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SOME REMARKS ON MÖBIUS STRUCTURES

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Abstract. Some elementary relations between semimetrics in a Möbius structure allow us to give a correct definition of subspaces of a Möbius space and obtain the solutions in the problem of junction for Ptolemaic Möbius spaces as well as the problem of extending the Möbius structure from a subset to the whole set.

Keywords: Möbius structure, Möbius space, Ptolemaic semimetric, Ptolemaic Möbius structure, Möbius equivalence of semimetrics.

1. Möbius structures

Let us remind that a *semimetric* $\rho(.,.)$ on a set X is a real non-negative function on $X \times X$ such that $\rho(x,y) \equiv \rho(y,x)$ and $\rho(x,y) = 0$ iff $x = y$.

The concept of Möbius structure has been introduced by S.V. Buyalo (see [1]) as a usefull tool in the theory of hyperbolic spaces and has the following definition:

Definition 1. *Let the set X contain at least four distinct points. Two semimetrics $\rho(x,y)$ and $\sigma(x,y)$ on X are Möbius equivalent to each other iff the equality*

$$\frac{\rho(x_1,x_2)\rho(x_3,x_4)}{\rho(x_1,x_3)\rho(x_2,x_4)} = \frac{\sigma(x_1,x_2)\sigma(x_3,x_4)}{\sigma(x_1,x_3)\sigma(x_2,x_4)} \tag{1.1}$$

holds for each four of distinct points x_1, x_2, x_3, x_4 in X. The family of all semimetrics on X falls into the classes of equivalence under the relation of Möbius equivalency, and each of them is called Möbius structure on X.

If $\mathbf{M}$ is a Möbius structure on X and $\rho \in \mathbf{M}$, we say that $\mathbf{M}$ is generated by ρ. And the pair $(X, \mathbf{M})$ will be called *Möbus space.*

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Proposition 1. *Let semimetrics ρ and σ on X be Möbius equivalent, and (a,b) be a pair of distinct points in X. Then the constant*

$$k = \frac{\rho(b,c)\sigma(a,c)}{\rho(a,c)\sigma(b,c)} \tag{1.2}$$

does not depend on the choise of point $c \in X \setminus \{a,b\}$, and the equality

$$\frac{\rho(x,b)}{\rho(x,a)} = k\frac{\sigma(x,b)}{\sigma(x,a)} \tag{1.3}$$

holds for every $x \in X \setminus \{a\}$.

Proof. Applying the condition of Möbius equivalence (1.1) to distinct points $\{x_1,x_2,x_3,x_4\} = \{x,b,a,c\}$ we obtain the desired equality (1.3):

$$\frac{\rho(x,b)}{\rho(x,a)} = \left(\frac{\rho(x,b)\rho(a,c)}{\rho(x,a)\rho(b,c)}\right)\frac{\rho(b,c)}{\rho(a,c)} = \left(\frac{\sigma(x,b)\sigma(a,c)}{\sigma(x,a)\sigma(b,c)}\right)\frac{\rho(b,c)}{\rho(a,c)} = \frac{\sigma(x,b)}{\sigma(x,a)}\cdot k$$

for $x \in X \setminus \{a,b,c\}$. For $x = c$ the equality (1.3) follows from (1.2).

Thus (1.3) is valid for all $x \in X \setminus \{a,b\}$, and so

$$k = \frac{\rho(x,b)\sigma(x,a)}{\rho(x,a)\sigma(x,b)}$$

does not depend on $x \in X \setminus \{a,b\}$. Besides, (1.3) is true for $x = b$, for it turns to $0 = 0$. □

The following theorem gives a slightly more convenient definition of Möbius equivalence.

Theorem 1. *Two semimetrics ρ and σ on X are Möbius equivalent if and only if there exists a positive real function $f(x)$ on X such that*

$$\rho(x,y) = f(x)f(y)\sigma(x,y) \tag{1.4}$$

for all $x,y \in X$. Besides, the function f in (1.4) is unique and may be represented by formula

$$f(x) = \frac{\rho(x,a)}{\sigma(x,a)}\sqrt{k\cdot\frac{\sigma(a,b)}{\rho(a,b)}} \quad \text{for } x \neq a;$$

$$f(a) = \sqrt{\frac{1}{k}\cdot\frac{\rho(a,b)}{\sigma(a,b)}} \tag{1.5}$$

with arbitrarily chosen pair (a,b) of distinct points, where

$$k = \frac{\rho(c,b)\sigma(c,a)}{\rho(c,a)\sigma(c,b)} \tag{1.6}$$

does not depend on $c \in X \setminus \{a,b\}$.

Proof. The sufficiency. The immediate calculation shows that (1.4) involves (1.1) for any four (x_1,x_2,x_3,x_4) of distinct points in X, i.e. the Möbius equivalence of σ and ρ.

The necessity. Let a,b be distinct points in X. By Proposition 1 the number k in (1.6) does not depend on $c \in X \setminus \{a,b\}$, so

$$k = \frac{\rho(b,x)\sigma(a,x)}{\rho(a,x)\sigma(b,x)} \tag{1.7}$$

for every $x \in X \setminus \{a, b\}$. Then, taking into account (1.7) and Möbius equivalence of ρ and σ, we have for any distinct $x, y \in X \setminus \{a, b\}$ the desired equality (1.4) with function f from (1.5):

$$f(x)f(y)\sigma(x,y) = \frac{\rho(x,a)}{\sigma(x,a)} \cdot \frac{\rho(y,a)}{\sigma(y,a)} \cdot \left(\frac{\rho(b,x)\sigma(a,x)}{\rho(a,x)\sigma(b,x)}\right) \frac{\sigma(a,b)}{\rho(a,b)}\sigma(x,y) =$$

$$\left(\frac{\sigma(a,b)\sigma(x,y)}{\sigma(x,b)\sigma(y,a)}\right) \frac{\rho(y,a)\rho(b,x)}{\rho(a,b)} = \left(\frac{\rho(a,b)\rho(x,y)}{\rho(x,b)\rho(y,a)}\right) \frac{\rho(y,a)\rho(b,x)}{\rho(a,b)} = \rho(x,y) \ .$$

In the case where $y = b$, $x \in X \setminus \{a\}$ the equality (1.4) is also true:

$$f(x)f(b)\sigma(x,b) = \frac{\rho(x,a)}{\sigma(x,a)} \left(k\frac{\sigma(a,b)}{\rho(a,b)}\right) \frac{\rho(b,a)}{\sigma(b,a)}\sigma(x,b) = \frac{\rho(x,a)}{\sigma(x,a)} \cdot k \cdot \sigma(x,b) =$$

$$\frac{\rho(x,a)}{\sigma(x,a)} \frac{\rho(b,x)}{\rho(a,x)} \frac{\sigma(a,x)}{\sigma(b,x)}\sigma(x,b) = \rho(x,b) \ .$$

Finally, if $y = a$ and $x \neq a$ we also obtain (1.4):

$$f(x)f(a)\sigma(x,a) = \frac{\rho(x,a)}{\sigma(x,a)} \sqrt{\frac{k \cdot \sigma(a,b)}{\rho(a,b)}} \sqrt{\frac{\rho(a,b)}{k \cdot \sigma(a,b)}}\sigma(x,a) = \rho(x,a) \ .$$

In the case $x = y$ (1.4) is trivially true.

The uniqueness. Now let us assume the existence of some other positive real function $g(x)$ for which the equality $\rho(x,y) = g(x)g(y)\sigma(x,y)$ is true for all $x, y \in X$. Then

$$\rho(x,y) = f(x)f(y)\sigma(x,y) = g(x)g(y)\sigma(x,y)$$

for all $x, y \in X$. For any arbitrarily chosen pair (x_0, y_0) of distinct points in X we have $\sigma(x_0, y_0) \neq 0$ and

$$\frac{f(x_0)}{g(x_0)} = \frac{g(y_0)}{f(y_0)} = \lambda$$

Then for all $x \neq y_0$ we have $\sigma(x, y_0) \neq 0$ and

$$\frac{f(x)}{g(x)} = \frac{g(y_0)}{f(y_0)} = \lambda$$

Simultaneously, for all $x \neq x_0$ we have $\sigma(x, x_0) \neq 0$ and

$$\frac{f(x)}{g(x)} = \frac{g(x_0)}{f(x_0)} = \frac{1}{\lambda}$$

It means that $\lambda = 1$ and $f(x) = g(x)$ for all $x \in X$. Thus the uniqueness of f in (1.4) is proved. □

Corollary 1. *Let the subset X_0 of the set X contain at least four distinct points, and $\mathbf{M}$ be a Möbius structure on X. Then the family of all restrictions $\rho_0 = \rho|X_0$ of semimetrics $\rho \in \mathbf{M}$ is the Möbius structure on X_0.*

Proof. Obviously, for any semimetrics $\rho, \sigma \in \mathbf{M}$ their restrictions $\rho_0 = \rho|X_0$ and $\sigma_0 = \sigma|X_0$ to X_0 are Möbius equivalent to each other. Let λ be arbitrarily chosen semimetric in $\mathbf{M}$ and $\lambda_0 = \lambda|X_0$. Let $\mathbf{M}_0$ be Möbius structure on X_0 generated by λ_0. We have to show that every semimetric $\mu_0 \in \mathbf{M}_0$ is the restriction to X_0 of some semimetric $\mu \in \mathbf{M}$.

Since μ_0 is Möbius equivalent to λ_0 there exists (by Theorem 1) a real positive function f_0 on X_0 for which the equiality $\mu_0(x,y) = f_0(x)f_0(y)\lambda_0(x,y)$ is valid for all $x, y \in X_0$. Let $f(x)$ be an arbitrarily constructed extension of f_0 from X_0 to the set X. (For example, we may asume $f(x) = 1$ for $x \in X \setminus X_0$.) Then the semimetric $\mu(x,y) = f(x)f(y)\lambda(x,y)$ belongs to $\mathbf{M}$ and $\mu_0 = \mu|X_0$. Thus the Möbius structure $\mathbf{M}_0$ on X_0 coinsides with the family of all restrictions $\rho|X_0$ of semimetrics from $\mathbf{M}$. □

Corollary 1 provides the correctness of the following definition

Definition 2. *Let the subset X_0 of the set X have at least four distinct points and $\mathbf{M}$ be a Möbius structure on X. Then there exists the unique Möbius structure $\mathbf{M}_0$ on X_0 consisting of all restrictions to X_0 of semimetrics from $\mathbf{M}$. This Möbius structure is said to be the restriction to X_0 of the Möbius structure $\mathbf{M}$, and we shall use the notation $\mathbf{M}_0 = \mathbf{M}|X_0$. In that case the Möbius space $(X_0, \mathbf{M}_0)$ is said to be the subspace of the Möbius space $(X, \mathbf{M})$.*

2. Normalization of semimetrics

We shall consider a special subfamily of semimetrics in a Möbius structure.

Definition 3. *Let $(X, \mathbf{M})$ be a Möbius space. Given a pair of distinct points (a,b) in X, the semimetric $\mu \in \mathbf{M}$ is called $(a,b)-$normalized iff*

$$\mu(x,a) + \mu(x,b) = 1 \tag{2.1}$$

for all $x \in X$. In this case, it's clear that $\mu(a,b) = 1$.

The $(a,b)-$normalization of any semimetric $\rho \in \mathbf{M}$ is realized by formula

$$\rho_0(x,y) := \frac{\rho(x,y)\rho(a,b)}{(\rho(x,a)+\rho(x,b))(\rho(y,a)+\rho(y,b))} \tag{2.2}$$

Since $\rho_0(x,y) = g(x)g(y)\rho(x,y)$ with $g(x) = \sqrt{\rho(a,b)}/(\rho(x,a)+\rho(x,b))$, then by Theorem 1 the semimetric ρ_0 is Möbus equivalent to ρ, i.e (2.2) defines the operator of $(a,b)-$normalization $\rho \mapsto \rho_0$ on $\mathbf{M}$. For a $(a,b)-$normalized semimetric ρ_0 we have $\rho_0(a,b) = 1$.

Recall that a semimetric $\mu(x,y)$ on a set X is called to be *Ptolemaic* if the Ptolemy's Inequality

$$\mu(x_1,x_2)\mu(x_3,x_4) \le \mu(x_1,x_3)\mu(x_2,x_4) + \mu(x_1,x_4)\mu(x_2,x_3)$$

is valid for all $x_1, x_2, x_3, x_4 \in X$. It is easy to check up that if any one of Möbius equivalent semimetrics is Ptolemaic then each of them is Ptolemaic as well.

A Möbius structure $\mathbf{M}$ on X is called *Ptolemaic* if any one (and then every one) of $\rho \in \mathbf{M}$ is a Ptolemaic semimetric. In that case $(X, \mathbf{M})$ is said to be Ptolemaic Möbius space.

It was proved in [2] that any $(a,b)-$normalized Ptolemaic semimetric μ on X satisfies the Triangle Inequality, i.e. is a metric on X. Besides, $\mu(x,y) \le 1$ for all $x, y \in X$. Indeed, from (2.1) we have $\mu(a,b) = 1$, and $\mu(z,a) \le 1$, $\mu(z,b) \le 1$ for all $z \in X$. Then Ptolemy's Inequality gives: $\rho_0(x,y) = \rho_0(x,y)\rho_0(a,b) \le \rho_0(x,a)\rho_0(y,b) + \rho_0(x,b)\rho_0(y,a) \le \rho_0(x,a) + \rho_0(x,b) = 1$.

Now we can reformulate the Theorem 1 for the case of $(a,b)-$normalized semimetrics.

Theorem 2. *Let a pair of distinct points (a,b) be chosen in a Möbius space $(X, \mathbf{M})$. Given $(a,b)-$normalized semimetrics ρ_0 and σ_0 in $\mathbf{M}$, the equality*

$$\rho_0(x,y) = f_0(x)f_0(y)\sigma_0(x,y) \tag{2.3}$$

holds for all $x, y \in X$, where

$$f_0(x) = \frac{\sqrt{k}}{\sigma_0(x,a) + k \cdot \sigma_0(x,b)} \tag{2.4}$$

and the constant

$$k = \frac{\rho_0(b,c)\sigma_0(a,c)}{\rho_0(a,c)\sigma_0(b,c)} \tag{2.5}$$

does not depend on the choise of a point $c \in X \setminus \{a,b\}$.

Proof. By Theorem 1 there exists a unique real positive function $f_0(x)$ on X such that (2.3) holds for all $x, y \in X$, and the function f_0 is

$$f_0(x) = \frac{\rho_0(x,a)}{\sigma_0(x,a)}\sqrt{k} \;\text{ for } x \neq a; \quad f_0(a) = \sqrt{1/k} \tag{2.6}$$

with a constant k defined by (2.5). By Proposition 1 the equality

$$\frac{\rho_0(x,b)}{\rho_0(x,a)} = k\frac{\sigma_0(x,b)}{\sigma_0(x,a)}$$

is valid on $X \setminus \{a\}$. It means that for all $x \neq a$

$$k \cdot \frac{\sigma_0(x,b)}{\sigma_0(x,a)} = \frac{1 - \rho_0(x,a)}{\rho_0(x,a)} = \frac{1}{\rho_0(x,a)} - 1,$$

and therefore

$$\frac{1}{\rho_0(x,a)} = 1 + k\frac{\sigma_0(x,b)}{\sigma_0(x,a)} = \frac{\sigma_0(x,a) + k \cdot \sigma_0(x,b)}{\sigma_0(x,a)} \, .$$

The substitution

$$\rho_0(x,a) = \frac{\sigma_0(x,a)}{\sigma_0(x,a) + k \cdot \sigma_0(x,b)}$$

into (2.6) gives the required formula (2.4) for all $x \neq a$. In the case where $x = a$ we obtain the true equality $f_0(a) = \sqrt{k}/k$ as well. □

Corollary 2. *Given a pair of distinct points (a,b) in a Möbius space $(X, \mathbf{M})$, if $(a,b)-$normalized semimetrics $\rho_0, \sigma_0 \in \mathbf{M}$ coinside to each other on a triple $\{a,b,c\}$, $c \in X \setminus \{a,b\}$, then $\rho_0(x,y) \equiv \sigma_0(x,y)$ on X.*

Proof. In that case $k = 1$ in formula (2.4), and because of the semimetrics ρ_0 and σ_0 being $(a,b)-$normalized, the function $f_0(x)$ in (2.4) is

$$f_0(x) = \frac{1}{\sigma_0(x,a) + \sigma_0(x,b)} = 1 \; ,$$

i.e. $\rho_0(x,y) = \sigma_0(x,y)$ for all $x, y \in X$. □

3. Disjuncted union of Möbius spaces

Let J be a set of indices, and $\{(X_j, \mathbf{M}_j)\}_{j \in J}$ be a collection of Möbius spaces. Considering the disjoint union $X := \sqcup_{j \in J} X_j$ we are to define a Möbius structure $\mathbf{M}$ on X in order to call $(X, \mathbf{M})$ the *disjoint union* of a given collection of Möbius spaces. This problem has two following settings.

Problem 1. To find a Möbius space $(X, \mathbf{M})$ such that $\mathbf{M}|X_j = \mathbf{M}_j$ for every $j \in J$.

Problem 2. To find a Möbius space $(X,\mathbf{M})$ such that for any collection $\{\rho_j\}_{j\in J}$ of semimetrics $\rho_j \in \mathbf{M}_j$ there exists a semimetric $\rho \in \mathbf{M}$ such that $\rho|X_j = \rho_j$ for every $j \in J$.

Besides, in the case where all Möbius structures $\mathbf{M}_j$ $(j \in J)$ are Ptolemaic, the Möbius space $(X,\mathbf{M})$ must be Ptolemaic as well.

Proposition 2. *The Problems 1 and 2 are equivalent to each other. That is, every solution $(X,\mathbf{M})$ for any one of these Problems will be the solution of another one.*

Proof. Let $(X,\mathbf{M})$ be a solution to Problem 2. For an arbitrarily chosen collection of semimetrics $\{\rho_j \in \mathbf{M}_j\}_{j\in J}$ there exists $\rho \in \mathbf{M}$ such that $\rho_j = \rho|X_j$ $(j \in J)$. Since both Möbius structures $\mathbf{M}_j$ and $\mathbf{M}|X_j$ on X_j are generated by semimetric $\rho_j = \rho|X_j$, we have $\mathbf{M}_j = \mathbf{M}|X_j$. Thus $(X,\mathbf{M})$ is a solution to Problem 1.

Now let $(X,\mathbf{M})$ be a solution to Problem 1 and $\{\rho_j\}$ be a given collection of semimetrics $\rho_j \in \mathbf{M}_j$ $(j \in J)$. For an arbitrarily chosen semimetric $\sigma \in \mathbf{M}$ we have $\sigma_j := \sigma|X_j \in \mathbf{M}|X_j = \mathbf{M}_j$ $(j \in J)$. By Theorem 1 there exists a real positive function f_j on X_j such that $\rho_j(x,y) = f_j(x)f_j(y)\sigma(x,y)$ for all $x,y \in X_j$ $(j \in J)$. Consider the real positive function f which is defined on X by formula $f(x) = f_j(x)$ as $x \in X_j$ $(j \in J)$. The semimetric $\rho(x,y) := f(x)f(y)\sigma(x,y)$ is Möbius equivalent to semimetric $\sigma \in \mathbf{M}$. Thus we obtain $\rho \in \mathbf{M}$ such that $\rho|X_i = \rho_j$ for all $j \in J$. It means that $(X,\mathbf{M})$ is a solution to Problem 2.

□

Proposition 3. *In the class of Ptolemaic M(̈o)bis structures, there exists a solution (not unique) to both Problems 1 and 2.*

Proof. Choose in each X_j $(j \in J)$ an arbitrary pair (a_j,b_j) of distinct points and a $(a_j,b_j)-$normalized semimetric $\rho_j \in \mathbf{M}_j$. Let the semimetric ρ on $X = \sqcup_{j\in J}X_j$ be defined by formula: $\rho(x,y) = \rho_j(x,y)$ as $x,y \in X_j$; $\rho(x,y) = 1$ as $x \in X_j$, $y \in X_i$, $i \neq j$. Since each of Möbius spaces $(X_j,\mathbf{M}_j)$ is Ptolemaic, it suffices to verify the Ptolemaic inequality

$$\alpha(a,b;c,d) := \frac{\rho(a,b)\rho(c,d)}{\rho(a,c)\rho(b,d)+\rho(a,d)\rho(b,c)} \leq 1 \tag{3.1}$$

in the case where the points x_1,x_2,x_3,x_4 are distinct and do not belong to the same X_j.

Case 1. Let one of points a,b,c,d is in X_i while three other points are in X_j, $i \neq j$. Since

$$\alpha(a,b;c,d) = \alpha(b,a;d,c) = \alpha(c,d;a,b) = \alpha(d,c;b,a) \tag{3.2}$$

we may assume that $a \in X_i$ while $b,c,d \in X_j$. Since any normalised Ptolemaic semimetric is metric, we obtain the desired inequality (3.1):

$$\alpha(a,b;c,d) = \frac{1\cdot\rho_j(c,d)}{1\cdot\rho_j(b,d)+1\cdot\rho_j(b,c)} \leq 1$$

Case 2.1. Let $a,b \in X_i$ while $c \in X_j$,$d \in X_k$ where $i \neq j$, $i \neq k$. Since any normalized Ptolemaic semimetric is ≤ 1, we have

$$\alpha(a,b;c,d) = \frac{\rho_i(a,b)\rho(c,d)}{1+1} < 1\ .$$

Case 2.2. Let $a, c \in X_i$ while $b \in X_j$, $d \in X_k$ where $i \neq j$, $i \neq k$. Then

$$\alpha(a,b;c,d) = \frac{1 \cdot 1}{\rho_i(a,c)\rho(b,d) + 1 \cdot 1} < 1 \ .$$

Case 2.3. Let $a, d \in X_i$ while $b \in X_j$, $c \in X_k$ where $i \neq j$, $i \neq k$. Then

$$\alpha(a,b;c,d) = \frac{1 \cdot 1}{1 \cdot 1 + \rho_i(a,d)\rho(b,c)} < 1 \ .$$

Case 3. If points a, b, c, d are in distinct X_i, X_j, X_k, X_s then $\alpha(a,b,c,d) = 1/2 < 1$. Thus the semimetric ρ on X is Ptolemaic. Consider the Möbius structure $\mathbf{M}$ on X generated by ρ. Since the restriction $\mathbf{M}|X_j$ is a Möbius structure on X_j (by Corollary 1) and $\rho_j \in \mathbf{M}|X_j \cap \mathbf{M}_j$, we have the equality $\mathbf{M}_j = \mathbf{M}|X_j$. So $(X, \mathbf{M})$ is a desired solution to Problem 1. According to Proposition 2 it is also the solution to Problem 2. □

4. The extension of Ptolemaic Möbius space

Next consider the

Problem 3. Given a Ptolemaic Möbius space $(X, \mathbf{M})$ and a point $q \notin X$, to find a Ptolemaic Möbius structure $\mathbf{M}^*$ on $X^* = X \cup \{q\}$ such that $\mathbf{M} = \mathbf{M}^*|X$.

An arbitrary pair (a,b) of distinct points in X being chosen, consider $(a,b)-$normalized metric $\sigma_0 \in \mathbf{M}$ and define it's extension to X^* by formula: $\sigma_0^*(x,y) = \sigma_0(x,y)$ if $x, y \in X$ and $\sigma_0^*(x,q) = 1$ for all $x \in X$. Since $\sigma_0^*|X = \sigma_0$ is Ptolemaic, we have to check up the Ptolemy's inequality only for a tetrad (x_1, x_2, x_3, q). In that case we have

$$\frac{\sigma_0^*(x_1,x_2)\sigma_0^*(x_3,q)}{\sigma_0^*(x_1,x_3)\sigma_0^*(x_2,q) + \sigma_0^*(x_1,q)\sigma_0^*(x_2,x_3)} = \frac{\sigma_0(x_1,x_2)}{\sigma_0(x_1,x_3) + \sigma_0(x_2,x_3)} \leq 1,$$

since σ_0 is a metric on X. So the semimetric σ_0^* on X^* is Ptolemaic. Then σ_0^* generate a Ptolemaic Möbius structure $\mathbf{M}^*$ on X^*, which gives a solution to Problem 3.

Now we can obtain the extention ρ^* to X^* for any semimetric $\rho \in \mathbf{M}$. Since

$$\rho^*(x,y) = \begin{cases} \rho(x,y) & \text{as } x, y \in X \\ \varphi(x) & \text{as } x \in X, y = q \end{cases}$$

we are to find a positive function $\varphi(x)$ on X which provides that $\rho^* \in \mathbf{M}^*$.

For $x \neq a, b$ the equality

$$\frac{\rho^*(x,q)\rho^*(a,b)}{\rho^*(x,a)\rho^*(b,q)} = \frac{\sigma_0^*(x,q)\sigma_0^*(a,b)}{\sigma_0^*(x,a)\sigma_0^*(b,q)} = \frac{1}{\sigma_0(x,a)}$$

gives

$$\varphi(x) = \varphi(b)\frac{\rho(x,a)}{\sigma_0(x,a)\rho(a,b)} \ .$$

The constant

$$k = \frac{\rho(c,b)\sigma_0(c,a)}{\rho(c,a)\sigma_0(c,b)}$$

does not depend on $c \in X \setminus \{a,b\}$. Then the equality

$$\frac{\rho^*(a,q)\rho^*(c,b)}{\rho^*(a,c)\rho^*(b,q)} = \frac{\sigma_0^*(a,q)\sigma_0^*(c,b)}{\sigma_0^*(a,c)\sigma_0^*(b,q)} = \frac{\sigma_0(c,b)}{\sigma_0(c,a)}$$

gives

$$\varphi(a) = \varphi(b)\frac{\rho(a,c)\sigma_0(b,c)}{\rho(b,c)\sigma_0(a,c)} = \frac{\varphi(b)}{k}\ .$$

Thus we obtain the desired extention of ρ from X to X^*:

$$\rho^*(x,q) = \varphi(x) = \begin{cases} C & \text{as } x = b \\ C/k & \text{as } x = a \\ C \cdot \frac{\rho(x,a)}{\rho(a,b)\sigma_0(x,a)} & \text{as } x \in X \setminus \{a,b\} \end{cases}$$

where $C > 0$ may be arbitrarily chosen.

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References

[1] S.V. Buyalo, *Möbius and submöbius structures*, St. Petersburg Math. J, **28**:5 (2017), 555-568. MR3637585

[2] V.V.Aseev, *Generalized angles in Ptolemaic Möbius structures*, Siberian Math. J, **59**:2 (2018), 241-256. MR3869981

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