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THREE-DIMENSIONAL HOMOGENEOUS SPACES OF NONSOLVABLE LIE GROUPS WITH EQUIAFFINE CONNECTIONS OF NONZERO CURVATURE

N.P. MOZHEY

Abstract. The problem of establishing links between the curvature and the structure of a manifold is one of the important problems in geometry. The purpose of the work is a local description of the equiaffine (locally equiaffine) connections on three-dimensional homogeneous spaces that only admit invariant connections of nonzero curvature. We have considered the case of the nonsolvable Lie group of transformations. Basic notions are defined, such as an isotropically-faithful pair, an (invariant) affine connection, curvature and torsion tensors, Ricci tensor, equiaffine (locally equiaffine) connection. In the main part of the work, for three-dimensional homogeneous spaces of nonsolvable Lie groups (that only admit invariant connections of nonzero curvature) equiaffine (locally equiaffine) connections and their Ricci tensors are found and written out in explicit form. A special feature of the methods presented in the work is an application of a purely algebraic approach to a description of manifolds and structures on them. Results obtained in the work can be applied to solving problems in differential geometry, differential equations, topology, as well as in other areas of mathematics and physics. The algorithms for finding connections can be computerized and used to solve similar problems in large dimensions.

Keywords: equiaffine connection, Lie group, homogeneous space,curvature tensor, Ricci tensor.

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1. Introduction

The main structures in differential geometry of manifolds are the structures of connections, which are highly applicable in mathematics and physics. Even Felix Klein [1] stated that the most useful way to study geometric structures is to study symmetries, that is, the groups of transformations that preserve the structural features. This approach was revolutionary in studies on geometry and still effects its development nowadays. In Klein's approach, the notion of a homogeneous space is fundamental, among which an important subclass is formed by isotropically-faithful homogeneous spaces. In particular, this subclass includes all the homogeneous spaces that admit invariant connection. "The necessity to compare various kinds of geometric quantities in distinct points of a «curved» space makes the notion of connectivity one of the most important ones in geometry and physics" [2]. The theory of connections and its application in studying of homogeneous spaces takes an important place in differential geometry. The invariant connections on homogeneous spaces have been studied by P. K. Raschewski, M. Kurita, E. B. Vinberg, S. Kobayashi, K. Nomizu, B. Dubrov, B. Komrakov, Yu. Chempkovskiy and others. An affine connection is an equiaffine one if it permits a parallel volume form (see [3]). The notion of an equiaffine curvature can be encountered in Blaschke [4]; an alternative approach is provided in [5].

Homogeneous spaces of nonsolvable Lie groups, that only admit affine connections of nonzero curvature, are listed in the work [6]. In our paper, we study the equiaffine (locally equiaffine) connections on the mentioned spaces and their Ricci tensors.

2. Main definitions

Suppose that M is a differentiable manifold and a group $\bar{G}$ acts transitively on it, $G = \bar{G}_x$ is a stabilizer of an arbitrary point $x \in M$. Let $\bar{\mathfrak{g}}$ be the Lie algebra of a Lie group $\bar{G}$, and $\mathfrak{g}$ be a subalgebra which corresponds to the subgroup G. The problem of classification of the homogeneous spaces $(M,\ \bar{G})$ is equivalent to classification (up to equivalency) of pairs of Lie groups $(\bar{G},\ G)$, where $G \subset \bar{G}$, since the manifold M can be identified to the manifold of left cosets $\bar{G}/G$ (for details, see [7]); a description of pairs $(\bar{G},\ G)$ that are associated to the given pair $(\bar{\mathfrak{g}},\ \mathfrak{g})$, is given by G. D. Mostow in [8]. A pair $(\bar{\mathfrak{g}},\ \mathfrak{g})$ is called *isotropically-faithful* if the isotropic representation $\mathfrak{g}$ is faithful. That means that the natural action of the stabilizer $\bar{G}_x$, $x \in M$ on T_xM has a zero kernel. Everywhere where it does not cause confusion, we will identify a subspace complement to $\mathfrak{g}$ in $\bar{\mathfrak{g}}$, to the factor-space $\mathfrak{m} = \bar{\mathfrak{g}}\ /\mathfrak{g}$. An *affine connection* on the pair $(\bar{\mathfrak{g}},\ \mathfrak{g})$ is a mapping $\Lambda : \bar{\mathfrak{g}} \to \mathfrak{gl}(\mathfrak{m})$, such that its restriction on $\mathfrak{g}$ is an isotropic representation of the subalgebra, and mapping is $\mathfrak{g}$-invariant. Invariant affine connections on the homogeneous space $(M,\ \bar{G})$ are in a one-to-one correspondence (see, for example, [9]) with the affine connections on the pair $(\bar{\mathfrak{g}},\ \mathfrak{g})$. The necessary condition for the existence of an affine connection is that the representation of the isotropy for G has to be faithful if $\bar{G}$ is effective on $\bar{G}/G$ [10]. The tensors of the curvature R and the torsion T of an invariant connection can be expressed in terms of Λ in the following way (for details, see [10]): $R(x_{\mathfrak{m}}, y_{\mathfrak{m}}) = [\Lambda(x), \Lambda(y)] - \Lambda([x, y])$, $T(x_{\mathfrak{m}}, y_{\mathfrak{m}}) = \Lambda(x)y_{\mathfrak{m}} - \Lambda(y)x_{\mathfrak{m}} - [x, y]_{\mathfrak{m}}$ for every $x, y \in \bar{\mathfrak{g}}$. We will say that Λ has a zero torsion or is a torsion-free connection, if $T = 0$. In this case, first Bianchi identity is valid: $R(x, y)z + R(y, z)x + R(z, x)y = 0$ for every $x, y, z \in \mathfrak{m}$.

We will define a *Ricci tensor* $\mathrm{Ric}(y,z) = \mathrm{tr}\{x \mapsto R(x,y)z\}$. We say that an affine connection Λ is *locally equiaffine*, if $\mathrm{tr}\Lambda([x,y]) = 0$ for every $x, y \in \bar{\mathfrak{g}}$ (that is, $\Lambda([\bar{\mathfrak{g}},\bar{\mathfrak{g}}]) \subset \mathfrak{sl}(\mathfrak{m})$). An affine connection with zero torsion has a symmetrical Ricci tensor if and only if it is locally equiaffine [3]. Indeed, $\mathrm{Ric}(y,z) - \mathrm{Ric}(z,y) = \mathrm{tr}\{x \mapsto R(x,y)z – R(x,z)y\}$. Taking into account the first Bianchi identity, we obtain $\mathrm{Ric}(y,z) - \mathrm{Ric}(z,y) = \mathrm{tr}\{x \mapsto –R(y,z)x\} = –\mathrm{tr}R(y,z)$. Since $\mathrm{Ric}(y,z)–\mathrm{Ric}(z,y) = –\mathrm{tr}(\Lambda(y)\Lambda(z)–\Lambda(z)\Lambda(y)) + \mathrm{tr}\Lambda([y,z]) = \mathrm{tr}\Lambda([y,z])$, the tensor Ric is symmetrical if and only if $\mathrm{tr}\Lambda([y,z]) = 0$ for every $y, z \in \bar{\mathfrak{g}}$. We will refer to an affine (torsion-free) connection Λ, for which $\mathrm{tr}\Lambda(x) = 0$ for every $x \in \bar{\mathfrak{g}}$, as an *equiaffine connection*. In this case, it is obvious that $\Lambda(\mathfrak{g}) \subset \mathfrak{sl}(\mathfrak{m})$.

We will describe the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ with the help of the multiplication table for the Lie algebra $\bar{\mathfrak{g}}$. By $\{e_1, ..., e_n\}$ we designate the basis $\bar{\mathfrak{g}}$ ($n = \dim\bar{\mathfrak{g}}$). We will assume that the Lie subalgebra $\mathfrak{g}$ is generated by the vectors $e_1, ..., e_{n-3}$, and $\{u_1 = e_{n-2}, u_2 = e_{n-1}, u_3 = e_n\}$ is a basis $\mathfrak{m}$. To enumerate the subalgebras, we use the notation $d.n$, and the notation $d.n.m$ will be used for numeration of pairs, corresponding to the ones provided in [6], here d is the dimension of the subalgebra, n is the number of the subalgebra in $\mathfrak{gl}(3,\mathbb{R})$, and m is the number of the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$. Since the restriction $\Lambda : \bar{\mathfrak{g}} \to \mathfrak{gl}(\mathfrak{m})$ on $\mathfrak{g}$ is an isotropic representation of a subalgebra, the connection is defined by its values on $\mathfrak{m}$. We will write it out in terms of images of the basic vectors $\Lambda(u_1)$, $\Lambda(u_2)$, $\Lambda(u_3)$, write the tensor of curvature R in its values $R(u_1,u_2)$, $R(u_1,u_3)$, $R(u_2,u_3)$, and torsion tensor T in its values $T(u_1,u_2)$, $T(u_1,u_3)$, $T(u_2,u_3)$.

3. The description of equiaffine connections on homogeneous spaces

In paper [6], the following result was obtained:

Theorem 1. *All the three-dimensional homogeneous spaces, defined by the pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$, such that $\bar{\mathfrak{g}}$ is nonsolvable and $\mathfrak{g}$ is solvable, that only admit connections of nonzero curvature, locally have the following form:*

2.1.2.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	u_1	$-u_2$	0
e_2	0	0	0	0	u_3
u_1	$-u_1$	0	0	e_1	0
u_2	u_2	0	$-e_1$	0	0
u_3	0	$-u_3$	0	0	0

,

2.3.2.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	$-u_2$	u_1	0
e_2	0	0	0	0	u_3
u_1	u_2	0	0	e_1	0
u_2	$-u_1$	0	$-e_1$	0	0
u_3	0	$-u_3$	0	0	0

,

2.3.3.	e_1	e_2	u_1	u_2	u_3
e_1	0	0	$-u_2$	u_1	0
e_2	0	0	0	0	u_3
u_1	u_2	0	0	$-e_1$	0
u_2	$-u_1$	0	e_1	0	0
u_3	0	$-u_3$	0	0	0

,

2.9.12.	e_1	e_2	u_1	u_2	u_3
e_1	0	$-e_2$	u_1	$-2u_2$	$2u_3$
e_2	e_2	0	0	0	u_1
u_1	$-u_1$	0	0	e_2	0
u_2	$2u_2$	0	$-e_2$	0	$-e_1$
u_3	$-2u_3$	$-u_1$	0	e_1	0

,

3.8.8.	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_3	u_1	0	0
e_2	0	0	e_3	0	u_2	$-u_3$
e_3	$-e_3$	$-e_3$	0	0	0	u_1
u_1	$-u_1$	0	0	0	e_3	0
u_2	0	$-u_2$	0	$-e_3$	0	$2e_2-e_1$
u_3	0	u_3	$-u_1$	0	e_1-2e_2	0

,

3.21.7.	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_3$	e_2	0	$-u_3$	u_2
e_2	e_3	0	0	0	u_1	0
e_3	$-e_2$	0	0	0	0	u_1
u_1	0	0	0	0	$-e_2$	$-e_3$
u_2	u_3	$-u_1$	0	e_2	0	$-e_1$
u_3	$-u_2$	0	$-u_1$	e_3	e_1	0

,

3.19.14.	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	e_3	0	u_2	$-u_3$
e_2	e_2	0	0	0	u_1	0
e_3	$-e_3$	0	0	0	0	u_1
u_1	0	0	0	0	e_3	e_2
u_2	$-u_2$	$-u_1$	0	$-e_3$	0	e_1
u_3	u_3	0	$-u_1$	$-e_2$	$-e_1$	0

,

3.21.6.	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_3$	e_2	0	$-u_3$	u_2
e_2	e_3	0	0	0	u_1	0
e_3	$-e_2$	0	0	0	0	u_1
u_1	0	0	0	0	e_2	e_3
u_2	u_3	$-u_1$	0	$-e_2$	0	e_1
u_3	$-u_2$	0	$-u_1$	$-e_3$	$-e_1$	0

,

4.11.2.	e_1	e_2	e_3	e_4	u_1	u_2	u_3
e_1	0	0	e_3	e_4	u_1	0	0
e_2	0	0	$-e_3$	e_4	0	u_2	$-u_3$
e_3	$-e_3$	e_3	0	0	0	u_1	0
e_4	$-e_4$	$-e_4$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	e_4	e_3
u_2	0	$-u_2$	$-u_1$	0	$-e_4$	0	e_2
u_3	0	u_3	0	$-u_1$	$-e_3$	$-e_2$	0

,

4.13.2.	e_1	e_2	e_3	e_4	u_1	u_2	u_3
e_1	0	e_2	e_3	0	u_1	0	0
e_2	$-e_2$	0	0	e_3	0	u_1	0
e_3	$-e_3$	0	0	$-e_2$	0	0	u_1
e_4	0	$-e_3$	e_2	0	0	$-u_3$	u_2
u_1	$-u_1$	0	0	0	0	e_2	e_3
u_2	0	$-u_1$	0	u_3	$-e_2$	0	e_4
u_3	0	0	$-u_1$	$-u_2$	$-e_3$	$-e_4$	0

,

4.13.3.	e_1	e_2	e_3	e_4	u_1	u_2	u_3
e_1	0	e_2	e_3	0	u_1	0	0
e_2	$-e_2$	0	0	e_3	0	u_1	0
e_3	$-e_3$	0	0	$-e_2$	0	0	u_1
e_4	0	$-e_3$	e_2	0	0	$-u_3$	u_2
u_1	$-u_1$	0	0	0	0	$-e_2$	$-e_3$
u_2	0	$-u_1$	0	u_3	e_2	0	$-e_4$
u_3	0	0	$-u_1$	$-u_2$	e_3	e_4	0

.

The proof of the theorem is provided in [6]. Using the obtained classification, we will find equiaffine (locally equiaffine) connections on the spaces of the mentioned form, and also their Ricci tensors. When $\Lambda(u_1) = \Lambda(u_2) = \Lambda(u_3) = 0$, the connection will be called a zero one.

Theorem 2. *Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a three-dimensional homogeneous space, such that $\bar{\mathfrak{g}}$ is nonsolvable and $\mathfrak{g}$ is solvable, that only admits connections of nonzero curvature (provided in Theorem 1). Locally equiaffine (torsion-free) connections on $(\bar{\mathfrak{g}}, \mathfrak{g})$ have the form shown in the table (here and after, $p_{i,j}, q_{i,j}, r_{i,j} \in \mathbb{R}$ $(i,j = \overline{1,3})$):*

Pair	*Locally equiaffine connection*
3.19.14	$\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&q_{1,3}\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&q_{1,3}&0\\0&0&0\\0&0&0 \end{pmatrix}$
3.21.6 3.21.7	$\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&q_{1,2}&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&q_{1,2}\\0&0&0\\0&0&0 \end{pmatrix}$
2.9.12, 3.8.8	*does not admit locally equiaffine connections*
4.11.2, 4.13.2, 4.13.3, 2.1.2, 2.3.2, 2.3.3	*zero*

Equiaffine connections have the form provided in the table:

Pair	*Equiaffine connection*
3.19.14	$\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&q_{1,3}\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&q_{1,3}&0\\0&0&0\\0&0&0 \end{pmatrix}$
3.21.6 3.21.7	$\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&q_{1,2}&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&q_{1,2}\\0&0&0\\0&0&0 \end{pmatrix}$
2.9.12, 3.8.8, *4.11.2, 4.13.2, 4.13.3,* *2.1.2, 2.3.2, 2.3.3*	*does not admit equiaffine connections*

Proof. We will find equiaffine (locally equiaffine) connections for the pairs mentioned in Theorem 1. In the book [11], classifications for all the subalgebras $\mathfrak{g}$ in $\mathfrak{gl}(3,\mathbb{R})$ up to a conjugation are provided, and all the isotropically-faithful pairs (up to equivalency) $(\bar{\mathfrak{g}},\mathfrak{g})$ are found, and in the work [6], among those pairs, the ones with an nonsolvable algebra $\bar{\mathfrak{g}}$ and a solvable one $\mathfrak{g}$, that only admit connections of nonzero curvature, are selected. Therefore, if $\mathfrak{g}$ is a solvable subalgebra of the Lie algebra $\mathfrak{gl}(3,\mathbb{R})$, such that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ defines a three-dimensional homogeneous space that only admits connections of nonzero curvature, and $\bar{\mathfrak{g}}$ is not solvable, then $\mathfrak{g}$ is conjugated to exactly one of the following subalgebras:

$$2.1. \boxed{\begin{matrix} x&&\\&-x&\\&&y \end{matrix}};\ 2.3. \boxed{\begin{matrix} &x&\\-x&&\\&&y \end{matrix}};\ 2.9. \boxed{\begin{matrix} x&&y\\&-2x&\\&&2x \end{matrix}};\ 3.8. \boxed{\begin{matrix} x&&z\\&y&\\&&-y \end{matrix}};$$

$$3.19. \boxed{\begin{matrix} &y&z\\&x&\\&&-x \end{matrix}};\ 3.21. \boxed{\begin{matrix} &y&z\\&&x\\&-x& \end{matrix}};\ 4.11. \boxed{\begin{matrix} x&z&u\\&y&\\&&-y \end{matrix}};\ 4.13. \boxed{\begin{matrix} x&y&z\\&&u\\&-u& \end{matrix}}.$$

Here to simplify the notation, instead of the standard designation for the subalgebra in $\mathfrak{gl}(3,\mathbb{R})$

$$\mathfrak{g} = \left\{ \begin{pmatrix} x&0&0\\0&-x&0\\0&0&y \end{pmatrix} \;\middle|\; x,y\in\mathbb{R} \right\},\ \text{the following notation is used:}\ \boxed{\begin{matrix} x&&\\&-x&\\&&y \end{matrix}}.$$

It is assumed that the variables are designated by the Latin letters and belong to $\mathbb{R}$; we will choose the basis of a subalgebra, by default, by ascribing the value of 1 to one of the Latin variables, and 0 value to the rest of them, and the numeration of the basic vectors corresponds to the alphabetic sequence.

Respectively, for the cases 2.1, 2.3, 2.9, 3.8, 4.11, and 4.13, the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ does not admit equiaffine connections, since even $\Lambda(\mathfrak{g})$ does not belong to $\mathfrak{sl}(\mathfrak{m})$.

The affine connections have the form

Pair	Affine connection
3.19.14	$\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&q_{1,3}\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&r_{1,2}&0\\0&0&0\\0&0&0 \end{pmatrix}$
3.21.6 3.21.7	$\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&q_{1,2}&q_{1,3}\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&-q_{1,3}&q_{1,2}\\0&0&0\\0&0&0 \end{pmatrix}$
2.9.12,4.11.2,4.13.2,4.13.3, 3.8.8, 2.1.2, 2.3.2, 2.3.3	zero

The torsion tensors on the mentioned homogeneous spaces are as follows:

Pair	Tensor of curvature
3.19.14	$(0,0,0),(0,0,0),(q_{1,3}-r_{1,2},0,0)$
3.21.6, 3.21.7	$(0,0,0),(0,0,0),(2q_{1,3},0,0)$
2.9.12,4.11.2, 4.13.2,4.13.3, 3.8.8, 2.1.2, 2.3.2, 2.3.3	zero

Therefore, for the case 3.19.14, $T=0$ when $r_{1,2}=q_{1,3}$, and in the cases 3.21.6 and 3.21.7 it is so when $q_{1,3}=0$. In these cases, the connection is locally equiaffine, since $\mathrm{tr}\Lambda([y,z])=0$ for every $y,z\in\bar{\mathfrak{g}}$; the connection is equiaffine given any values of the parameters, since $\mathrm{tr}\Lambda(x)=0$ for every $x\in\bar{\mathfrak{g}}$. In the rest of the cases, we have that $T=0$ given any values of the parameters in all the cases except for 2.9.12 (because $[u_2,u_3]=-e_1$, then $\mathrm{tr}\Lambda(e_1)\neq 0$, hence, the connection is not locally equiaffine) and 3.8.8 (because $[u_2,u_3]=2e_2-e_1$, then $\mathrm{tr}\Lambda(2e_2-e_1)\neq 0$, hence, the connection is not locally equiaffine), we have that $\mathrm{tr}\Lambda([y,z])=0$ for every $y,z\in\bar{\mathfrak{g}}$, hence, the connection is locally equiaffine; the connection is equiaffine given any values of the parameters (since $\mathrm{tr}\Lambda(x)=0$ for every $x\in\bar{\mathfrak{g}}$) in all the cases except for 2.1.2, 2.3.2, 2.3.3, 2.9.12, 3.8.8, 4.11.2, 4.13.2, 4.13.3, because in these cases (as it has been mentioned above) even $\Lambda(\mathfrak{g})$ does not belong to $\mathfrak{sl}(\mathfrak{m})$.

We will find Ricci tensors of the mentioned connections. For example, in the case 2.9.12, the tensor of curvature is as follows,

$$R(u_1,u_2)=\begin{pmatrix}0&0&-1\\0&0&0\\0&0&0\end{pmatrix},R(u_1,u_3)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix},R(u_2,u_3)=\begin{pmatrix}1&0&0\\0&-2&0\\0&0&2\end{pmatrix},$$

then the Ricci tensor $\mathrm{Ric}(y,z)=\mathrm{tr}\{x\mapsto R(x,y)z\}$ has the form

$$\begin{pmatrix}0&0&0\\0&0&-3\\0&-2&0\end{pmatrix}.$$

The rest of the cases are considered using the similar reasoning. Therefore, by direct calculations, we obtain Ricci tensors:

Pair	Ricci Tensor	Pair	Ricci Tensor
2.1.2	$\begin{pmatrix}0&-1&0\\-1&0&0\\0&0&0\end{pmatrix}$	2.3.2	$\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$
2.3.3	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$	2.9.12, 3.8.8	$\begin{pmatrix}0&0&0\\0&0&-3\\0&-2&0\end{pmatrix}$
3.19.14, 4.11.2	$\begin{pmatrix}0&0&0\\0&0&-2\\0&-2&0\end{pmatrix}$	3.21.6, 4.13.2	$\begin{pmatrix}0&0&0\\0&-2&0\\0&0&-2\end{pmatrix}$
3.21.7, 4.13.3	$\begin{pmatrix}0&0&0\\0&2&0\\0&0&2\end{pmatrix}$		

According to the obtained above, a Ricci tensor is symmetrical if and only if a connection is locally equiaffine, that is, in all the mentioned cases except for 2.9.12 and 3.8.8.

Therefore, for all the three-dimensional homogeneous spaces with an nonsolvable $\bar{\mathfrak{g}}$ and a solvable $\mathfrak{g}$, that only admit connections of nonzero curvature, we have found the equiaffine (locally equiaffine) connections and their Ricci tensors. □

In a similar way, in the paper [6], the following statement was obtained:

Theorem 3. *All the three-dimensional homogeneous spaces, defined by the pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$, such that $\bar{\mathfrak{g}}$ and $\mathfrak{g}$ are nonsolvable, that only admits connections of nonzero curvature, locally have the following form:*

3.5.2	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_3	$-e_2$	$-u_3$	0	u_1
e_2	$-e_3$	0	e_1	$-u_2$	u_1	0
e_3	e_2	$-e_1$	0	0	$-u_3$	u_2
u_1	u_3	u_2	0	0	e_2	e_1
u_2	0	$-u_1$	u_3	$-e_2$	0	e_3
u_3	$-u_1$	0	$-u_2$	$-e_1$	$-e_3$	0

,

3.4.3	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$-e_3$	u_1	0	$-u_3$
e_2	$-e_2$	0	e_1	0	u_1	u_2
e_3	e_3	$-e_1$	0	u_2	u_3	0
u_1	$-u_1$	0	$-u_2$	0	$-e_2$	e_1
u_2	0	$-u_1$	$-u_3$	e_2	0	e_3
u_3	u_3	$-u_2$	0	$-e_1$	$-e_3$	0

,

5.2.3	e_1	e_2	e_3	e_4	e_5	u_1	u_2	u_3
e_1	0	$2e_2$	$-2e_3$	e_4	$-e_5$	u_1	$-u_2$	0
e_2	$-2e_2$	0	e_1	0	e_4	0	u_1	0
e_3	$2e_3$	$-e_1$	0	e_5	0	u_2	0	0
e_4	$-e_4$	0	$-e_5$	0	0	0	0	$u_1+\alpha e_4$
e_5	e_5	$-e_4$	0	0	0	0	0	$u_2+\alpha e_5$
u_1	$-u_1$	0	$-u_2$	0	0	0	0	αu_1-e_4
u_2	u_2	$-u_1$	0	0	0	0	0	αu_2-e_5
u_3	0	0	0	$-u_1-\alpha e_4$	$-u_2-\alpha e_5$	$-\alpha u_1+e_4$	$-\alpha u_2+e_5$	0

, $\alpha \geq 0$,

6.1.3.	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3
e_1	0	$2e_2$	$-2e_3$	0	e_5	$-e_6$	u_1	$-u_2$	0
e_2	$-2e_2$	0	e_1	0	0	e_5	0	u_1	0
e_3	$2e_3$	$-e_1$	0	0	e_6	0	u_2	0	0
e_4	0	0	0	0	e_5	e_6	u_1	u_2	0
e_5	$-e_5$	0	$-e_6$	$-e_5$	0	0	0	0	u_1
e_6	e_6	$-e_5$	0	$-e_6$	0	0	0	0	u_2
u_1	$-u_1$	0	$-u_2$	$-u_1$	0	0	0	0	$-e_5$
u_2	u_2	$-u_1$	0	$-u_2$	0	0	0	0	$-e_6$
u_3	0	0	0	0	$-u_1$	$-u_2$	e_5	e_6	0

.

The proof of this theorem is also provided in [6].

Using the obtained classification, we will find equiaffine (locally equiaffine) connections on the spaces of the mentioned form and also their Ricci tensors.

Theorem 4. *Suppose that $(\bar{\mathfrak{g}}, \mathfrak{g})$ is a three-dimensional homogeneous space, such that $\bar{\mathfrak{g}}$ and $\mathfrak{g}$ are nonsolvable, that only admits connections of nonzero curvature (provided in Theorem 3). The locally equiaffine (torsion-free) connections on $(\bar{\mathfrak{g}}, \mathfrak{g})$ have the form provided in the table:*

Pair	*Locally equiaffine connection*
3.4.3, 3.5.2	*zero*
6.1.3	$\begin{pmatrix} 0 & 0 & p_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p_{1,3} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} p_{1,3} & 0 & 0 \\ 0 & p_{1,3} & 0 \\ 0 & 0 & 2p_{1,3} \end{pmatrix}$
5.2.3	$\begin{pmatrix} 0 & 0 & q_{2,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & q_{2,3} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} q_{2,3} & 0 & 0 \\ 0 & q_{2,3} & 0 \\ 0 & 0 & 2q_{2,3} \end{pmatrix}$

The equiaffine connections have the form provided in the table:

Pair	*Equiaffine connection*
3.4.3, 3.5.2	*zero*
6.1.3	*does not admit equiaffine connections*
5.2.3	$\begin{pmatrix} 0 & 0 & q_{2,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & q_{2,3} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} q_{2,3} & 0 & 0 \\ 0 & q_{2,3} & 0 \\ 0 & 0 & 2q_{2,3} \end{pmatrix}$

Proof. The reasoning is similar to the one in the proof of Theorem 2, but it is used for the pairs mentioned in Theorem 3. Taking into account the results of [11] and [6], by direct calculations, we obtain that if $\mathfrak{g}$ is an nonsolvable subalgebra of the Lie algebra $\mathfrak{gl}(3,\mathbb{R})$, such that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ only admits the connections of nonzero curvature, then $\mathfrak{g}$ is conjugated to exactly one of the subalgebras 3.4, 3.5, 5.2, 6.1:

$$3.4.\ \boxed{\begin{matrix} x & y & \\ z & & y \\ & z & -x \end{matrix}};\ 3.5.\ \boxed{\begin{matrix} & y & x \\ -y & & z \\ -x & -z & \end{matrix}};\ 5.2.\ \boxed{\begin{matrix} x & u & u \\ z & -x & v \\ & & \end{matrix}};\ 6.1.\ \boxed{\begin{matrix} x & z & w \\ u & y & v \\ & & \end{matrix}}.$$

Therefore, in the case 6.1, the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ does not admit equiaffine connections, since even $\Lambda(\mathfrak{g})$ does not belong to $\mathfrak{sl}(\mathfrak{m})$.

The affine connections have the form:

Pair	Affine connection
3.4.3	$\begin{pmatrix} 0 & p_{1,2} & 0 \\ 0 & 0 & p_{1,2} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -p_{1,2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p_{1,2} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ -p_{1,2} & 0 & 0 \\ 0 & -p_{1,2} & 0 \end{pmatrix}$
3.5.2	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p_{2,3} \\ 0 & -p_{2,3} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -p_{2,3} \\ 0 & 0 & 0 \\ p_{2,3} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & p_{2,3} & 0 \\ -p_{2,3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
6.1.3	$\begin{pmatrix} 0 & 0 & p_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p_{1,3} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} r_{1,1} & 0 & 0 \\ 0 & r_{1,1} & 0 \\ 0 & 0 & r_{1,1}+p_{1,3} \end{pmatrix}$
5.2.3	$\begin{pmatrix} 0 & 0 & q_{2,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & q_{2,3} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} r_{1,1} & 0 & 0 \\ 0 & r_{1,1} & 0 \\ 0 & 0 & r_{1,1}+q_{2,3} \end{pmatrix}$

The torsion tensors on the mentioned spaces are as follows:

Pair	Torsion tensors
3.4.3	$(2p_{1,2},0,0),(0,2p_{1,2},0),(0,0,2p_{1,2})$
3.5.2	$(0,0,-2p_{2,3}),(0,2p_{2,3},0),(-2p_{2,3},0,0)$
6.1.3	$(0,0,0),(-p_{1,3}+r_{1,1},0,0),(0,-p_{1,3}+r_{1,1},0)$
5.2.3	$(0,0,0),(q_{2,3}-r_{1,1},0,0),(0,q_{2,3}-r_{1,1},0)$

Hence, in the case 3.4.3, $T = 0$ when $p_{1,2} = 0$, in the case 3.5.2 it is so when $p_{2,3} = 0$, in the case 6.1.3 it is so when $r_{1,1} = p_{1,3}$, in the case 5.2.3 it is so when $r_{1,1} = q_{2,3}$. In these cases, the connection is locally equiaffine because $\mathrm{tr}\Lambda([y,z])=0$ for every $y, z \in \bar{\mathfrak{g}}$; the connection is equiaffine given any values of the parameters (since $\mathrm{tr}\Lambda(x)=0$ for every $x \in \bar{\mathfrak{g}}$) in all the cases except for the case 6.1.3, because in this case (as it has been mentioned above) even $\Lambda(\mathfrak{g})$ does not belong to $\mathfrak{sl}(\mathfrak{m})$.

We will find the Ricci tensors for the mentioned connections. For example, in the case 3.4.3, the tensor of curvature is as follows,

$$R(u_1,u_2) = \begin{pmatrix} 0 & p_{1,2}^2+1 & 0 \\ 0 & 0 & p_{1,2}^2+1 \\ 0 & 0 & 0 \end{pmatrix}, R(u_1,u_3) = \begin{pmatrix} -p_{1,2}^2-1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p_{1,2}^2+1 \end{pmatrix},$$

$$R(u_2,u_3) = \begin{pmatrix} 0 & 0 & 0 \\ -p_{1,2}^2-1 & 0 & 0 \\ 0 & -p_{1,2}^2-1 & 0 \end{pmatrix},$$

then the Ricci tensor $\mathrm{Ric}(y,z)=\mathrm{tr}\{x \mapsto R(x,y)z\}$ has the form

$$\begin{pmatrix} 0 & 0 & -2p_{1,2}^2-2 \\ 0 & 2p_{1,2}^2+2 & 0 \\ -2p_{1,2}^2-2 & 0 & 0 \end{pmatrix},$$

when $p_{1,2} = 0$, the Ricci tensor takes the form

$$\begin{pmatrix} 0 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{pmatrix}.$$

The rest of the cases are considered using the similar reasoning. Therefore, by direct calculations, we obtain the Ricci tensors for the rest of the cases:

Pair	Ricci Tensor	Ricci Tensor given $T = 0$
3.4.3	$\begin{pmatrix} 0 & 0 & -2p_{1,2}^2-2 \\ 0 & 2p_{1,2}^2+2 & 0 \\ -2p_{1,2}^2-2 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{pmatrix}$
3.5.2	$\begin{pmatrix} -2p_{2,3}^2-2 & 0 & 0 \\ 0 & -2p_{2,3}^2-2 & 0 \\ 0 & 0 & -2p_{2,3}^2-2 \end{pmatrix}$	$\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$
5.2.3	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2q_{2,3}^2+2 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2q_{2,3}^2+2 \end{pmatrix}$
6.1.3	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2p_{1,3}^2+2 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2p_{1,3}^2+2 \end{pmatrix}$

According to the obtained above, a Ricci tensor is symmetrical, if the connection is locally equiaffine, hence, it is so for all the mentioned cases.

□

4. Conclusion

Therefore, we have found and presented in an explicit form a local description of equiaffine (locally equiaffine) connections on three-dimensional homogeneous spaces with an nonsolvable group of transformations, that only admits invariant connections of nonzero curvature, and have also described the Ricci tensors of the mentioned connections. A distinctive feature of the methods presented in the paper is the application of a purely algebraic approach to the description of manifolds and structures on those. The obtained results can be applied to solving problems in differential geometry, differential equations, topology, and also in other areas of mathematics and physics. The algorithms for searching for connections can be computerized and used in solving similar problems in higher dimensions.

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Natalya Pavlovna Mozhey
Belarusian State University of Informatics and Radioelectronics,
6, P. Brovki str.,
Minsk, 220013, Belarus
Email address: mozheynatalya@mail.ru