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ON ALMOST OMEGA-CATEGORICITY OF WEAKLY O-MINIMAL THEORIES

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ABSTRACT. Here we prove that weakly o-minimal theories of finite convexity rank having less than 2^ω countable models are almost ω-categorical.

Keywords: almost ω-categoricity, weak o-minimality, convexity rank, binary theory.

1. PRELIMINARIES

Let L be a countable first order language. Throughout this paper we consider L-structures and suppose that L contains a binary relation symbol $<$ which is interpreted as a linear order in these structures. An *open interval* in such a structure M is a parametrically definable subset of M of the form $I = \{c \in M : M \models a < c < b\}$ for some $a, b \in M \cup \{-\infty, \infty\}$ with $a < b$. Similarly, we may define *closed, half open–half closed*, etc., *intervals* in M. We can also represent an arbitrary point $a \in M$ as a closed interval $[a, a]$. By an *interval* in M we shall mean, ambiguously, any of the above types of intervals in M. A subset A of a linearly ordered structure M is *convex* if for all $a, b \in A$ and $c \in M$ whenever $a < c < b$ we have $c \in A$.

This paper concerns the notion of *weak o-minimality* which was initially deeply studied by D. Macpherson, D. Marker and C. Steinhorn in [1]. A *weakly o-minimal structure* is a linearly ordered structure $M = \langle M, =, <, \ldots\rangle$ such that any definable (with parameters) subset of M is a union of finitely many convex sets in M. We recall that such a structure M is said to be *o-minimal* if any definable (with parameters) subset of M is a union of finitely many intervals and points in M.

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Thus, weak o-minimality generalizes the notion of o-minimality. Real closed fields with a proper convex valuation ring [2] provide an important example of weakly o-minimal (not o-minimal) structures.

Let A and B be arbitrary subsets of a linearly ordered structure M. Then the expression $A < B$ means that $a < b$ whenever $a \in A$ and $b \in B$, and $A < b$ means that $A < \{b\}$. For an arbitrary subset A of M we introduce the following notations: $A^+ := \{b \in M \mid A < b\}$ and $A^- := \{b \in M \mid b < A\}$. For an arbitrary one-type p we denote by $p(M)$ the set of realizations of p in M. For an arbitrary tuple $\bar{b} = \langle b_1, b_2, \ldots, b_n\rangle$ of length n we denote by $\bar{b}_i$ the tuple $\langle b_1, b_2, \ldots, b_i\rangle$ for each $1 \le i \le n-1$. If $B \subseteq M$ and E is an equivalence relation on M then we denote by B/E the set of equivalence classes (E–classes) which have representatives in B. If f is a function on M then we denote by $\mathrm{Dom}(f)$ the domain of f. A theory T is said to be *binary* if every formula of T is equivalent in T to a boolean combination of formulas with at most two free variables.

Further throughout the paper we consider an arbitrary complete theory T (if unless otherwise stated), where M is a sufficiently saturated model of T.

Definition 1. Let T be a weakly o-minimal theory, $M \models T$, $A \subseteq M$, $p, q \in S_1(A)$ be non-algebraic. We say that p is not *weakly orthogonal* to q (denoting this by $p \not\perp^w q$) if there exist an A–definable formula $H(x, y)$, $\alpha \in p(M)$ and $\beta_1, \beta_2 \in q(M)$ such that $\beta_1 \in H(M, \alpha)$ and $\beta_2 \notin H(M, \alpha)$.

In other words, p is *weakly orthogonal* to q (denoting this by $p \perp^w q$) if $p(x) \cup q(y)$ has a unique extension to a complete 2-type over A.

Lemma 1. [3] *Let T be a weakly o-minimal theory, $M \models T$, $A \subseteq M$. Then the relation of non weak orthogonality $\not\perp^w$ is an equivalence relation on $S_1(A)$.*

Definition 2. [4] We say that p is *quite orthogonal* to q ($p \perp^q q$) if there is no A–definable bijection $f : p(M) \to q(M)$. We say that a weakly o-minimal theory is *quite o-minimal* if the notions of weak and quite orthogonality coincide for 1-types over arbitrary sets of models of the given theory.

Obviously, any o-minimal theory is quite o-minimal. In [5], countably categorical quite o-minimal theories were completely described, and in [6], algebras of distributions of binary isolating formulas were described for quite o-minimal theories with a small number of countable models.

We extend the definition of the rank of convexity of a formula [7] on arbitrary (non-necessarily definable) sets:

Definition 3. Let T be a weakly o-minimal theory, $M \models T$, $A \subseteq M$. *The rank of convexity of the set A* ($RC(A)$) is defined as follows:

1) $RC(A) = -1$ if $A = \emptyset$.
2) $RC(A) = 0$ if A is finite and non-empty.
3) $RC(A) \ge 1$ if A is infinite.
4) $RC(A) \ge \alpha + 1$ if there exist a parametrically definable equivalence relation $E(x, y)$ and an infinite sequence of elements $b_i \in A$, $i \in \omega$, such that:
   - For every $i, j \in \omega$ whenever $i \ne j$ we have $M \models \neg E(b_i, b_j)$;
   - For every $i \in \omega$, $RC(E(x, b_i)) \ge \alpha$ and $E(M, b_i)$ is a convex subset of A.
5) $RC(A) \ge \delta$ if $RC(A) \ge \alpha$ for all $\alpha \le \delta$, where δ is a limit ordinal.

If $RC(A) = \alpha$ for some α, we say that $RC(A)$ is defined. Otherwise (i.e. if $RC(A)) \ge \alpha$ for all α), we put $RC(A) = \infty$.

The rank of convexity of a formula $\phi(x,\bar{a})$, where $\bar{a} \in M$, is defined as the rank of convexity of the set $\phi(M,\bar{a})$, i.e. $RC(\phi(x,\bar{a})) := RC(\phi(M,\bar{a}))$.

The rank of convexity of an 1-type p is defined as the rank of convexity of the set $p(M)$, i.e. $RC(p) := RC(p(M))$.

In particular, a theory has convexity rank 1 if there are no definable (with parameters) equivalence relations with infinitely many infinite convex classes. Clearly, each o-minimal theory has convexity rank 1.

We say that a theory T has κ (*less than* κ) *countable models* if T has κ (less than κ) countable pairwise non-isomorphic models. We say that a complete theory T is *Ehrenfeucht* if T is not countably categorical and T has finitely many countable models. Recall that a complete theory T is *small* if $|S(T)| = \omega$, where $S(T) = \bigcup_{k=1}^{\omega} S_k(\emptyset)$. It is known that any countable theory having less than 2^ω countable models (in particular, any countably categorical and any Ehrenfeucht theory) is small.

Definition 4. [8, 9] Let T be a complete theory, and $p_1(x_1), \ldots, p_n(x_n) \in S_1(\emptyset)$. A type $q(x_1, \ldots, x_n) \in S_n(\emptyset)$ is said to be a $(p_1, \ldots, p_n)$*-type* if $q(x_1, \ldots, x_n) \supseteq \bigcup_{i=1}^{n} p_i(x_i)$. The set of all $(p_1, \ldots, p_n)$-types of the theory T is denoted by $S_{p_1,\ldots,p_n}(T)$. A countable theory T is said to be *almost* ω*-categorical* if for any types $p_1(x_1), \ldots, p_n(x_n) \in S_1(\emptyset)$ there are only finitely many types $q(x_1, \ldots, x_n) \in S_{p_1,\ldots,p_n}(T)$.

Almost ω-categoricity is closely connected with the notion of Ehrenfeuchtness of a theory. So in [8] it was proved that if T is an almost ω-categorical theory with $I(T,\omega) = 3$ then a dense linear order is interpreted in T. In [10] the authors established almost ω-categoricity of Ehrenfeucht quite o-minimal theories and that Exchange Principle for algebraic closure holds in almost ω-categorical quite o-minimal theories. Recently binarity of both almost ω-categorical quite o-minimal theories and almost ω-categorical weakly o-minimal theories of convexity rank 1 was proved in [11] and [12] respectively.

Definition 5. [13] The *disjunct union* $\bigsqcup_{n\in\omega} \mathcal{M}_n$ of pairwise disjoint structures $\mathcal{M}_n$ with predicate signatures Σ_n, $n \in \omega$, such that for any $n_1, n_2 < \omega$ with $n_1 \neq n_2$ we have $\Sigma_{n_1} \cap \Sigma_{n_2} = \{=, <\}$, is said to be the structure of the signature $\bigcup_{n\in\omega} \Sigma_n \cup \{P_n^{(1)} \mid n \in \omega\}$ with the universe $\bigsqcup_{n\in\omega} M_n$, $P_n = M_n$, and interpretations of predicate symbols from Σ_n, coinciding with their interpretations in the structures $\mathcal{M}_n$, $n \in \omega$. The *disjunct union of theories* T_n of predicate signatures Σ_n, $n \in \omega$, such that for any $n_1, n_2 < \omega$ with $n_1 \neq n_2$ we have $\Sigma_{n_1} \cap \Sigma_{n_2} = \{=, <\}$, is said to be the theory $\bigsqcup_{n\in\omega} T_n \rightleftharpoons \mathrm{Th}\left(\bigsqcup_{n\in\omega} \mathcal{M}_n\right)$, where $\mathcal{M}_n \models T_n$, $n \in \omega$.

Clearly, the disjunct union of theories $\bigsqcup_{n\in\omega} T_n$ does not depend on the choice of models M_n of theories T_n.

Observe that almost ω-categorical weakly o-minimal theories are not Ehrenfeucht in general. As an example of such a theory we can take a disjunct union of countably many copies of the Ehrenfeucht example with three countable models, ordered by type ω. This theory has countably many pairwise weakly orthogonal non-isolated 1-types over $\emptyset$, and therefore has the maximal number of countable models.

Observe also that almost ω-categorical weakly o-minimal theories are not small in general. As an example of such a theory we can consider the structure $M = \langle \mathbb{Q}, <, q\rangle_{q\in\mathbb{Q}}$. Obviously, $Th(M)$ is o-minimal and has continuum of 1-types over $\emptyset$, i.e. it is not small.

At last, observe that there exist almost ω-categorical weakly o-minimal theories having a non-algebraic type $p \in S_1(\emptyset)$ with an infinite convexity rank. As an example of such a theory we can consider any ω-categorical 2-indiscernible non-3-indiscernible weakly o-minimal structure constructed in [14].

Recently, Vaught's conjecture was confirmed for some classes of complete theories: for quite o-minimal theories in [15], for weakly o-minimal theories of convexity rank 1 in [16], and for weakly o-minimal theories of finite convexity rank in [17]. Also, a criterion for the countable spectrum to be maximal in small binary quite o-minimal theories of finite convexity rank was obtained in [18]. Here we prove that weakly o-minimal theories of finite convexity rank having less than 2^ω countable models are almost ω-categorical.

2. Main Theorem

Further we need the notion of a (p, q)-splitting formula, which was introduced in [19] for non-algebraic isolated 1-types. Let $A \subseteq M$, $p, q \in S_1(A)$ be non-algebraic, $p \not\perp^w q$. Extending the notion of a (p, q)-splitting formula to the non-isolated case, we say that an A-definable formula $\phi(x, y)$ is a (p, q)*–splitting formula* if there is $a \in p(M)$ such that

$$\phi(a, M) \cap q(M) \neq \emptyset, \neg\phi(a, M) \cap q(M) \neq \emptyset,$$

$$\phi(a, M) \cap q(M) \text{ is convex, and } [\phi(a, M) \cap q(M)]^- = [q(M)]^-.$$

If $\phi_1(x, y)$, $\phi_2(x, y)$ are (p, q)-splitting formulas we say that $\phi_1(x, y)$ is *less than* $\phi_2(x, y)$ if there exists $a \in p(M)$ such that

$$\phi_1(a, M) \cap q(M) \subset \phi_2(a, M) \cap q(M).$$

It is obvious that if $p, q \in S_1(A)$ are non-algebraic and $p \not\perp^w q$, then there exists a (p, q)-splitting formula, and the set of all (p, q)-splitting formulas is linearly ordered. It is also obvious that for every (p, q)-splitting formula $\phi(x, y)$ the function $f(x) := \sup \phi(x, M)$ is not constant on $p(M)$.

Lemma 2. [17] *Let T be a weakly o-minimal theory having less than 2^ω countable models, $M \models T$, $p_1, p_2 \in S_1(\emptyset)$ be non-algebraic, $p_1 \not\perp^w p_2$. Then there are only finitely many (p_1, p_2)-splitting formulas.*

Theorem 1. *Any weakly o-minimal theory of finite convexity rank having less than 2^ω countable models is almost ω-categorical.*

Proof. Let T be a weakly o-minimal theory of finite convexity rank having less than 2^ω countable models, and let M be a countable saturated model for T.

We prove by induction on $n \geq 2$ that for any family of non-algebraic types $p_1, \dots, p_n \in S_1(\emptyset)$ there exist only finitely many $(p_1, \dots, p_n)$-types.

Step $n = 2$.

Case 1. $p_1 \perp^w p_2$. Then the set of formulas $p_1(x_1) \cup p_2(x_2)$ determines a complete 2-type over $\emptyset$, and consequently, the number of (p_1, p_2)-types is equal to 1.

Case 2. $p_1 \not\perp^w p_2$. Then by Lemma 2 there are only finitely many (p_1, p_2)-splitting formulas. Denote them by $\Phi_1(x, y), \Phi_2(x, y), \ldots, \Phi_s(x, y)$ for some $s < \omega$, and suppose that for any $a \in p_1(M)$ we have

$$\Phi_1(a, M) \cap p_2(M) \subset \Phi_2(a, M) \cap p_2(M) \subset \ldots \subset \Phi_s(a, M) \cap p_2(M).$$

Then we assert that uniquely possible extensions of the set $p_1(x_1) \cup p_2(x_2)$ are the following:

$$p_1(x_1) \cup p_2(x_2) \cup \{\Phi_1(x_1, x_2)\}$$
$$p_1(x_1) \cup p_2(x_2) \cup \{\Phi_2(x_1, x_2) \wedge \neg\Phi_1(x_1, x_2)\}$$
$$\ldots\ldots\ldots$$
$$p_1(x_1) \cup p_2(x_2) \cup \{\Phi_s(x_1, x_2) \wedge \neg\Phi_{s-1}(x_1, x_2)\}$$
$$p_1(x_1) \cup p_2(x_2) \cup \{\neg\Phi_s(x_1, x_2)\}$$

Indeed, we show that $p_1(x_1) \cup p_2(x_2) \cup \{\Phi_1(x_1, x_2)\}$ determines a complete type over $\emptyset$. If this is not true then there is an $\emptyset$-definable formula $\Psi(x_1, x_2)$ such that

$$M \models \Phi_1(a_1, a_2) \wedge \Psi(a_1, a_2) \wedge \Phi_1(a_1', a_2') \wedge \neg\Psi(a_1', a_2')$$

for some $a_1, a_1' \in p_1(M), a_2, a_2' \in p_2(M)$.

Show that there is $a_2'' \in p_2(M)$ such that

$$M \models \Phi_1(a_1, a_2'') \wedge \neg\Psi(a_1, a_2'').$$

Assume the contrary: for any $a_2'' \in p_2(M)$ with $\Phi_1(a_1, a_2'')$ we have $\Psi(a_1, a_2'')$.

Let $q(x_1, x_2) := tp(\langle a_1, a_2\rangle/\emptyset)$. Consider for any $\theta(y) \in p_2$ and $\phi(x, y) \in q$ the following formula:

$$K_{\theta,\phi}(x) := \forall y[\theta(y) \wedge \phi(x, y) \to \Psi(x, y)].$$

Since we assumed that for any $a_2'' \in p_2(M)$ with $\Phi_1(a_1, a_2'')$ we have $\Psi(a_1, a_2'')$, we have $M \models K_{\theta,\Phi_1}(a_1)$ for any $\theta(y) \in p_2$. Since $a_1, a_1' \in p_1(M)$, we have $M \models K_{\theta,\Phi_1}(a_1')$ for any $\theta(y) \in p_2$. This contradicts $\neg\Psi(a_1', a_2')$.

Consequently, there is $a_2'' \in p_2(M)$ such that $M \models \Phi_1(a_1, a_2'') \wedge \neg\Psi(a_1, a_2'')$.

Without loss of generality, suppose $p_1(M) < p_2(M)$ and $a_2 < a_2''$. Also, by weak o-minimality we can assume that $\Psi(a_1, M)$ is convex.

Consider the following formula:

$$\Theta(a_1, y) := a_1 \leq y \wedge \exists t[\Psi(a_1, t) \wedge y \leq t].$$

Obviously, $\Theta(x, y)$ is a (p_1, p_2)-splitting formula and

$$\Theta(a_1, M) \cap p_2(M) \subset \Phi_1(a_1, M) \cap p_2(M).$$

The last contradicts that $\Phi(x, y)$ is the least (p_1, p_2)-splitting formula. Similarly, we can prove that the remaining extensions of the set $p_1(x_1) \cup p_2(x_2)$ determine a complete type over $\emptyset$.

Thus, there are only finitely many (p_1, p_2)-types.

Suppose that we already have established finiteness of the number of $(p_1, \ldots, p_k)$-types for all $k \leq n$ and we prove it for $n + 1$.

Step $n + 1 > 2$. Consider arbitrary non-algebraic types $p_1, \ldots, p_n, p_{n+1} \in S_1(\emptyset)$. By Theorem 4.4 of [17] T is binary, and therefore we have only the following cases.

Case 1. $p_{n+1} \perp^w p_i$ for every $1 \leq i \leq n$. In this case the number of $(p_1, \ldots, p_n, p_{n+1})$-types coincides with the number of $(p_1, \ldots, p_n)$-types.

Case 2. $p_{n+1} \not\perp^w p_i$ for every $1 \leq i \leq n$. In this case by Lemma 1 the types $p_1, \ldots, p_n, p_{n+1}$ are pairwise non-weakly orthogonal. By the induction hypothesis the number of $(p_1,\ldots,p_n)$-types is finite. Denote this number by $S_{p_1,\cdots,p_n}$. Since $p_i \not\perp^w p_{n+1}$ for every $1\le i\le n$, by Lemma 2 there exist only finitely many (p_i,p_{n+1})-splitting formulas. Denote this number by m_i, and let $\{\Phi_1^i(x,y), \Phi_2^i(x,y), \ldots, \Phi_{m_i}^i(x,y)\}$ be a complete list of (p_i,p_{n+1})-splitting formulas so that for any $a_i\in p_i(M)$ the following holds:

$$\Phi_1^i(a_i,M)\cap p_{n+1}(M)\subset \Phi_2^i(a_i,M)\cap p_{n+1}(M)\subset\ldots\subset \Phi_{m_i}^i(a_i,M)\cap p_{n+1}(M).$$

Obviously, for every $1\le i<j\le n$ and for any $1\le s\le m_i$, $1\le l\le m_j$ we have that either

$$\Phi_s^i(a_i,M)\cap p_{n+1}(M)\subseteq \Phi_l^j(a_j,M)\cap p_{n+1}(M)$$
$$\text{or } \Phi_l^j(a_j,M)\cap p_{n+1}(M)\subset \Phi_s^i(a_i,M)\cap p_{n+1}(M).$$

Consequently, (if necessary) there is some re-notation of all (p_1,p_{n+1})-splitting formulas $\Phi_1^1(x,y)$, …, $\Phi_{m_1}^1(x,y)$, (p_2,p_{n+1})-splitting formulas $\Phi_1^2(x,y)$, …, $\Phi_{m_2}^2(x,y)$, …, and (p_n,p_{n+1})-splitting formulas $\Phi_1^n(x,y)$, …, $\Phi_{m_n}^n(x,y)$ by $F_1(x,y)$, $F_2(x,y)$, …, $F_{m_1+m_2+\ldots+m_n}(x,y)$ so that

$$F_1(c_1,M)\cap p_{n+1}(M)\subseteq F_2(c_2,M)\cap p_{n+1}(M)\subseteq$$
$$\ldots\subseteq F_{m_1+m_2+\ldots+m_n}(c_{m_1+m_2+\ldots+m_n},M)\cap p_{n+1}(M),$$

where for any $1\le k\le m_1+m_2+\ldots+m_n$ there is $1\le i_k\le n$ such that $c_k=a_{i_k}$.

Take an arbitrary $q(x_1,x_2,\ldots,x_n)\in S_{p_1,\ldots,p_n}(T)$. We assert that there exist at most $m_1+m_2+\ldots+m_n$ extensions of the set of formulas $q(x_1,x_2,\ldots,x_n)\cup p(x_{n+1})$:

$$q(x_1,x_2,\ldots,x_n)\cup p(x_{n+1})\cup\{F_1(x^{(1)},x_{n+1})\}$$
$$q(x_1,x_2,\ldots,x_n)\cup p(x_{n+1})\cup\{F_2(x^{(2)},x_{n+1})\wedge\neg F_1(x^{(1)},x_{n+1})\}$$
$$\cdots\cdots\cdots$$
$$q(x_1,x_2,\ldots,x_n)\cup p(x_{n+1})\cup\{F_{m_1+m_2+\ldots+m_n}(x^{(m_1+m_2+\ldots+m_n)},x_{n+1})\wedge$$
$$\neg F_{m_1+m_2+\ldots+m_n-1}(x^{(m_1+m_2+\ldots+m_n-1)},x_{n+1})\}$$
$$q(x_1,x_2,\ldots,x_n)\cup p(x_{n+1})\cup\{\neg F_{m_1+m_2+\ldots+m_n}(x^{(m_1+m_2+\ldots+m_n)},x_{n+1})\},$$

where for every $1\le i\le m_1+m_2+\ldots+m_n$ there exists $1\le i_k\le n$ such that $x^{(i)}=x_{i_k}$.

Show that $q(x_1,x_2,\ldots,x_n)\cup p(x_{n+1})\cup\{F_1(x^{(1)},x_{n+1})\}$ determines a complete type over $\emptyset$.

Without loss of generality, suppose that $x^{(1)}=x_1$, i.e. for every $2\le j\le n$

$$\Phi_1^1(a_1,M)\cap p_{n+1}(M)\subseteq \Phi_1^j(a_j,M)\cap p_{n+1}(M).$$

Assume the contrary: there exists an $\emptyset$-definable formula $\psi(x_1,\ldots,x_n,x_{n+1})$ such that both

$$q(x_1,x_2,\ldots,x_n)\cup p(x_{n+1})\cup\{F_1(x^{(1)},x_{n+1})\}\cup\{\psi(x_1,\ldots,x_n,x_{n+1})\}$$
$$\text{and } q(x_1,x_2,\ldots,x_n)\cup p(x_{n+1})\cup\{F_1(x^{(1)},x_{n+1})\}\cup\{\neg\psi(x_1,\ldots,x_n,x_{n+1})\}$$

are consistent.

Consequently, there are $a_1,a_1'\in p_1(M), a_2,a_2'\in p_2(M),\ldots,a_n,a_n'\in p_n(M)$ and $a_{n+1},a_{n+1}'\in p_{n+1}(M)$ such that $\langle a_1,\ldots,a_n\rangle,\langle a_1',\ldots,a_n'\rangle\in q(M^n)$ and

$$M\models F_1(a_1,a_{n+1})\wedge F_1(a_1',a_{n+1}')\wedge\psi(a_1,\ldots,a_n,a_{n+1})\wedge\neg\psi(a_1',\ldots,a_n',a_{n+1}').$$

By binarity of T we have that

$$\psi(x_1,\ldots,x_n,x_{n+1})\equiv\vee_{l=1}^s[\wedge_{1\le i<j\le n+1}\psi_{ij}^l(x_i,x_j)].\qquad(*)$$

Since $tp(\langle a_1, \ldots, a_n\rangle/\emptyset) = tp(\langle a'_1, \ldots, a'_n\rangle/\emptyset)$, for every $1 \le l \le s$ the following holds:

$$M \models \wedge_{1\le i<j\le n}\psi^l_{ij}(a_i, a_j) \Leftrightarrow M \models \wedge_{1\le i<j\le n}\psi^l_{ij}(a'_i, a'_j).$$

Obviously, by $(*)$ we have

$$\neg\psi(x_1, \ldots, x_n, x_{n+1}) \equiv \wedge^s_{l=1}[\vee_{1\le i<j\le n+1}\neg\psi^l_{ij}(x_i, x_j)], \text{ i.e.}$$

$$\neg\psi(x_1, \ldots, x_n, x_{n+1}) \equiv \wedge^s_{l=1}[\vee_{1\le i<j\le n}\neg\psi^l_{ij}(x_i, x_j) \vee \vee^n_{m=1}\neg\psi^l_{m,n+1}(x_m, x_{n+1})],$$

and consequently,

$$\neg\psi(x_1, \ldots, x_n, x_{n+1}) \equiv \wedge^s_{l=1}[\wedge_{1\le i<j\le n}\psi^l_{ij}(x_i, x_j) \to \vee^n_{m=1}\neg\psi^l_{m,n+1}(x_m, x_{n+1})].$$

By similar arguments as in Step $n = 2$ (Case 2) we can show that there is $a''_{n+1} \in p_{n+1}(M)$ such that

$$M \models F_1(a_1, a''_{n+1}) \wedge \neg\psi(a_1, \ldots, a_n, a''_{n+1}).$$

Consequently, there are $1 \le l \le s$ and $1 \le j \le n$ such that $M \models \neg\psi^l_{j,n+1}(a_j, a''_{n+1})$. Then we obtain a contradiction with the condition that $\Phi^j_1(x_j, x_{n+1})$ is the least (p_j, p_{n+1})-splitting formula.

Similar arguments show that each of the extensions of the set of formulas $q(x_1, \ldots, x_n, x_{n+1}) \cup p(x_{n+1})$ determines a complete type over $\emptyset$.

Case 3. $p_{n+1} \not\perp^w p_i$ and $p_{n+1} \perp^w p_j$ for some distinct $1 \le i, j \le n$. Then there exist (if necessary) some renumbering of the set $\{p_1, \ldots, p_n\}$ and some $1 \le k < n$ such that $p_{n+1} \perp^w p_j$ for all $1 \le j \le k$ and $p_{n+1} \not\perp^w p_l$ for all $k+1 \le l \le n$. By induction hypothesis both the number of $(p_1, \ldots, p_k, p_{n+1})$-types and the number of $(p_{k+1}, \ldots, p_n, p_{n+1})$-types are finite, and the number of $(p_1, \ldots, p_k, p_{n+1})$-types coincides with the number of $(p_1, \ldots, p_k)$-types. Denote these numbers by $S_{p_1,\cdots,p_k}$ and $S_{p_{k+1},\cdots,p_n,p_{n+1}}$ respectively. Then we assert that the number of $(p_1, \ldots, p_n, p_{n+1})$-types is at most $S_{p_1,\cdots,p_k} \times S_{p_{k+1},\cdots,p_n,p_{n+1}}$, i.e. it is also finite. □

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