial x_1}\, dx_1 = -\int\limits_{R^n} \varphi \frac{\partial v_j}{\partial x_1}\, dx_1 \tag{32}$$

for all $\varphi \in C_0^\infty(R^n)$. Now note that in (32) it is possible to redefine the values of v_j, $\frac{\partial v_j}{\partial x_1}$ on $e = e_j$. Then we get $g_j = v_j$ on $R^n \setminus e$, $g_j = 0$ on e, and $\frac{\partial g_j}{\partial x_1} = \frac{\partial v_j}{\partial x_1}$ on $R^n \setminus e_j$, $\frac{\partial g_j}{\partial x_1} = 0$ on e_j, $j = 1, \dots, n$.

Similarly, let's make sure that the function g_j has a weak derivative $\frac{\partial g_j}{\partial x_i}$ in R^n for $i = 2, \dots, n$. This implies, by construction, $g_j \in W_\omega^{1,1}(R^n)$, $g_j = 0$ on $R^n \setminus \overline{B'_j}$, $g_j = u\varphi_j$ on $\Omega \setminus E$. Set $g = \sum\limits_j g_j$ on Ω. Using technique from the proof of Theorem 2 (see (22)–(24)), we deduce that $\|u - g\|_{W_\omega^{1,1}(\Omega' \setminus E)} = 0$ for all open $\Omega' \Subset \Omega$, $\|u - g\|_{L^1_{1,\omega}(\Omega \setminus E)} = 0$, $\|u\|_{L^1_{1,\omega}(\Omega \setminus E)} = \|g\|_{L^1_{1,\omega}(\Omega)}$.

In other words, $g \in L^1_{1,\omega}(\Omega)$ and $g|_{\Omega \setminus E} = u$.

Step 2. Now let u be an arbitrary function in $L^1_{1,\omega}(\Omega \setminus E)$ and $\{D_k\}$ be a sequence (maybe finite) of all mutually disjoint connected components D_k of Ω. Then, by the definition of E, $D_k \setminus E$ is the connected component of $\Omega \setminus E$, $k \geq 1$, and $\Omega \setminus E = \bigcup\limits_k (D_k \setminus E)$. By Proposition 7, there exists a sequence of bounded functions $u_j \in L^1_{1,\omega}(\Omega \setminus E) \cap C^\infty(\Omega \setminus E)$ such that

$$\lim_{j \to \infty} \|u_j - u\|_{L^1_{1,\omega}(\Omega \setminus E)} = 0, \tag{33}$$

$$\lim_{j \to \infty} \|u_j - u\|_{L_{1,\omega}(\Omega')} = 0 \text{ for all } \Omega' \Subset \Omega \setminus E. \tag{34}$$

Accordingly to Step 1, we assume that $u_j \in L^1_{1,\omega}(\Omega)$ for all $j \geq 1$. In view of (33) and $m_n(E) = 0$ it follows that $\{u_j\}$ is the Cauchy sequence $L^1_{1,\omega}(\Omega)$. Then, by Proposition 2, $\{u_j\}$ converges in $L^1_{1,\omega}(D_k)$ to some function v_k, $k \geq 1$, as $j \to \infty$.

Moreover, from (33) $|\nabla(u - v_k)| = 0$ a.e. on $D_k \setminus e$ and therefore $u = v_k + c_k$ (see [5, Sec. 1.1.5]) on $D_k \setminus E$. Using (34), it is easily to show that $c_k = 0$, $k \geq 1$. For all $x \in \Omega$ we set $v(x) = v_k(x)$ if $x \in D_k$. By construction,

$$\|u\|_{L^1_{1,\omega}(\Omega\setminus E)} = \|v\|_{L^1_{1,\omega}(\Omega)}, \quad v(x)|_{\Omega\setminus E} = u(x).$$

Thus, E is a removable set for $L^1_{1,\omega}(\Omega)$, that completes the proof of the Theorem. □

A simple modification of the arguments in the proof of Theorem 3 gives another assertion.

Theorem 4. *If E is an $NC_{1,\omega}$-set in Ω then $L^m_{1,\omega}(\Omega\setminus E) = L^m_{1,\omega}(\Omega)$, $W^m_{1,\omega}(\Omega\setminus E) = W^m_{1,\omega}(\Omega)$, $W^{m,1}_{\omega}(\Omega \setminus E) = W^{m,1}_{\omega}(\Omega)$.*

6. $(1, 1, \omega)$-equivalent domains in R^n

Following Vodop'yanov and Gol'dstein [8] domains G_1 and G_2 ($G_1 \supset G_2$) in R^n will be called $(1, 1, \omega)$-equivalent, if the restriction operator $\theta : L^1_{1,\omega}(G_1) \to L^1_{1,\omega}(G_2)$ ($\theta u = u|_{G_2}$) is the isomorphism of the vector spaces $L^1_{1,\omega}(G_1)$ and $L^1_{1,\omega}(G_2)$.

Theorem 5. *Domain G_1 and G_2 ($G_1 \supset G_2$) are $(1, 1, \omega)$-equivalent iff the set $E = G_1 \setminus G_2$ is an $NC_{1,\omega}$-set in G_1.*

Proof. Necessity. Let the spaces $L^1_{1,\omega}(G_1)$ and $L^1_{1,\omega}(G_2)$ be isomorphic as linear spaces for the restriction isomorphism $\theta u = u|_{G_2}$ and $u \in L^1_{1,\omega}(G_1)$. Passing to the factor-spaces $\check{L}^1_{1,\omega}(G_1)$ and $\check{L}^1_{1,\omega}(G_2)$ (see Sec. 2.3 and Proposition 3, 4) and using the Banach theorem, we obtain the boundedness of the operator θ^{-1}. Let us prove that $m_n(G_1 \setminus G_2) = 0$. Assume the converse. Then the set $G_1 \setminus G_2$ has at least one density point x_0, which is also the Lebesgue point for weight ω.

Let us consider a sequence of open coordinate cubes $Q_m = Q\left(x_0, \dfrac{1}{m}\right)$ with the center x_0 and the edge of length $\dfrac{1}{m}$. Let us consider the function $u_m(x) = \operatorname{dist}(x, R^n \setminus Q_m)$ on R^n. It is known (see [7]) that $|\nabla u_m| = 1$ a.e. on Q_m and $|\nabla u_m| = 0$ a.e. on $R^n \setminus Q_m$, $|u_m(x') - u_m(x'')| \leq |x' - x''|$ for any $x', x'' \in R^n$. Hence $u_m \in L^1_{1,\omega}(Q_m) \cap L^1_{1,\omega}$ for all $m \geq 1$. From the boundedness of the operator θ^{-1} we have

$$\int\limits_{Q_m} \omega\, dx \leq \int\limits_{G_1} |\nabla u_m| \omega\, dx \leq \|\theta^{-1}\| \int\limits_{G_2} |\nabla u_m| \omega\, dx \leq \|\theta^{-1}\| \int\limits_{G_2\cap Q_m} \omega\, dx.$$

This implies

$$\frac{1}{|Q_m|} \int\limits_{Q_m} \omega\, dx \leq \frac{\|\theta^{-1}\|}{|Q_m|} \int\limits_{G_2\cap Q_m} \omega\, dx.$$

For $m \to \infty$ the inequality is not valid. Consequently, $m_n(G_1 \setminus G_2) = 0$ and every function $v \in L^1_{1,\omega}(G_2)$ may be extended to function $u \in L^1_{1,\omega}(G_1)$ for which $u|_{G_2} = v$. Hence $E = G_1 \setminus G_2$ is a removable set for $L^1_{1,\omega}(G_1)$, and, by Theorem 3, E is an $NC_{1,\omega}$-set in G_1.

The sufficiency condition in the Theorem follows from the Theorem 3. Indeed, let $E = G_1 \setminus G_2$ be an $NC_{1,\omega}$-set in G_1. By Theorem 3, every function $v \in L^1_{1,\omega}(G_2)$ is extended to the function $u \in L^1_{1,\omega}(G_1)$ such that $u|_{G_2} = v$. In view of $m_n(E) = 0$, the extension is only one (the functions $u_1, u_2 \in L^1_{1,\omega}(G_1)$ with $m_n(\{x \in G_1 : u_1(x) \neq u_2(x)\}) = 0$ will be identified).

Consequently, $\theta : L^1_{1,\omega}(G_1) \to L^1_{1,\omega}(G_2)$ is the isomorphism of $L^1_{1,\omega}(G_1)$ and $L^1_{1,\omega}(G_2)$. The Theorem is proved. □

Remark 3. *In the case $1 < p < \infty$ and $\omega \equiv 1$ the criterion of $(1,p)$-equivalence of domains G_1 and G_2 ($G_1 \supset G_2$) in terms of NC_p-sets was established in* [18].

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