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MULTISCALE ANALYSIS OF A MODEL PROBLEM OF A THERMOELASTIC BODY WITH THIN INCLUSIONS

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Abstract. A model statical problem for a thermoelastic body with thin inclusions is studied. This problem incorporates two small positive parameters δ and ε, which describe the thickness of an individual inclusion and the distance between two neighboring inclusions, respectively. Relying on the variational formulation of the problem, by means of the modern methods of asymptotic analysis, we investigate the behavior of solutions as δ and ε tend to zero. As the result, we construct two models corresponding to the limiting cases. At first, as $\delta \to 0$, we derive a limiting model in which inclusions are thin (of zero diameter). Then, from this limiting model, as $\varepsilon \to 0$, we derive a homogenized model, which describes effective behavior on the macroscopic scale, i.e., on the scale where there is no need to take into account each individual inclusion. The limiting passage as $\varepsilon \to 0$ is based on the use of homogenization theory. The final section of the article presents a series of numerical experiments for the established limiting models.

Key words: linear thermoelasticity, composite material, thin inclusion, homogenization, two-scale convergence, generalized solution, numerical experiment.

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Introduction

The properties and behavior of real mechanisms and constructions significantly depend on the heterogeneity of the materials from which they are made. Composite materials stand out among other heterogeneous materials: they are composed of dissimilar components that, when combined, result in a new material with improved characteristics. The family of composites includes fibrous composites — bodies consisting of a binder matrix stitched with reinforcing fibers (threads), which actually behave like thin inclusions incorporated into the matrix. The constantly growing demand for fibrous composites over the past decades motivates the development and implementation of mathematical tools for their effective design and description in the framework of high-level mathematical models.

The present article is devoted to a model problem of description of a thermoelastic body incorporating thin deformable inclusions. In this introduction, we briefly explain the structure of the article and the essence of the study.

A two-dimensional model problem that describes statical equilibrium of a linearly thermoelastic body is the starting point of the research. The peculiarity of this problem is that we distinguish a rectangular subdomain of a small dimensionless thickness $\delta > 0$ in the entire domain occupied by the thermoelastic body, so that this subdomain corresponds to an inclusion whose thermomechanical properties drastically differ from those of the rest of the body. Value δ plays the role of a small parameter in this formulation. The original problem, called **Problem A**, is set up in variational form, and its well-posedness is established for all fixed values $\delta > 0$, see in Secs. 1.1 and 1.2. Also, in Sec. 1.1 we give the differential formulation of Problem A. This formulation is a boundary value problem for the system of two elliptic equations. We call it **Problem A-diff**, and Problem A is its weak formulation in the sense of distributions.

In Secs. 1.3 and 1.4 we study the asymptotic behavior of solutions of Problem A as δ tends to zero. Relying on the techniques, developed earlier in the articles [14] and [31], the limiting passage as $\delta \to 0$ is fulfilled and the variational formulation of the limiting problem is constructed. This variational formulation is called **Problem B** and its equivalent in the sense of distributions differential formulation is called **Problem B-diff**. Problem B (equivalently, Problem B-diff) describes statical equilibrium of the thermoelastic body with one thin inclusion. In this problem, the governing equations for the inclusion are set up on the subset of co-dimension one (with respect to the domain, occupied by the entire body). Worth noticing that similar formulations for thin inclusions in elastic bodies have been intensively studied recently, see, for example, articles [13, 17, 19–21, 23–25, 30, 32, 35].

In Sec. 1.5 we make a simple but very important observation that all considerations of Secs. 1.2-1.4 are naturally generalized to the case of an arbitrary finite number of inclusions: this generalization of Problem B is formulated in Sec. 2.1 and is called **Problem** $\mathbf{B}_{\boldsymbol{\varepsilon}}$. The problem comprising many thin inclusions incorporates the small dimensionless parameter $\varepsilon > 0$, which describes the distance between two neighboring inclusions. Despite the fact that the formulation of the problem for the case of multiple inclusions is mathematically well-posed, its application in practice is very difficult, since for small values of the parameter ε (that is, for a large number of inclusions), the thermomechanical properties rapidly oscillate. This circumstance leads to the idea to conduct a study of Problem B_ε as $\varepsilon \to 0$ by the homogenization method with the aim to derive the homogenized model whose solution is close to solutions of Problem B_ε for small $\varepsilon > 0$ and at the same time whose form is rather simple and does not require to consider each inclusion separately. This study is fulfilled in Secs. 2.2 and 2.3. It consists in performing the homogenization procedure, i.e., carrying out and rigorously justifying the passage to the limit as $\varepsilon \to 0$. The corresponding constructions are based on the original Allaire-Nguetseng method of two-scale convergence [5,29] and its version for homogenization on curves and surfaces, proposed by G. Allaire, A. Damlamian, and U. Hornung in [6]. As the result of the homogenization procedure, the well-posed homogenized model, called **Problem H**, is derived. Its solution is the L^2-strong limit of the family of solutions to Problem B_ε as $\varepsilon \to 0$. The differential form of Problem H, called **Problem H-diff**, is the boundary value problem for the system of thermoelasticity equations with homogenized coefficients of elasticity, heat conductivity and thermal expansion. These homogenized coefficients depend on microscopic data, i.e., on physical characteristics of the inclusions and the binder matrix.

The final section of the article (Sec. 3) presents a series of numerical experiments for the established limiting models. The results of these experiments confirm the correctness of the theoretical study, and also clearly demonstrate that, in the numerical calculation, it is advantageous to use Problems B and H in comparison with Problems A and B_ε, respectively.

From a general point of view, the presence of two independent small parameters δ and ε in the model under study and the fact that the passages to the limit along them are performed one after another indicate the presence of three well-separated scales: the characteristic scale of the body as a whole is the large (macroscopic) scale, the characteristic scale of the cross section of the layer between two neighboring inclusions is the intermediate scale, and the characteristic scale of the cross section of an individual inclusion is the smallest (microscopic) scale. In this view, the problem considered in the article is indeed a problem of multiscale analysis, as noted in the title of the article.

Generally speaking, various methods of multiscale modeling are widely used in the design of composite materials (in a broader context: reinforced media) and the study of their effective characteristics is described in detail, for example, in the classical monographs by V. V. Bolotin, Yu. N. Novichkov [9], A. M. Skudra, F. Ya. Bulavs [36], and R. M. Christensen [11] and in the recently published monographs by R. M. Jones [18] (the newest edition) and S. K. Golushko and Yu. V. Nemirovsky [16] (see also survey [15]). In the present article, we apply two original methods in this direction. As noted above, the passage to the limit as $\delta \to 0$ is fulfilled and mathematically strictly justified in Sec. 1 using a new technique developed in detail in the articles [14,31]. This technique makes it possible to model thin inclusions in elastic bodies by passing to the limit as the volume (or area, in the two-dimensional case) of the cross-section of the original 'bulk' inclusion tends to zero. This technique goes back to the classical works of E. Sanchez-Palencia [33], [34, chapter XIII]. Worth to note that by now there are many works in which the derivation of models with thin inclusions from models with 'bulk' inclusions is carried out by formal methods of mechanics, for example, by the method of formal asymptotic expansions. As well, there is a number of works in which rigorous mathematical justification of the asymptotics is provided, see, for example, articles [7, 8] and references therein. The results of Sec. 1 of the present article make an additional contribution to this theory.

In Sec. 2 the method of two-scale convergence is used for the passage to the limit as $\varepsilon \to 0$. This method arose from the pioneering works [5,29] and in the last thirty years has shaped in a consistent theory and received some modifications and generalizations. To date, a huge array of scientific texts — articles, monographs, textbooks — has been created on the homogenization of bulk inclusions, see, for example, [10,12,28]; but the problems of averaging thin inclusions have been studied much poorer. In addition to the already mentioned work by G. Allaire, A. Damlamian, and U. Hornung [6], we can also mention the series of articles by V. V. Zhikov [37,38], A. Ainouz [1–4] and V. A. Kovtunenko and A. V. Zubkova [26,27]. In [6] the authors successfully applied the method of two-scale convergence on surfaces to the study of some homogenization problems in poromechanics. In [37,38] a modification of the Allaire-Nguetseng two-scale convergence method for Radon measures is constructed and with its help a number of questions of the theory of elasticity of singular and fine structures are studied. In [1–4], with the help of the technical tools from [6], some new averaged models for filtration flows of a liquid with an admixture are built with account of the effects of molecular diffusion and sedimentation. In [26,27], using the methods of scale transformation and Zhikov-Kozlov-Oleinik asymptotic correctors, the authors succeeded to homogenize surfaces in the framework of homogenization problems for the system of Poisson-Nernst-Planck equations of nonlinear diffusion describing electrochemical processes in a porous medium. At the same time, the homogenization problems for composites with thin inclusions were not studied by the two-scale convergence methods so far.

Concluding this introduction, firstly, we note that any physically meaningful model of a thermoelastic composite reinforced with thin fiber-inclusions is significantly complicated: in it, the fields of the sought quantities are represented by high-rank tensors connected by complex systems of differential equations. The model problems considered in this article are simpler. At the same time, they convey the main feature of fibrous composites, which is the difference in the scale of the composite as a whole, the thickness of a single fiber-inclusion, and the distance between neighboring inclusions. Thus, the authors hope that the approach used here in the article for model problems will be further developed and will be fruitful for more complex models of fibrous composites used in practice. Secondly, we make a technical note that the article contains many notations, which is a usual thing in studies of homogenization problems, so for the convenience of readers, an appendix with a fairly complete list of notations is placed before the list of references.

1. Justification of the model with thin inclusions

Synopsis. This section is devoted to justification of well-posedness of the antiplane shear problem describing the statical state of a thermoelastic body with thin inclusions. As the starting point of the research, we formulate and consider the statical problem for an isotropic nonhomogeneous thermoelastic body consisting of the two two-dimensional components with distinct thermomechanical properties. Primarily, we set up this problem as the variational formulation such that the solution of this formulation is a weak generalized solution of the corresponding boundary value problem for the system of partial differential equations of elasticity with account of heat transfer. This formulation incorporates a small parameter δ: we assume that one of the components occupies a rectangular domain Ω_m^δ of a small height δ, so that the two parts Ω_+^δ and Ω_-^δ of the second component are adjacent to Ω_m^δ at the top and bottom, as shown in Fig. 1. As well, we assume that the thermomechanical characteristics of the component occupying Ω_m^δ also depend on δ in a special way. The form of dependence is clarified in Sec. 1.3.

Further in this section, the precise formulation of the problem is given and its unique solvability for fixed values $\delta > 0$ is proved. After this, the limiting passage in the family of solutions is fulfilled as δ tends to zero, and the resulting limiting problem is analyzed. As will be seen, the solution of the limiting problem exactly corresponds to the statical state of a thermoelastic body with a thin inclusion.

1.1. **Geometric structure of the body.** Let us give a detailed description of geometric structure of the thermoelastic body under consideration, followed by a variational formulation of the equilibrium problem for it.

Let $\Omega \subset \mathbb{R}_y^2$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. Set the Cartesian coordinate system Oy_1y_2 so that Ω is intersected by Oy_1 axis along the segment

$$\gamma = \big(\Omega \cap \{y_2 = 0\}\big) = \big\{(y_1, y_2) \in \mathbb{R}^2 \colon\ 0 < y_1 < y_1^*,\ y_2 = 0\big\}, \quad y_1^* = \text{const} > 0,$$

and is divided by this intersection into two non-empty subdomains $\Omega_\pm$ with Lipschitz boundaries $\partial\Omega_\pm$. Introduce the following geometrical objects associated with $\Omega_\pm$ and γ:

$$\begin{aligned}
&\Omega_m^\delta = \{0 < y_1 < y_1^*\} \times \{-\delta/2 < y_2 < \delta/2\},\\
&\Omega_\pm^\delta = \big\{(y_1, y_2) \in \mathbb{R}^2 \colon\ (y_1, y_2 \mp \delta/2) \in \Omega_\pm\big\},\\
&\gamma_\pm^\delta = \gamma \pm (0, \delta/2) =\\
&\qquad = \big\{(y_1, y_2) \in \mathbb{R}^2 \colon\ (y_1, y_2) \in \big(\{0 < y_1 < y_1^*\} \times \{y_2 = \pm\delta/2\}\big)\big\},
\end{aligned}$$

i.e., Ω_m^δ is the rectangle located symmetrically relative to the abscissa axis, Ω_+^δ is the translation of the subdomain Ω_+ upwards parallel to the ordinate axis by height $\delta/2$ and, correspondingly, Ω_-^δ is the translation of the subdomain Ω_- downwards parallel to the ordinate axis also by $\delta/2$. Clearly, the union of the sets Ω_m, $\Omega_\pm^\delta$, and $\gamma_\pm^\delta$ is a domain with a Lipschitz boundary. Denote it by Ω^δ:

$$\Omega^\delta = \Omega_+^\delta \cup \Omega_-^\delta \cup \Omega_m^\delta \cup \gamma_+^\delta \cup \gamma_-^\delta.$$

Also, we define the sets $\Gamma_\pm^\delta = \partial\Omega^\delta \cap \partial\Omega_\pm^\delta$, which are the parts of the outer boundary $\partial\Omega^\delta$ of the domain Ω^δ.

Resuming, we have that the plain two-component thermoelastic body under consideration occupies the domain Ω^δ, with one component located in the rectangle Ω_m^δ. The second component consists of two parts Ω_+^δ and Ω_-^δ adjacent to the first component at the top and bottom, see on Fig. 1.

1.2. **Basic formulation and its solvability.** Let us proceed to formulation of the equilibrium problem for a body with the geometry described above.

Introduce the functional space

$$\begin{aligned}
V^\delta = \big\{&(\boldsymbol{u}, \boldsymbol{\theta}) \in [H_\Gamma^1(\Omega_+^\delta) \times H_\Gamma^1(\Omega_-^\delta) \times H^1(\Omega_m^\delta)]^2 \colon\\
&\boldsymbol{u} = (u_+, u_-, u_m), \quad \boldsymbol{\theta} = (\theta_+, \theta_-, \theta_m); \quad u_\pm = u_m,\ \theta_\pm = \theta_m \text{ on } \gamma_\pm^\delta\big\},
\end{aligned}$$

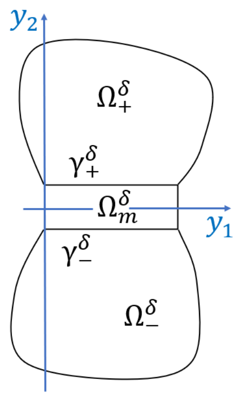

FIG. 1. Geometric structure of the body

where

$$H^1_\Gamma(\Omega^\delta_\pm) = \{v \in H^1(\Omega^\delta_\pm) \colon\ v = 0 \text{ on } \Gamma^\delta_\pm\},$$

and by $H^1(\mathcal{O})$, $\mathcal{O} := \Omega^\delta_+, \Omega^\delta_-, \Omega^\delta_m$, we denote the Sobolev space equipped with the standard norm $\|w\|_{H^1(\mathcal{O})} = \big(\|w\|^2_{L^2(\mathcal{O})} + \|\nabla_y w\|^2_{L^2(\mathcal{O})}\big)^{1/2}$.

For $(\boldsymbol{u}, \boldsymbol{\theta}), (\boldsymbol{v}, \boldsymbol{\vartheta}) \in V^\delta$ define the bilinear forms

$$b_1^\delta(\boldsymbol{u}, \boldsymbol{v}) = \int\limits_{\Omega^\delta_+} a_\delta \nabla_y u_+ \cdot \nabla_y v_+ \, d\boldsymbol{y} + \int\limits_{\Omega^\delta_-} a_\delta \nabla_y u_- \cdot \nabla_y v_- \, d\boldsymbol{y} + \int\limits_{\Omega^\delta_m} a_\delta \nabla_y u_m \cdot \nabla_y v_m \, d\boldsymbol{y},$$

$$b_2^\delta(\boldsymbol{\theta}, \boldsymbol{v}) = \int\limits_{\Omega^\delta_+} \big(a_\delta \beta_\delta \theta_+\big)\, \mathbf{1} \cdot \nabla_y v_+ \, d\boldsymbol{y} + \int\limits_{\Omega^\delta_-} \big(a_\delta \beta_\delta \theta_-\big)\, \mathbf{1} \cdot \nabla_y v_- \, d\boldsymbol{y} + \int\limits_{\Omega^\delta_m} \big(a_\delta \beta_\delta \theta_m\big)\, \mathbf{1} \cdot \nabla_y v_m \, d\boldsymbol{y},$$

$$b_3^\delta(\boldsymbol{\theta}, \boldsymbol{\vartheta}) = \int\limits_{\Omega^\delta_+} \lambda_\delta \nabla_y \theta_+ \cdot \nabla_y \vartheta_+ \, d\boldsymbol{y} + \int\limits_{\Omega^\delta_-} \lambda_\delta \nabla_y \theta_- \cdot \nabla_y \vartheta_- \, d\boldsymbol{y} + \int\limits_{\Omega^\delta_m} \lambda_\delta \nabla_y \theta_m \cdot \nabla_y \vartheta_m \, d\boldsymbol{y}.$$

Here and further functions $a_\delta, \beta_\delta, \lambda_\delta \in L^\infty(\Omega^\delta)$ are given and, moreover, a_δ and λ_δ are uniformly positive, i.e., $a_\delta, \lambda_\delta \geq \text{const} > 0$ in Ω^δ. Also, $\mathbf{1} = (1,1)^T$ is the vector in $\mathbb{R}^2$ with the components equal to unity. We use the standard notations for the gradient operator $\nabla_y = (\partial_{y_1}, \partial_{y_2})^T$ and for the scalar product in $\mathbb{R}^2$:

$$\boldsymbol{\phi} \cdot \boldsymbol{\psi} = \phi_1 \psi_1 + \phi_2 \psi_2, \quad \forall\, \boldsymbol{\phi}, \boldsymbol{\psi} \in \mathbb{R}^2.$$

In particular,

$$\nabla_y u \cdot \nabla_y v = \partial_{y_1} u\, \partial_{y_1} v + \partial_{y_2} u\, \partial_{y_2} v, \quad \mathbf{1} \cdot \nabla_y v = \partial_{y_1} v + \partial_{y_2} v.$$

Next, introduce the linear functionals $\boldsymbol{v} \mapsto l_1^\delta(\boldsymbol{v})$ and $\boldsymbol{\vartheta} \mapsto l_2^\delta(\boldsymbol{\vartheta})$ by the formulas

$$l_1^\delta(\boldsymbol{v}) = \int\limits_{\Omega_+^\delta} \tilde{f} v_+ \, d\boldsymbol{y} + \int\limits_{\Omega_-^\delta} \tilde{f} v_- \, d\boldsymbol{y}, \quad l_2^\delta(\boldsymbol{\vartheta}) = \int\limits_{\Omega_+^\delta} \tilde{g} \vartheta_+ \, d\boldsymbol{y} + \int\limits_{\Omega_-^\delta} \tilde{g} \vartheta_- \, d\boldsymbol{y},$$

where $\tilde{f}, \tilde{g} \in L^2(\Omega^\delta)$ are given functions vanishing on Ω_m^δ.

For every fixed $\delta > 0$ we consider the following variational problem.

Problem A. *Find a pair of vector-functions* $(\boldsymbol{u}, \boldsymbol{\theta}) \in V^\delta$ *satisfying the integral equalities*

$$b_1^\delta(\boldsymbol{u}, \boldsymbol{v}) - b_2^\delta(\boldsymbol{\theta}, \boldsymbol{v}) = l_1^\delta(\boldsymbol{v}), \tag{1}$$

$$b_3^\delta(\boldsymbol{\theta}, \boldsymbol{\vartheta}) = l_2^\delta(\boldsymbol{\vartheta}), \tag{2}$$

for all pairs of test vector-functions $(\boldsymbol{v}, \boldsymbol{\vartheta}) \in V^\delta$.

Remark 1. Problem A admits a more simple and explicit form as follows:

find a pair of functions $(u, \theta) \in H^\delta \times H^\delta$ *such that*

$$\int\limits_{\Omega^\delta} a_\delta \nabla_y u \cdot \nabla_y v \, d\boldsymbol{y} - \int\limits_{\Omega^\delta} \big(a_\delta \beta_\delta \theta\big) \, \mathbf{1} \cdot \nabla_y v \, d\boldsymbol{y} = \int\limits_{\Omega^\delta} \tilde{f} v \, d\boldsymbol{y}, \tag{3}$$

$$\int\limits_{\Omega^\delta} \lambda_\delta \nabla_y \theta \cdot \nabla_y \vartheta \, d\boldsymbol{y} = \int\limits_{\Omega^\delta} \tilde{g} \vartheta \, d\boldsymbol{y}, \tag{4}$$

for all pairs of test scalar functions $(v, \vartheta) \in H^\delta \times H^\delta$.

Notation 1. Here by H^δ we denote the space

$$\big\{v \in H^1(\Omega^\delta) \colon \; v = 0 \text{ on } \Gamma_+^\delta \cup \Gamma_-^\delta\big\}.$$

The components of vectors $\boldsymbol{u} = (u_+, u_-, u_m)$ and $\boldsymbol{\theta} = (\theta_+, \theta_-, \theta_m)$ are the restrictions of the functions u and θ to the subdomains Ω_+^δ, Ω_-^δ, and Ω_m^δ, respectively.

As compared to the formulation given in Remark 1, the original formulation of Problem A looks more cumbersome. However, the original formulation appears to be more convenient for studying and therefore it is used as the basic one for analysis of the limiting passage as $\delta \to 0+$.

Remark 2. In the sense of distributions, Problem A is the weak formulation of the following boundary value problem.

Problem A-diff. *In* Ω^δ, *find functions* $u = u(\boldsymbol{y})$ *and* $\theta = \theta(\boldsymbol{y})$ *satisfying the following system of equations and boundary conditions:*

$$-\nabla_y \cdot \big(a_\delta (\nabla_y u - \beta_\delta \, \theta \, \mathbf{1})\big) = \tilde{f} \quad \text{in } \Omega^\delta, \tag{5}$$

$$-\nabla_y \cdot (\lambda_\delta \nabla_y \theta) = \tilde{g} \quad \text{in } \Omega^\delta, \tag{6}$$

$$u = 0, \quad \theta = 0 \quad \text{on } \Gamma_+^\delta \text{ and } \Gamma_-^\delta, \tag{7}$$

$$\big(a_\delta (\nabla_y u - \beta_\delta \theta \, \mathbf{1})\big) \cdot \boldsymbol{n}^\delta = 0 \quad \text{on } \partial\Omega^\delta \setminus (\Gamma_+^\delta \cup \Gamma_-^\delta), \tag{8}$$

$$(\lambda_\delta \nabla_y \theta) \cdot \boldsymbol{n}^\delta = 0 \quad \text{on } \partial\Omega^\delta \setminus (\Gamma_+^\delta \cup \Gamma_-^\delta), \tag{9}$$

where $\boldsymbol{n}^\delta$ *is the unit outward normal to* $\partial\Omega^\delta$.

The equivalence assertion for Problems A and A-diff in the sense of distributions is easy to verify. To this end, it is sufficient to establish the formal equivalence of the systems (1)-(2) and (5)-(9) on a class of rather smooth solutions by means of the formula of integration by parts. At the same time, the system (5)-(9) can be identified as the simplified model statical problem for a thermoelastic body. Indeed, the equations (5) and (6) describe the equilibrium state of a body under the action of a given mass force $\tilde{f}$ and a given heat source $\tilde{g}$; in this case, u and θ are the sought displacement and temperature fields, and the coefficients a_δ, β_δ and λ_δ are the given elastic modulus, thermal expansion coefficient, and thermal conductivity, respectively. The conditions (7) characterize the immobility and isothermicity of the body on one of the parts of the boundary, more certainly, on $\Gamma_+^\delta \cup \Gamma_-^\delta$. The conditions (8) and (9) mean that the surface stresses and heat flow on the other part of the boundary, i.e., on $\partial\Omega^\delta \setminus (\Gamma_+^\delta \cup \Gamma_-^\delta)$, are equal to zero. Note that the equation (6) omits the inertia term $a_\delta \beta_\delta \mathbf{1} \cdot (\nabla_y \partial_t u)$ coupling it with the equation (5). (Here, by ∂_t we mean the time derivative.) For the fully coupled thermoelastic model with an interface (thin inclusion), moreover, when unilateral interface conditions are taken into consideration, we refer to Sec. 3.3 in the monograph [22] and references therein.

Now we prove the assertion on existence and uniqueness of Problem A.

Theorem 1. *Let $\tilde{f}, \tilde{g} \in L^2(\Omega^\delta)$ be given functions vanishing on Ω_m^δ. Then, for every fixed $\delta > 0$, Problem A has a unique solution.*

Proof. On the strength of Remark 1, it is sufficient to prove existence and uniqueness of solution to the problem (3)-(4). Note that the left hand side of (4) is the continuous bilinear form of the arguments θ and ϑ. This form is coercive due to the Poincaré-Friedrichs inequality. Also note that, for a fixed $\tilde{g}$, the right hand side of (4) defines the linear continuous functional. This allows to apply the Lax-Milgram theorem and conclude that, for any $\vartheta \in H^\delta$, there exists the unique function $\theta \in H^\delta$ satisfying (4). Thus, the problem (3)-(4) reduces to the following one:

find $u \in H^\delta$ satisfying the integral equality

$$\int\limits_{\Omega^\delta} a_\delta \nabla_y u \cdot \nabla_y v \, d\boldsymbol{y} = \int\limits_{\Omega^\delta} \left(a_\delta \beta_\delta \theta\right) \mathbf{1} \cdot \nabla_y v \, d\boldsymbol{y} + \int\limits_{\Omega^\delta} \tilde{f} v \, d\boldsymbol{y}, \quad \forall\, v \in H^\delta, \tag{10}$$

where the function θ is already known.

In turn, existence and uniqueness of solution to this problem also follows from the Lax-Milgram theorem. Indeed, it is easy to verify that the left hand side of (10) is the coercive continuous bilinear form of u and v and the right hand side of (10) is the linear continuous functional of v. As the result, we conclude that there exists the unique solution to (3)-(4), which completes the proof of Theorem 1. □

1.3. **Refinement of the formulation of Problem A.**

Assumption A. Starting from here, we assume that the coefficients in the differential equations (5) and (6) and in the corresponding bilinear forms in the formulation of Problem A have the following special representation:

$$
\begin{cases} a_\delta = a_\pm \\ \beta_\delta = \beta_\pm \\ \lambda_\delta = \lambda_\pm \end{cases} \text{in } \Omega^\delta_\pm, \qquad \begin{cases} a_\delta = \delta^{-1} a_m \\ \beta_\delta = \beta_m \\ \lambda_\delta = \delta^{-1}\lambda_m \end{cases} \text{in } \Omega^\delta_m, \tag{11}
$$

where $a_\pm$, $\beta_\pm$, $\lambda_\pm$, a_m, β_m, and λ_m are given constants such that $a_\pm, \lambda_\pm, a_m, \lambda_m > 0$. Thus the geometrical shape of the body and the coefficients in the equations in Problem A depend on the small parameter δ. Obviously, this means that the solution a priori depends on δ.

Notation 2. In line with this, the notation $(\boldsymbol{u}^\delta, \boldsymbol{\theta}^\delta)$ will be used further for solutions of Problem A, where $\boldsymbol{u}^\delta = (u^\delta_+, u^\delta_-, u^\delta_m)$ and $\boldsymbol{\theta}^\delta = (\theta^\delta_+, \theta^\delta_-, \theta^\delta_m)$.

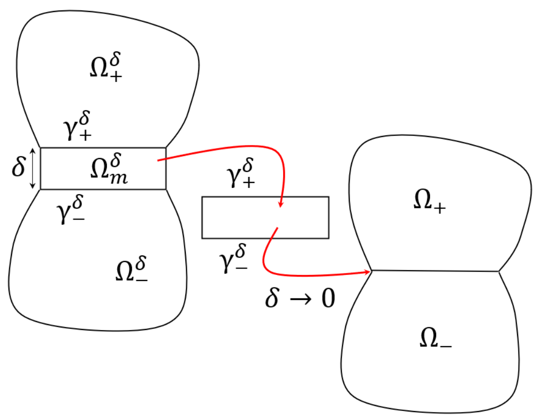

FIG. 2. Passage to the problem with a thin inclusion

Our goal now is to fulfill and justify the limiting passage in Problem A and, as the result, to derive the formulation of the limiting problem as δ tends to zero. With this aim, in $\mathbb{R}^2$, alongside Oy_1y_2 we introduce two more coordinate systems: Ox_1x_2 and Oz_1z_2. We introduce the coordinate transformations by the formulas

$$
\begin{cases} y_1 = x_1, \\ y_2 = 0 \text{ for } x_2 = 0, \\ y_2 = x_2 \pm \delta/2 \text{ for } \pm x_2 > 0, \end{cases} \qquad \begin{cases} y_1 = z_1, \\ y_2 = \delta z_2. \end{cases} \tag{12}
$$

Note that the pre-images of the subdomains $\Omega^\delta_+ \subset \mathbb{R}^2_y$ and $\Omega^\delta_- \subset \mathbb{R}^2_y$ under the coordinate transformation (12) are exactly the subdomains $\Omega_+ \subset \mathbb{R}^2_x$ and $\Omega_- \subset \mathbb{R}^2_x$ introduced in Sec. 1.1. Let us denote the pre-image of the domain $\Omega^\delta_m \subset \mathbb{R}^2_y$ by $\Omega_m \subset \mathbb{R}^2_z$. We emphasize that the coordinate transformations (12) establish a diffeomorphism between $\Omega^\delta_\pm$ and $\Omega_\pm$ and between Ω^δ_m and Ω_m, and that the description of $\Omega_\pm$ and Ω_m in the coordinate systems Ox_1x_2 and Oz_1z_2 does not depend on δ, in particular, $\Omega_m = \{0 < z_1 < y_1^*\} \times \{-1/2 < z_2 < 1/2\}$.

In order to study the limiting passage in Problem A, it is convenient to formulate this problem in terms of domains whose description is independent of δ. To this end, in (1) and (2) we fulfill the inverse change of variables to change (12). Notice that

(12) establishes a one-to-one correspondence between the triple of spaces $H^1_\Gamma(\Omega^\delta_+)$, $H^1_\Gamma(\Omega^\delta_-)$, $H^1(\Omega^\delta_m)$ and the triple of spaces $H^1_\Gamma(\Omega_+)$, $H^1_\Gamma(\Omega_-)$, $H^1(\Omega_m)$ and that the image of the space V^δ is the space

$$\begin{aligned} V_0 = \big\{ (\boldsymbol{u},\boldsymbol{\theta}) \in [H^1_\Gamma(\Omega_+) \times H^1_\Gamma(\Omega_-) \times H^1(\Omega_m)]^2 :\\ \boldsymbol{u} = (u_+, u_-, u_m),\ \boldsymbol{\theta} = (\theta_+, \theta_-, \theta_m); \quad u_\pm|_\gamma = u_m|_{\gamma^\pm},\ \theta_\pm|_\gamma = \theta_m|_{\gamma^\pm} \big\}, \end{aligned}$$

where $\gamma_\pm = \{(y_1, y_2) : \ (y_1, y_2 \mp 1/2) \in \gamma\}$. Furthermore, changing variables in the bilinear forms and linear functionals in equations (1) and (2), we arrive at the representations

$$\begin{aligned} b^\delta_1(\boldsymbol{u}^\delta, \boldsymbol{v}) &= b_1(\boldsymbol{u}_\delta, \boldsymbol{v}) + b_1(\delta; \boldsymbol{u}_\delta, \boldsymbol{v}),\\ b^\delta_2(\boldsymbol{\theta}^\delta, \boldsymbol{v}) &= b_2(\boldsymbol{\theta}_\delta, \boldsymbol{v}) + b_2(\delta; \boldsymbol{\theta}_\delta, \boldsymbol{v}),\\ b^\delta_3(\boldsymbol{\theta}^\delta, \boldsymbol{\vartheta}) &= b_3(\boldsymbol{\theta}_\delta, \boldsymbol{\vartheta}) + b_3(\delta; \boldsymbol{\theta}_\delta, \boldsymbol{\vartheta}),\\ l^\delta_1(\boldsymbol{v}) &= l_1(f; \boldsymbol{v}), \quad l^\delta_2(\boldsymbol{\vartheta}) = l_2(g; \boldsymbol{\vartheta}), \end{aligned}$$

where

$$f(\boldsymbol{x}) = f(x_1,x_2) = \tilde f(x_1, x_2 \pm \delta/2), \quad g(\boldsymbol{x}) = g(x_1,x_2) = \tilde g(x_1, x_2 \pm \delta/2) \quad \text{in} \ \ \Omega_\pm,$$

$$\begin{aligned} &\boldsymbol{u}_\delta = (u_{\delta+}, u_{\delta-}, u_{\delta m}), && \boldsymbol{\theta}_\delta = (\theta_{\delta+}, \theta_{\delta-}, \theta_{\delta m}),\\ &u_{\delta\pm}(x_1,x_2) = u^\delta_\pm(x_1, x_2 \pm \delta/2), && \theta_{\delta\pm}(x_1,x_2) = \theta^\delta_\pm(x_1, x_2 \pm \delta/2) \ \text{ in } \ \Omega_\pm,\\ &u_{\delta m}(z_1,z_2) = u^\delta_m(z_1, \delta z_2), && \theta_{\delta m}(z_1,z_2) = \theta^\delta_m(z_1, \delta z_2) \ \text{ PI } \ \Omega_m, \end{aligned}$$

and

$$\begin{aligned} b_1(\boldsymbol{u},\boldsymbol{v}) &= a_+ \int\limits_{\Omega_+} \nabla_x u_+ \cdot \nabla_x v_+ \, d\boldsymbol{x} + a_- \int\limits_{\Omega_-} \nabla_x u_- \cdot \nabla_x v_- \, d\boldsymbol{x}\\ &\qquad + a_m \int\limits_{\Omega_m} \partial_{z_1} u_m \partial_{z_1} v_m \, d\boldsymbol{z},\\ b_2(\boldsymbol{\theta},\boldsymbol{v}) &= a_+\beta_+ \int\limits_{\Omega_+} \theta_+ \mathbf{1} \cdot \nabla_x v_+ \, d\boldsymbol{x} + a_-\beta_- \int\limits_{\Omega_-} \theta_- \mathbf{1} \cdot \nabla_x v_- \, d\boldsymbol{x}\\ &\qquad + a_m\beta_m \int\limits_{\Omega_m} \theta_m \partial_{z_1} v_m \, d\boldsymbol{z},\\ b_3(\boldsymbol{\theta},\boldsymbol{\vartheta}) &= \lambda_+ \int\limits_{\Omega_+} \nabla_x \theta_+ \cdot \nabla_x \vartheta_+ \, d\boldsymbol{x} + \lambda_- \int\limits_{\Omega_-} \nabla_x \theta_- \cdot \nabla_x \vartheta_- \, d\boldsymbol{x}\\ &\qquad + \lambda_m \int\limits_{\Omega_m} \partial_{z_1} \theta_m \partial_{z_1} \vartheta_m \, d\boldsymbol{z},\\ b_1(\delta; \boldsymbol{u},\boldsymbol{v}) &= \frac{a_m}{\delta^2} \int\limits_{\Omega_m} \partial_{z_2} u_m \partial_{z_2} v_m \, d\boldsymbol{z}, \qquad b_2(\delta; \boldsymbol{\theta},\boldsymbol{v}) = \frac{a_m\beta_m}{\delta} \int\limits_{\Omega_m} \theta_m \partial_{z_2} v_m \, d\boldsymbol{z},\\ b_3(\delta; \boldsymbol{\theta},\boldsymbol{\vartheta}) &= \frac{\lambda_m}{\delta^2} \int\limits_{\Omega_m} \partial_{z_2} \theta_m \partial_{z_2} \vartheta_m \, d\boldsymbol{z}, \end{aligned}$$

$$l_1(f;\boldsymbol{v}) = \int\limits_{\Omega_+} f v_+ \, d\boldsymbol{x} + \int\limits_{\Omega_-} f v_- \, d\boldsymbol{x}, \qquad l_2(g;\boldsymbol{\vartheta}) = \int\limits_{\Omega_+} g\vartheta_+ \, d\boldsymbol{x} + \int\limits_{\Omega_-} g\vartheta_- \, d\boldsymbol{x}.$$

Taking into account the above considerations, we conclude that Problem A is equivalent to the following one:

find a pair of vector-functions $(\boldsymbol{u}_\delta, \boldsymbol{\theta}_\delta) \in V_0$ *satisfying the relations*

$$b_1(\boldsymbol{u}_\delta,\boldsymbol{v}) + b_1(\delta;\boldsymbol{u}_\delta,\boldsymbol{v}) - b_2(\boldsymbol{\theta}_\delta,\boldsymbol{v}) - b_2(\delta;\boldsymbol{\theta}_\delta,\boldsymbol{v}) = l_1(f;\boldsymbol{v}), \tag{13}$$

$$b_3(\boldsymbol{\theta}_\delta,\boldsymbol{\vartheta}) + b_3(\delta;\boldsymbol{\theta}_\delta,\boldsymbol{\vartheta}) = l_2(g;\boldsymbol{\vartheta}), \tag{14}$$

for all pairs of test vector-functions $(\boldsymbol{v},\boldsymbol{\vartheta}) \in V_0$.

Further, we pass to the limit as $\delta \to 0+$ in this very formulation.

1.4. **Passage to the limit as $\boldsymbol{\delta \to 0+}$. The limiting formulation with a thin inclusion.** Let us prove that it is possible to extract a convergent subsequence from the family of solutions of the problem (13)-(14). Choosing $(\boldsymbol{v},\boldsymbol{\vartheta}) = (\boldsymbol{u}_\delta,\boldsymbol{\theta}_\delta)$ as the pair of test vector-functions in (13) and (14), we get the identities

$$b_1(\boldsymbol{u}_\delta,\boldsymbol{u}_\delta) + b_1(\delta;\boldsymbol{u}_\delta,\boldsymbol{u}_\delta) - b_2(\boldsymbol{\theta}_\delta,\boldsymbol{u}_\delta) - b_2(\delta;\boldsymbol{\theta}_\delta,\boldsymbol{u}_\delta) = l_1(f;\boldsymbol{u}_\delta),$$

$$b_3(\boldsymbol{\theta}_\delta,\boldsymbol{\theta}_\delta) + b_3(\delta;\boldsymbol{\theta}_\delta,\boldsymbol{\theta}_\delta) = l_2(g;\boldsymbol{\theta}_\delta).$$

With the help of the Poincaré-Friedrichs and Cauchy-Bunyakovsky inequalities, from these identities we deduce the uniform in δ ($\delta \in (0,\delta_0]$, where δ_0 is some sufficiently small fixed value) bounds

$$\begin{aligned} &\|u_{\delta\pm}\|_{H^1_\Gamma(\Omega_\pm)} \le c_0, \quad \|\partial_{z_1} u_{\delta m}\|_{L^2(\Omega_m)} \le c_0, \quad \|\delta^{-1}\partial_{z_2} u_{\delta m}\|_{L^2(\Omega_m)} \le c_0, \\ &\|\theta_{\delta\pm}\|_{H^1_\Gamma(\Omega_\pm)} \le c_0, \quad \|\partial_{z_1} \theta_{\delta m}\|_{L^2(\Omega_m)} \le c_0, \quad \|\delta^{-1}\partial_{z_2} \theta_{\delta m}\|_{L^2(\Omega_m)} \le c_0 \end{aligned} \tag{15}$$

with a constant $c_0 \ge 0$ independent of δ. At the same time, from [31, lemma 1] it follows that the inequality

$$\|q\|^2_{L^2(\Omega_m)} \le c_1\big(\|\partial_{z_1} q\|^2_{L^2(\Omega_m)} + \|q\|^2_{L^2(\gamma_\pm)}\big) \quad (c_1 = \mathrm{const} > 0)$$

holds true for any $q \in H^1(\Omega_m)$.

Taking into account this inequality, the bounds (15), and continuity of the trace operator (on γ), we establish the estimates

$$\|u_{\delta m}\|_{L^2(\Omega_m)} \le c_2, \quad \|\theta_{\delta m}\|_{L^2(\Omega_m)} \le c_2, \tag{16}$$

where c_2 is a constant independent of m. Now, the estimates (15) and (16) allow us to choose such subsequence of solutions (still labeled by δ) that

$$\begin{aligned} &u_{\delta\pm} \to u_\pm \text{ weakly in } H^1_\Gamma(\Omega_\pm), && \theta_{\delta\pm} \to \theta_\pm \text{ weakly in } H^1_\Gamma(\Omega_\pm), \\ &u_{\delta m} \to u_m \text{ weakly in } L^2(\Omega_m), && \theta_{\delta m} \to \theta_m \text{ weakly in } L^2(\Omega_m), \\ &\partial_{z_1} u_{\delta m} \to \partial_{z_1} u_m \text{ weakly in } L^2(\Omega_m), && \partial_{z_1} \theta_{\delta m} \to \partial_{z_1} \theta_m \text{ weakly in } L^2(\Omega_m), \\ &\|\partial_{z_2} u_{\delta m}\|_{L^2(\Omega_m)} \overset{(15)_3}{\le} c_0\,\delta \to 0, && \|\partial_{z_2} \theta_{\delta m}\|_{L^2(\Omega_m)} \overset{(15)_6}{\le} c_0\,\delta \to 0, \end{aligned} \tag{17}$$

as $\delta \to 0+$. From the limiting relations (17) and the definition of the space V_0 it follows that the limiting point $(\boldsymbol{u},\boldsymbol{\theta})$ belongs to the space

$$V_1 := \big\{(\boldsymbol{u},\boldsymbol{\theta}) \in V_0 \colon\ \partial_{z_2} u_m = 0,\ \partial_{z_2}\theta_m = 0 \text{ in } \Omega_m\big\}.$$

Finally, fixing arbitrarily a pair of vector-functions $(\boldsymbol{v},\boldsymbol{\vartheta})\in V_1$ in the problem (13)-(14), we pass to the limit as $\delta\to 0+$ using the limiting relations (17). As the result, we arrive at the following formulation:

find a pair of vector-functions $(\boldsymbol{u},\boldsymbol{\theta})\in V_1$ *satisfying the equalities*

$$b_1(\boldsymbol{u},\boldsymbol{v})-b_2(\boldsymbol{\theta};\boldsymbol{v})=l_1(f;\boldsymbol{v}), \tag{18}$$

$$b_3(\boldsymbol{\theta},\boldsymbol{\vartheta})=l_2(g;\boldsymbol{\vartheta}), \tag{19}$$

for all pairs of test vector-functions $(\boldsymbol{v},\boldsymbol{\vartheta})\in V_1$.

Thus we arrive at one of the main results of the article:

Theorem 2. *As* $\delta\to 0+$*, the family* $\{\boldsymbol{u}_\delta,\boldsymbol{\theta}_\delta\}_{\delta>0}$ *of solutions of the problem* (13)-(14) *(equivalently, the solutions of Problem A) converges to the solution of the problem* (18)-(19) *in the sense of the limiting relations* (17).

Let us simplify the presentation of the obtained limiting problem (18)-(19).

Notice that any pair $(\boldsymbol{u},\boldsymbol{\theta})\in V_1$ corresponds to an element $(u,\theta)\in H_0^1(\Omega)\times H_0^1(\Omega)$ such that

$$u\big|_\gamma\in H_0^1(\gamma),\quad \theta\big|_\gamma\in H_0^1(\gamma), \tag{20}$$

and, simultaneously,

$$\begin{array}{llll} & u_\pm=u, & \theta_\pm=\theta & \text{in } \Omega_\pm,\\ (21) & u\big|_\gamma=u\big|_\gamma(x_1), & \theta\big|_\gamma=\theta\big|_\gamma(x_1) & \text{on } \gamma,\\ & u_m(z_1,z_2)=u\big|_\gamma(z_1), & \theta_m(z_1,z_2)=\theta\big|_\gamma(z_1) & \text{in } \Omega_m.\end{array}$$

Here and further we standardly denote $H_0^1(\Omega):=\{v\in H^1(\Omega)\colon\ v=0 \text{ on } \partial\Omega\}$.

Notation 3. Also, in (20) and further, $H_0^1(\gamma):=H_0^1(0,y_1^*)$ is the Sobolev space of functions defined on a one-dimensional set (interval) $[0,y_1^*]\times\{y_2=0\}$ and vanishing at the endpoints $\partial\gamma=\{(0,0)\}\cup\{(y_1^*,0)\}$ of this interval. This space is supplemented with the standard norm

$$\|\phi\|_{H_0^1(\gamma)}=\Big(\|\phi\|^2_{L^2(\gamma)}+\|\partial_{x_1}\phi\|^2_{L^2(\gamma)}\Big)^{1/2},\quad \forall\,\phi\in H_0^1(\gamma),$$

$$\text{where}\quad \|\psi\|^2_{L^2(\gamma)}=\int_0^{y_1^*}|\psi(x_1,0)|^2dx_1.\quad (\text{Here } \psi:=\phi,\ \partial_{x_1}\phi.)$$

Guided by the formulas (20), let us introduce the space

$$V=\big\{(u,\theta)\in H_0^1(\Omega)\times H_0^1(\Omega)\colon\ u\big|_\gamma\in H_0^1(\gamma),\quad \theta\big|_\gamma\in H_0^1(\gamma)\big\}.$$

Notice that the relations (21) establish a one-to-one correspondence between V_1 and V. Moreover, by means of rather simple arguments we conclude that the relations

$$\begin{aligned} b_1(\boldsymbol{u},\boldsymbol{v})-b_2(\boldsymbol{\theta};\boldsymbol{v})&=\int\limits_\Omega a\nabla_x u\cdot\nabla_x v\,d\boldsymbol{x}-\int\limits_\Omega(a\beta\theta)\,\mathbf{1}\cdot\nabla_x v\,d\boldsymbol{x}\\ &\quad+a_m\int\limits_\gamma(\partial_{x_1}u-\beta_m\theta)\partial_{x_1}v\,dx_1\end{aligned}$$

and

$$b_3(\boldsymbol{\theta},\boldsymbol{\vartheta}) = \int\limits_\Omega \lambda \nabla_x \theta \cdot \nabla_x \vartheta \, d\boldsymbol{x} + \lambda_m \int\limits_\gamma \partial_{x_1}\theta \, \partial_{x_1}\vartheta \, dx_1,$$

hold, where

$$a = \begin{cases} a_+ \text{ in } \Omega_+, \\ a_- \text{ in } \Omega_-, \end{cases} \qquad \beta = \begin{cases} \beta_+ \text{ in } \Omega_+, \\ \beta_- \text{ in } \Omega_-, \end{cases} \qquad \lambda = \begin{cases} \lambda_+ \text{ in } \Omega_+, \\ \lambda_- \text{ in } \Omega_-. \end{cases}$$

On the strength of the above considerations, we arrive at the variational formulation as follows.

Problem B. *Find a pair of functions $(u,\theta) \in V$ satisfying the integral equalities*

$$\int\limits_\Omega a \nabla_x u \cdot \nabla_x v \, d\boldsymbol{x} - \int\limits_\Omega (a\beta\theta)\, \mathbf{1} \cdot \nabla_x v \, d\boldsymbol{x} + a_m \int\limits_\gamma (\partial_{x_1} u - \beta_m \theta) \partial_{x_1} v \, dx_1 = \int\limits_\Omega f v \, d\boldsymbol{x}, \tag{22}$$

$$\int\limits_\Omega \lambda \nabla_x \theta \cdot \nabla_x \vartheta \, d\boldsymbol{x} + \lambda_m \int\limits_\gamma \partial_{x_1}\theta \, \partial_{x_1}\vartheta \, dx_1 = \int\limits_\Omega g \vartheta \, d\boldsymbol{x}, \tag{23}$$

for all pairs of test-functions $(v,\vartheta) \in V$.

Thus, the following assertion is valid.

Proposition 1. *The two limiting formulations, namely, the problem* (18)-(19) *and Problem B, are equivalent to each other.*

Remark 3. In the sense of distributions, Problem B is the weak formulation of the following boundary value problem.

Problem B-diff. *In the bounded domain $\Omega \subset \mathbb{R}^2_x$, find the functions $u = u(\boldsymbol{x})$ and $\theta = \theta(\boldsymbol{x})$ such that*

$$-\nabla_x \cdot \big(a_\pm(\nabla_x u - \beta_\pm \theta \mathbf{1})\big) = f \quad \text{in } \Omega_\pm, \tag{24}$$

$$-\lambda_\pm \Delta_x \theta = g \quad \text{in } \Omega_\pm, \tag{25}$$

$$u = 0, \quad \theta = 0 \quad \text{on } \Gamma_\pm, \tag{26}$$

$$u = 0, \quad \theta = 0 \quad \text{on } \partial\gamma, \tag{27}$$

$$a_+\big(\nabla_x u - \beta_+\theta\mathbf{1}\big)\cdot \boldsymbol{n}_+ + a_-\big(\nabla_x u - \beta_-\theta\mathbf{1}\big)\cdot \boldsymbol{n}_- = a_m \partial_{x_1}(\partial_{x_1} u - \beta_m \theta) \quad \text{on } \gamma, \tag{28}$$

$$\lambda_+ \partial_{n_+}\theta + \lambda_- \partial_{n_-}\theta = \lambda_m \partial^2_{x_1}\theta \quad \text{on } \gamma, \tag{29}$$

where $\boldsymbol{n}_\pm$ are the unit outward normal vectors to $\partial\Omega_\pm$ and $\partial_{n_\pm} = \boldsymbol{n}_\pm \cdot \nabla_x$ are the derivatives in the directions of $\boldsymbol{n}_\pm$.

Problem B-diff and its equivalent variational formulation — Problem B — can be identified as the model problem for description of a statical state of a thermoelastic body with thin inclusion. In its formulation, functions u and θ are the sought displacement and temperature, equations (24) and (25) are the equations of statical equilibrium of the body, relations (26) and (27) are the conditions of immobility and isothermicity on the exterior boundary, and equations (28) and (29) are the equations of statical equilibrium of the thin inclusion.

Clearly, we have proved the existence of solutions to Problem B directly by the limiting passage from Problem A, as $\delta \to 0+$. It is important to note that the existence theorem for Problem B can be established in an alternative way, independently of Problem A. More certainly, we have:

Theorem 3. *For any given $f, g \in L^2(\Omega)$ Problem B has a unique solution.*

Proof. Similarly to the proof of Theorem 1, both existence and uniqueness assertions follow from the Lax-Milgram theorem. □

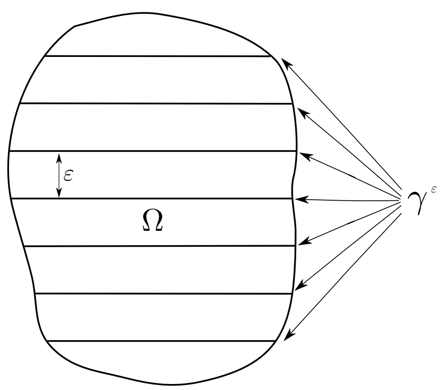

FIG. 3. Several thin inclusions

1.5. **Generalization to any finite number of thin inclusions.** The results of Secs. 1.2-1.4 can be generalized (in a natural way) to any finite number of two-dimensional inclusions of the form Ω_m in the original Problem A and, correspondingly, to any finite number of thin inclusions of the form γ in the limiting Problem B.

In particular, using the similar arguments (with natural modifications), as in Secs. 1.2-1.4, we construct the well-posed model of thermoelastic body incorporating a family of thin inclusions $\gamma^\varepsilon = \Omega \cap \{x_2 = j\varepsilon,\ j \in \mathbb{Z}\}$, which are parallel to each other and spaced apart from each other at a distance of $\varepsilon > 0$, as in Fig. 3. In this case, the essential requirement is only that each of the subdomains, into which the domain Ω is divided by the set γ^ε, has a Lipschitz boundary.

In the next section, the precise formulation of the problem with a finite number of periodically situated inclusions is set (see Problem B_ε in Sec. 2.1) and then the homogenization procedure is fulfilled for it as $\varepsilon \to 0+$.

2. Homogenization by the number of thin inclusions

2.1. **Equations of microstructure and their solvability.** The precise formulation of the problem with a finite number of periodically situated inclusions is as follows.

Let thin inclusions occupy the set of physical positions

$$\gamma^\varepsilon = \Omega \cap \{x_2 = j\varepsilon,\ j \in \mathbb{Z}\}, \tag{30}$$

where $\varepsilon > 0$ is a dimensionless parameter characterizing the distance between two neighboring inclusions. We suppose that ε is small enough so that γ^ε is nonempty.

Let us define $L^2(\gamma^\varepsilon)$ and $H_0^1(\gamma^\varepsilon)$ in the standard way.

We say that a function $w\colon \gamma^\varepsilon \mapsto \mathbb{R}$ belongs to $L^2(\gamma^\varepsilon)$ if it is Lebesgue-measurable and has a finite norm

$$\|w\|_{L^2(\gamma^\varepsilon)} := \left(\int_{\gamma^\varepsilon} |w(\boldsymbol{x})|^2 d\sigma^\varepsilon(\boldsymbol{x}) \right)^{1/2} < +\infty.$$

By $d\sigma^\varepsilon$ here and further we denote the one-dimensional Lebesgue measure on γ^ε, i.e., $d\sigma^\varepsilon(x) = dx_1$ on any segment $\{x_* < x_1 < x_{**},\ x_2 = j\varepsilon,\ j \in \mathbb{Z}\} \subset \gamma^\varepsilon$.

We say that a function w: $\gamma^\varepsilon \mapsto \mathbb{R}$ belongs to $H^1_0(\gamma^\varepsilon)$ if w, $\partial_{x_1} w \in L^2(\gamma^\varepsilon)$ and

$$\int\limits_{\gamma^\varepsilon} \partial_{x_1} w\, \phi\, d\sigma^\varepsilon(\boldsymbol{x}) = -\int\limits_{\gamma^\varepsilon} w\, \partial_{x_1} \phi\, d\sigma^\varepsilon(\boldsymbol{x}), \quad \forall\, \phi \in C^1(\overline{\Omega}). \tag{31}$$

This integral equality expresses the fact that w vanishes on $\partial\gamma^\varepsilon = \overline{\gamma^\varepsilon} \cap \partial\Omega$ in the trace sense. The norm in $H^1_0(\gamma^\varepsilon)$ is defined by the canonical formula

$$\|w\|_{H^1_0(\gamma^\varepsilon)} = \left(\|w\|^2_{L^2(\gamma^\varepsilon)} + \|\partial_{x_1} w\|^2_{L^2(\gamma^\varepsilon)}\right)^{1/2}.$$

Introduce into considerations the Sobolev space

$$V^\varepsilon = \left\{(u,\theta) \in H^1_0(\Omega) \times H^1_0(\Omega):\ u\big|_{\gamma^\varepsilon} \in H^1_0(\gamma^\varepsilon),\ \theta\big|_{\gamma^\varepsilon} \in H^1_0(\gamma^\varepsilon)\right\}$$

supplemented with the standard norm and scalar product.

With account of these notations, we set the following variational formulation.

Problem $\mathbf{B}_{\boldsymbol{\varepsilon}}$. *For any fixed $\varepsilon > 0$ and $p \in \mathbb{R}$, find a pair of functions $(u^\varepsilon, \theta^\varepsilon) \in V^\varepsilon$ satisfying the integral equalities*

$$\int\limits_\Omega a\nabla_x u^\varepsilon \cdot \nabla_x v\, d\boldsymbol{x} - \int\limits_\Omega (a\beta\theta^\varepsilon)\, \mathbf{1} \cdot \nabla_x v\, d\boldsymbol{x} + a_m \varepsilon^p \int\limits_{\gamma^\varepsilon} (\partial_{x_1} u^\varepsilon - \beta_m \theta^\varepsilon) \partial_{x_1} v\, d\sigma^\varepsilon(\boldsymbol{x}) = \int\limits_\Omega f v\, d\boldsymbol{x}, \tag{32}$$

$$\int\limits_\Omega \lambda \nabla_x \theta^\varepsilon \cdot \nabla_x \vartheta\, d\boldsymbol{x} + \lambda_m \varepsilon^p \int\limits_{\gamma^\varepsilon} \partial_{x_1} \theta^\varepsilon \partial_{x_1} \vartheta\, d\sigma^\varepsilon(\boldsymbol{x}) = \int\limits_\Omega g\vartheta\, d\boldsymbol{x}, \tag{33}$$

for all pairs of test functions $(v, \vartheta) \in V^\varepsilon$.

Remark 4. As well as ε, the exponent $p \in \mathbb{R}$ in Problem B_ε is considered a given parameter. As is obvious from the formulation, p corresponds to the 'contrast' of the thermomechanical properties of the components $(\Omega \setminus \gamma^\varepsilon)$ and γ^ε. Based on the vast body of results in homogenization theory, one should expect that different values of p will lead to essentially different homogenized models for families of solutions to Problem B_ε as $\varepsilon \to 0+$. In this article, we restrict ourselves to considering the case $p = 1$, see further Assumption B in Sec. 2.3.

According to the note made in Sec. 1.5, for Problem $\mathrm{P'}_\varepsilon$ the following well-posedness result holds true.

Proposition 2. *For any fixed value of ε, for all given $f, g \in L^2(\Omega)$, Problem P'_ε has a unique solution.*

In Secs. 2.2-2.3, we fulfill and rigorously justify the homogenization procedure for Problem $\mathrm{P'}_\varepsilon$ as $\varepsilon \to 0+$. As the result, we derive the homogenized model whose solution is the limit of the family $\{(u^\varepsilon, \theta^\varepsilon)\}_{\varepsilon>0}$ of solutions to Problem B_ε.

2.2. **The toolbox of the method of two-scale convergence.** The homogenization procedure for Problem $\mathrm{P'}_\varepsilon$ as $\varepsilon \to 0+$, i.e., the limiting passage in the integral equalities (32) and (33), is based on implementation of the standard Allaire-Nguetseng method of two-scale convergence and its modification for homogenization on manifolds of minor dimension, proposed by G. Allaire, A. Damlamian, and U. Hornung. In order to formulate the provisions of this method necessary for further considerations, we first introduce some spaces of periodic functions.

Notation 4. In the space $\mathbb{R}^2_{\xi}$ of variables $\boldsymbol{\xi} = (\xi_1, \xi_2)$, by Ξ we denote the unit square $[0,1)^2$. Following the commonly accepted terminology in homogenization theory, we call Ξ the *periodicity cell* and say that the variables $\boldsymbol{\xi} = (\xi_1, \xi_2)$ are the *fast (or microscopic) variables.*

Definition 1. Function $f = f(\boldsymbol{\xi})$, defined on $\mathbb{R}^2_{\xi}$ and satisfying the equalities

$$f(\boldsymbol{\xi} + \boldsymbol{e}_1) = f(\boldsymbol{\xi}), \quad f(\boldsymbol{\xi} + \boldsymbol{e}_2) = f(\boldsymbol{\xi}), \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^2_{\xi},$$

is called Ξ-*periodic* or 1-*periodic in* $\boldsymbol{\xi}$.

Here and further, $\boldsymbol{e}_1 = (1,0)$ and $\boldsymbol{e}_2 = (0,1)$ are Cartesian basis vectors in $\mathbb{R}^2$. Using this definition we standardly introduce the spaces of periodic functions as follows.

Definition 2. Let $C^\infty_\sharp(\Xi)$ be the subset of $C^\infty(\mathbb{R}^2_\xi)$, consisting of Ξ-periodic functions. By $C_\sharp(\Xi)$, $C^1_\sharp(\Xi)$, and $H^1_\sharp(\Xi)$ we denote the closures of $C^\infty_\sharp(\Xi)$ in the norms of the spaces $C(\Xi)$, $C^1(\Xi)$, and $H^1(\Xi)$, respectively.

We outline the notion of two-scale convergence and the basic properties of two-scale convergent sequences following the original works [5, 29].

Definition 3. Let $\{v^\varepsilon\}_{\varepsilon\to 0+}$ be a sequence in $L^2(\Omega)$. We say that $\{v^\varepsilon\}_{\varepsilon\to 0+}$ *two-scale converges* to a function $v_0 \in L^2(\Omega \times \Xi)$ if the limiting relation

$$\int\limits_\Omega v^\varepsilon(\boldsymbol{x})\, \varphi\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\Big)\, d\boldsymbol{x} \underset{\varepsilon\to 0+}{\longrightarrow} \int\limits_\Omega \int\limits_\Xi v_0(\boldsymbol{x}, \boldsymbol{\xi})\, \varphi(\boldsymbol{x}, \boldsymbol{\xi})\, d\boldsymbol{\xi}\, d\boldsymbol{x}$$

holds for all $\varphi \in C(\overline{\Omega}; C_\sharp(\Xi))$.

Proposition 3. (Existence of two-scale convergent sequences.) *Assume $\{v^\varepsilon\}_{\varepsilon>0}$ is a bounded family in $L^2(\Omega)$; then there is a sequence $\{v^{\varepsilon'}\}$ and a function $v_0 \in L^2(\Omega \times \Xi)$ such that $\{v^{\varepsilon'}\}$ two-scale converges to v_0 as $\varepsilon' \to 0+$, in the sense of Definition 3.*

Proposition 4. (Two-scale convergence of gradients.) *Assume $\{v^\varepsilon\}_{\varepsilon\to 0+}$ is a sequence in $H^1(\Omega)$ such that $v^\varepsilon \underset{\varepsilon\to 0+}{\longrightarrow} v_0$ weakly in $H^1(\Omega)$; then*

- **(i)** *$\{v^\varepsilon\}$ two-scale converges to v_0 in the sense of Definition 3;*
- **(ii)** *there exist a subsequence $\{\varepsilon' \to 0+\}$ and a function $v_1 = v_1(\boldsymbol{x}, \boldsymbol{\xi})$ belonging to $L^2(\Omega; H^1_\sharp(\Xi))$ such that*

$$\nabla v^{\varepsilon'} \underset{\varepsilon'\to 0+}{\longrightarrow} \nabla_x v_0 + \nabla_\xi v_1 \ \textit{two-scale in the sense of Definition 3.}$$

We will also use the notion and a number of properties of two-scale convergence for sequences of functions defined on thin inclusions, which are straight line segments in the plane in our problem.

First, we give a description of thin inclusions in a form suitable for using the two-scale convergence toolbox, and then present the necessary concepts and results on two-scale convergence on thin inclusions.

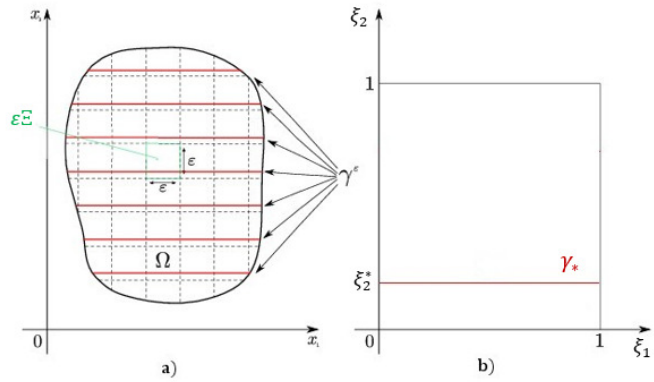

FIG. 4. Covering Ω with a regular ε-net

Set the structure of the pattern periodicity cell Ξ, as shown on Fig. 4(b). The formal description of this structure is that, inside Ξ, the thin inclusion γ_* is the segment parallel to abscissa axis $O\xi_1$ spaced from $O\xi_1$ at a distance $\xi_2^* = \text{const} \in (0,1)$, i.e.,

$$\gamma_* = \big\{\boldsymbol{\xi} = (\xi_1, \xi_2) \in \Xi\colon\ \xi_2 = \xi_2^*\big\}.$$

Then construct the periodic repetition of Ξ all over $\mathbb{R}^2_\xi$ and set $\gamma_*^k := \gamma_* + \boldsymbol{k}$, $\boldsymbol{k} \in \mathbb{Z}^2_\xi$. Clearly, the union of inclusions $\gamma_{\rm utd} = \bigcup\limits_{\boldsymbol{k}\in\mathbb{Z}^2} \gamma_*^k$ is an infinite number of straight lines in $\mathbb{R}^2_\xi$ parallel to the $O\xi_1$ axis and spaced by a unit distance from each other:

$$\gamma_{\rm utd} = \big\{\boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathbb{R}^2\colon\ -\infty < \xi_1 < +\infty,\ \xi_2 = \xi_2^* + k_2,\ k_2 \in \mathbb{Z}\big\}.$$

Based on this construction, we introduce into consideration the regular ε-net covering Ω, see Fig. 4(a). Each cell of the net is a cube Ξ_i^ε with edge length equal to ε. Each cube Ξ_i^ε, $i = 1, 2, \dots, N$ ($N = N(\varepsilon) = O(1/\varepsilon^2)$ for $\varepsilon \ll 1$), is obtained from Ξ by means of the linear homeomorphism Π_i^ε consisting of compressing $1/\varepsilon$ times and parallel translation to the vector $\varepsilon\boldsymbol{k}$, where $\boldsymbol{k}$ ranges over all values from $\mathbb{Z}^2$ such that $\Xi_i^\varepsilon \cap \Omega \neq \emptyset$. Thus, the set $\gamma_{\rm utd}$ is compressed $1/\varepsilon$ times, and the intersection of this compression with Ω exactly forms the set γ^ε, defined by the formula (30).

The following basic notions and results regarding two-scale convergence on thin inclusions are given following the original work [6] in the form adapted for the purposes of the present article.

Definition 4. Let $\{w^\varepsilon\}_{\varepsilon\to0+}$ be a sequence in $L^2(\gamma^\varepsilon)$. We call it *two-scale convergent to* $w_0 \in L^2(\Omega\times\gamma_*)$ (we have $w_0 = w_0(\boldsymbol{x}, \xi_1, \xi_2^*)$) if the limiting relation

$$\varepsilon \int\limits_{\gamma^\varepsilon} w^\varepsilon(\boldsymbol{x})\varphi\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\Big)\, d\sigma^\varepsilon(\boldsymbol{x}) \underset{\varepsilon\to0+}{\longrightarrow} \int\limits_{\Omega}\int\limits_{\gamma_*} w_0(\boldsymbol{x}, \xi_1, \xi_2^*)\, \varphi(\boldsymbol{x}, \xi_1, \xi_2^*)\, d\xi_1\, d\boldsymbol{x}$$

holds for all $\varphi \in C(\overline{\Omega}; C_\sharp(\Xi))$.

Evidently, in this definition and further, integral on γ_* is an integral in $d\xi_1$ on the interval $\{0 < \xi_1 < 1\}$.

The following fundamental result on existence of two-scale convergent sequences holds true.

Proposition 5. *Assume* $\{w^\varepsilon\}_{\varepsilon\to 0+}$ *is a sequence in* $L^2(\gamma^\varepsilon)$ *such that*

$$\varepsilon^{1/2}\|w^\varepsilon\|_{L^2(\gamma^\varepsilon)} \le c_3,$$

where $c_3 > 0$ *is independent of* ε*; then there exist a subsequence from* $\{\varepsilon \to 0+\}$*, still labeled by* ε*, and a limiting function* $w_0 \in L^2(\Omega \times \gamma_*)$ $(w_0 = w_0(\boldsymbol{x}, \xi_1, \xi_2^*))$ *such that the limiting relation*

$$w^\varepsilon \underset{\varepsilon\to 0+}{\longrightarrow} w_0 \quad \text{two-scale in the sense of Definition 4}$$

takes place.

The next two assertions allow us to pass to the limit in integrals incorporating derivatives and traces on γ^ε, if the necessary estimates are available.

Proposition 6. **(i)** *Assume* $\{w^\varepsilon\}_{\varepsilon\to 0+}$ *is a sequence in* $H^1(\Omega)$ *such that*

$$\|w^\varepsilon\|_{L^2(\Omega)} + \varepsilon\|\nabla_x w^\varepsilon\|_{L^2(\Omega)} \le c_4,$$

where $c_4 > 0$ *is independent of* ε*; then, for* $\varepsilon > 0$*, the trace of* w^ε *on* γ^ε *does exist and satisfies the bound*

$$\varepsilon \int\limits_{\gamma^\varepsilon} |w^\varepsilon(\boldsymbol{x})|^2 d\sigma^\varepsilon(\boldsymbol{x}) \le c_5,$$

where $c_5 > 0$ *is independent of* ε.

(ii) *Let, in addition to hypotheses of item (i), the limiting relation*

$$\int\limits_\Omega w^\varepsilon(\boldsymbol{x})\,\varphi\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\Big)\,d\boldsymbol{x} \underset{\varepsilon\to 0+}{\longrightarrow} \int\limits_\Omega \int\limits_\Xi w_0(\boldsymbol{x}, \boldsymbol{\xi})\,\varphi(\boldsymbol{x}, \boldsymbol{\xi}) d\boldsymbol{\xi}\,d\boldsymbol{x}, \quad \forall\,\varphi \in C(\overline{\Omega}; C_\sharp(\Xi)),$$

hold true with some function $w_0 \in L^2(\Omega; H^1_\sharp(\Xi))$*, in line with Proposition 3. In other words, let* $\{w^\varepsilon\}_{\varepsilon\to 0+}$ *converge to* w_0 *two-scale in the usual sense, i.e., in the sense of Definition 3.*

Then there exists a subsequence $\{\varepsilon' \to 0+\}$ *of* $\{\varepsilon \to 0+\}$ *such that the sequence of traces of* $w^{\varepsilon'}$ *on* $\gamma^{\varepsilon'}$ *converges to the trace of* w_0 *on* γ *two-scale in the sense of Definition 4 as* $\varepsilon' \to 0+$*, i.e.,*

$$\varepsilon' \int\limits_{\gamma^{\varepsilon'}} w^{\varepsilon'}(\boldsymbol{x})\,\varphi\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon'}\Big)\,d\sigma^{\varepsilon'}(\boldsymbol{x}) \underset{\varepsilon'\to 0+}{\longrightarrow} \int\limits_\Omega \int\limits_{\gamma_*} w_0(\boldsymbol{x}, \xi_1, \xi_2^*)\,\varphi(\boldsymbol{x}, \xi_1, \xi_2^*)\,d\xi_1\,d\boldsymbol{x},$$
$$\forall\,\varphi \in C(\overline{\Omega}; C_\sharp(\Xi)).$$

(iii) *Furthermore, in hypotheses of items (i) and (ii), the limiting relation for the gradients holds true:*

$$\varepsilon' \int\limits_\Omega \nabla_x w^{\varepsilon'}(\boldsymbol{x}) \cdot \boldsymbol{\Phi}\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon'}\Big)\,d\boldsymbol{x} \underset{\varepsilon'\to 0+}{\longrightarrow} \int\limits_\Omega \int\limits_\Xi \nabla_\xi w_0(\boldsymbol{x}, \boldsymbol{\xi}) \cdot \boldsymbol{\Phi}(\boldsymbol{x}, \boldsymbol{\xi})\,d\boldsymbol{\xi}\,d\boldsymbol{x},$$
$$\forall\,\boldsymbol{\Phi} \in C(\overline{\Omega}; C_\sharp(\Xi))^2.$$

Proposition 7. *Assume $\{w^\varepsilon\}$ is a sequence in $H^1(\gamma^\varepsilon)$ such that*

$$\varepsilon^{1/2}\|w^\varepsilon\|_{L^2(\gamma^\varepsilon)} + \varepsilon^{3/2}\|\partial_{x_1} w^\varepsilon\|_{L^2(\gamma^\varepsilon)} \le c_6, \tag{34}$$

where $c_6 > 0$ is independent of ε; then there exist a subsequence $\{\varepsilon' \to 0+\}$ from $\{\varepsilon \to 0+\}$ and a function $w_1 \in L^2(\Omega; H^1_\sharp(\gamma_*))$ ($w_1 = w_1(\boldsymbol{x}, \xi_1)$) such that, in the sense of Definition 4, the subsequences $\{w^{\varepsilon'}\}$ and $\{\varepsilon' \partial_{x_1} w^{\varepsilon'}\}$ two-scale converge to w_1 and $\partial_{\xi_1} w_1$, respectively, as $\varepsilon' \to 0+$.

2.3. **Homogenization of Problem B_ε.** Now we pass to the limit in Problem B_ε as $\varepsilon \to 0+$. Let us start with derivation of uniform estimates.

Lemma 1. *Assume $\varepsilon_0 > 0$ is fixed and sufficiently small; then there exists a constant $c_7 \ge 0$, which depends on a, a_m, β, β_m, λ, λ_m, $\|f\|_{L^2(\Omega)}$, and $\|g\|_{L^2(\Omega)}$ and is independent of ε and p such that the family $\big\{(u^\varepsilon, \theta^\varepsilon)\big\}_{\varepsilon \in (0, \varepsilon_0]}$ of solutions to Problem B_ε satisfies the estimates*

$$\|u^\varepsilon\|_{H^1(\Omega)} \le c_7, \qquad \|\theta^\varepsilon\|_{H^1(\Omega)} \le c_7, \tag{35}$$

$$\varepsilon^{p/2}\|u^\varepsilon\|_{H^1(\gamma^\varepsilon)} \le c_7, \quad \varepsilon^{p/2}\|\theta^\varepsilon\|_{H^1(\gamma^\varepsilon)} \le c_7. \tag{36}$$

Proof. Taking $(v, \vartheta) = (u^\varepsilon, \theta^\varepsilon)$ in (32) and (33), we get the energy identities

$$\int\limits_\Omega a|\nabla_x u^\varepsilon|^2\,d\boldsymbol{x} + a_m\varepsilon^p \int\limits_{\gamma^\varepsilon} |\partial_{x_1} u^\varepsilon|^2\,d\sigma^\varepsilon(\boldsymbol{x})$$
$$= \int\limits_\Omega f u^\varepsilon\,d\boldsymbol{x} + \int\limits_\Omega (a\beta\theta^\varepsilon)\,\mathbf{1}\cdot\nabla_x u^\varepsilon\,d\boldsymbol{x} + a_m\varepsilon^p \int\limits_{\gamma^\varepsilon} \beta_m\theta^\varepsilon\partial_{x_1}u^\varepsilon\,d\sigma^\varepsilon(\boldsymbol{x}),$$

$$\int\limits_\Omega \lambda|\nabla_x\theta^\varepsilon|^2\,d\boldsymbol{x} + \lambda_m\varepsilon^p \int\limits_{\gamma^\varepsilon} |\partial_{x_1}\theta^\varepsilon|^2\,d\sigma^\varepsilon(\boldsymbol{x}) = \int\limits_\Omega g\theta^\varepsilon\,d\boldsymbol{x}.$$

On the strength of the Poincaré-Friedrichs and Cauchy inequalities, we derive the uniform estimates (35) and (36) from these energy identities by rather simple standard arguments. □

Assumption B. Set $p = 1$ further in the article.

On the strength of Assumption B, the uniform estimates (35) and (36), and the provisions of the method of two-scale convergence from Sec. 2.2, we establish the limiting relations for a subsequence of solutions to Problem B_ε.

Lemma 2. *Let Assumption B hold.*

Then there exist a subsequence $\{(u^\varepsilon, \theta^\varepsilon)\}_{\varepsilon\to 0+} \subset V^\varepsilon$ in the family of solutions to Problem B_ε and limiting functions u^*, $\theta^* \in H^1_0(\Omega)$ and u^1, $\theta^1 \in L^2(\Omega; H^1_\sharp(\Xi))$ such that the following limiting relations hold:

$$\lim_{\varepsilon\to 0+} \int\limits_\Omega \nabla_x u^\varepsilon(\boldsymbol{x}) \cdot \boldsymbol{\varphi}\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\Big)\,d\boldsymbol{x}$$
$$= \int\limits_\Omega \int\limits_\Xi \big(\nabla_x u^*(\boldsymbol{x}) + \nabla_\xi u^1(\boldsymbol{x}, \boldsymbol{\xi})\big) \cdot \boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{\xi})\,d\boldsymbol{\xi}\,d\boldsymbol{x}, \quad \forall \boldsymbol{\varphi} \in C^1(\overline{\Omega}; C^1_\sharp(\Xi))^2, \tag{37}$$

$$\lim_{\varepsilon\to 0+} \Bigg(\varepsilon^2 \int\limits_{\gamma^\varepsilon} \partial_{x_1} u^\varepsilon(\boldsymbol{x})\,\varphi\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\Big)\,d\sigma^\varepsilon(\boldsymbol{x})\Bigg) = 0, \quad \forall \varphi \in C^1(\overline{\Omega}; C^1_\sharp(\Xi)), \tag{38}$$

$$\lim_{\varepsilon\to 0+}\int\limits_{\Omega}\nabla_x\theta^{\varepsilon}(\boldsymbol{x})\cdot\boldsymbol{\varphi}\Big(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\Big)\,d\boldsymbol{x} = \int\limits_{\Omega}\int\limits_{\Xi}\big(\nabla_x\theta^*(\boldsymbol{x})+\nabla_\xi\theta^1(\boldsymbol{x},\boldsymbol{\xi})\big)\cdot\boldsymbol{\varphi}(\boldsymbol{x},\boldsymbol{\xi})\,d\boldsymbol{\xi}\,d\boldsymbol{x},\quad \forall\,\boldsymbol{\varphi}\in C^1(\overline{\Omega};C^1_\sharp(\Xi))^2, \tag{39}$$

$$\lim_{\varepsilon\to 0+}\Bigg(\varepsilon\int\limits_{\gamma^\varepsilon}u^\varepsilon(\boldsymbol{x})\,\varphi\Big(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\Big)\,d\sigma^\varepsilon(\boldsymbol{x})\Bigg) = \int\limits_{\Omega}\int\limits_{\gamma_*}u^*(\boldsymbol{x})\,\varphi(\boldsymbol{x},\xi_1,\xi_2^*)\,d\xi_1\,d\boldsymbol{x},\quad \forall\,\varphi\in C^1(\overline{\Omega};C^1_\sharp(\Xi)), \tag{40}$$

$$\lim_{\varepsilon\to 0+}\Bigg(\varepsilon\int\limits_{\gamma^\varepsilon}\theta^\varepsilon(\boldsymbol{x})\,\varphi\Big(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\Big)\,d\sigma^\varepsilon(\boldsymbol{x})\Bigg) = \int\limits_{\Omega}\int\limits_{\gamma_*}\theta^*(\boldsymbol{x})\,\varphi(\boldsymbol{x},\xi_1,\xi_2^*)\,d\xi_1\,d\boldsymbol{x},\quad \forall\,\varphi\in C^1(\overline{\Omega};C^1_\sharp(\Xi)), \tag{41}$$

$$\lim_{\varepsilon\to 0+}\Bigg(\varepsilon^2\int\limits_{\gamma^\varepsilon}\partial_{x_1}\theta^\varepsilon(\boldsymbol{x})\,\varphi\Big(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\Big)\,d\sigma^\varepsilon(\boldsymbol{x})\Bigg) = 0,\quad \forall\,\varphi\in C^1(\overline{\Omega};C^1_\sharp(\Xi)). \tag{42}$$

Proof. On the strength of the estimates (35), there exist subsequences $\{u^{\varepsilon'}\}\subset\{u^\varepsilon\}$ and $\{\theta^{\varepsilon'}\}\subset\{\theta^\varepsilon\}$ and limiting functions $u^*\in H^1_0(\Omega)$ and $\theta^*\in H^1_0(\Omega)$ such that

$$u^{\varepsilon'}\underset{\varepsilon\to 0+}{\longrightarrow}u^*\ \text{weakly in}\ H^1(\Omega), \tag{43}$$

$$\theta^{\varepsilon'}\underset{\varepsilon\to 0+}{\longrightarrow}\theta^*\ \text{weakly in}\ H^1(\Omega). \tag{44}$$

Then, from assertion (ii) of Proposition 4 it follows that the limiting relations (37) and (39) (with $\varepsilon := \varepsilon'$) are valid for $u^{\varepsilon'}$ and $\theta^{\varepsilon'}$, and from assertion (i) of Proposition 4 it follows that the limiting relations

$$\lim_{\varepsilon'\to 0+}\int\limits_{\Omega}u^{\varepsilon'}(\boldsymbol{x})\,\varphi\Big(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon'}\Big)\,d\boldsymbol{x} = \int\limits_{\Omega}\int\limits_{\Xi}u^*(\boldsymbol{x})\,\varphi(\boldsymbol{x},\boldsymbol{\xi})\,d\boldsymbol{\xi}\,d\boldsymbol{x} \tag{45}$$

and

$$\lim_{\varepsilon'\to 0+}\int\limits_{\Omega}\theta^{\varepsilon'}(\boldsymbol{x})\,\varphi\Big(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon'}\Big)\,d\boldsymbol{x} = \int\limits_{\Omega}\int\limits_{\Xi}\theta^*(\boldsymbol{x})\,\varphi(\boldsymbol{x},\boldsymbol{\xi})\,d\boldsymbol{\xi}\,d\boldsymbol{x} \tag{46}$$

hold true.

Further, the subsequences $\{u^{\varepsilon'}\}$ and $\{\theta^{\varepsilon'}\}$ satisfy the estimates (36) (with $p=1$). Due to this and the relations (45) and (46), on the strength of assertion (i) of Proposition 6, there exist subsequences $\{u^{\varepsilon''}\}\subset\{u^{\varepsilon'}\}$ and $\{\theta^{\varepsilon''}\}\subset\{\theta^{\varepsilon'}\}$ such that the limiting relations (40) and (41) hold true (with $\varepsilon := \varepsilon''$).

The subsequences $\{u^{\varepsilon''}\}$ and $\{\theta^{\varepsilon''}\}$, as well as the subsequences $\{u^{\varepsilon'}\}$ and $\{\theta^{\varepsilon'}\}$, also satisfy the estimates (36) (with $p=1$). Due to this and the relations (40) and (41) (with $\varepsilon=\varepsilon''$), on the strength of Proposition 7, there exist subsequences $\{u^{\varepsilon'''}\}\subset\{u^{\varepsilon''}\}$ and $\{\theta^{\varepsilon'''}\}\subset\{\theta^{\varepsilon''}\}$ such that the following limiting relations hold:

$$\lim_{\varepsilon'''\to 0+}\Bigg(\varepsilon'''\int\limits_{\gamma^{\varepsilon'''}}\varepsilon'''\partial_{x_1}u^{\varepsilon'''}(\boldsymbol{x})\,\varphi\Big(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon'''}\Big)\,d\sigma^{\varepsilon'''}(\boldsymbol{x})\Bigg)$$

$$= \int\limits_{\Omega}\int\limits_{\gamma_*} \partial_{\xi_1} u^*(\boldsymbol{x})\, \varphi(\boldsymbol{x}, \xi_1, \xi_2^*)\, d\xi_1\, d\boldsymbol{x},$$

$$\lim_{\varepsilon'''\to 0+}\left(\varepsilon''' \int\limits_{\gamma^{\varepsilon'''}} \varepsilon'''\partial_{x_1}\theta^{\varepsilon'''}(\boldsymbol{x})\, \varphi\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon'''}\Big)\, d\sigma^{\varepsilon'''}(\boldsymbol{x})\right)$$

$$= \int\limits_{\Omega}\int\limits_{\gamma_*} \partial_{\xi_1} \theta^*(\boldsymbol{x})\, \varphi(\boldsymbol{x}, \xi_1, \xi_2^*)\, d\xi_1\, d\boldsymbol{x}.$$

Here, in the right hand sides, the both limiting integrals vanish:

$$\int\limits_{\Omega}\int\limits_{\gamma_*} \partial_{\xi_1} u^*(\boldsymbol{x})\, \varphi(\boldsymbol{x}, \xi_1, \xi_2^*)\, d\xi_1\, d\boldsymbol{x} = 0, \quad \int\limits_{\Omega}\int\limits_{\gamma_*} \partial_{\xi_1} \theta^*(\boldsymbol{x})\, \varphi(\boldsymbol{x}, \xi_1, \xi_2^*)\, d\xi_1\, d\boldsymbol{x} = 0,$$

since u^* and θ^* do not depend on ξ_1. Thus, the limiting relations (38) and (42) hold (with $\varepsilon := \varepsilon'''$), which completes the proof of the lemma. □

Now we formulate and prove the second main result of this paper, namely, the homogenization result for Problem B_ε as $\varepsilon \to 0+$.

Theorem 4. *Let Assumption B hold, i.e., $p = 1$. Then, as $\varepsilon \to 0+$, the family $\{(u^\varepsilon, \theta^\varepsilon) \in V^\varepsilon\}_{\varepsilon>0}$ of solutions to Problem B_ε converges weakly in $H^1(\Omega)\times H^1(\Omega)$ to the solution $(u^*, \theta^*) \in H_0^1(\Omega)\times H_0^1(\Omega)$ of Problem H formulated below. Moreover, (u^*, θ^*) is* the unique *solution of Problem H.*

Problem H. (The effective homogenized model.) *Find a pair of functions $(u^*, \theta^*) \in H_0^1(\Omega)\times H_0^1(\Omega)$ satisfying the variational equations*

$$\int\limits_{\Omega} \mathbb{A}^*(\nabla_x u^* - \boldsymbol{a}_\beta^*\,\theta^*)\cdot\nabla_x\psi\, d\boldsymbol{x} = \int\limits_{\Omega} f\,\psi\, d\boldsymbol{x}, \quad \forall\,\psi \in H_0^1(\Omega), \tag{47}$$

$$\int\limits_{\Omega} \mathbb{L}^*\nabla_x\theta^*\cdot\nabla_x\phi\, d\boldsymbol{x} = \int\limits_{\Omega} g\,\phi\, d\boldsymbol{x}, \quad \forall\,\phi \in H_0^1(\Omega), \tag{48}$$

where

$\mathbb{A}^* = \begin{pmatrix} a + a_m & 0 \\ 0 & a \end{pmatrix}$ *is the matrix of effective elasticity moduli,*

$\boldsymbol{a}_\beta^* = \left(\dfrac{a\beta + a_m\beta_m}{a + a_m}, \beta\right)^T$ *is the vector of effective coefficients of linear thermal expansion,*

$\mathbb{L}^* = \begin{pmatrix} \lambda + \lambda_m & 0 \\ 0 & \lambda \end{pmatrix}$ *is the matrix of effective thermal conduction coefficients.*

Remark 5. In the sense of distributions, Problem H is the weak formulation of the following boundary value problem.

Problem H-diff. *In the bounded domain $\Omega \subset \mathbb{R}_x^2$, find the distributions of displacements $u^* = u^*(\boldsymbol{x})$ and temperature $\theta^* = \theta^*(\boldsymbol{x})$ satisfying the equilibrium equation*

$$-\nabla_x\cdot\big(\mathbb{A}^*(\nabla_x u^* - \boldsymbol{a}_\beta^*\,\theta^*)\big) = f, \quad \boldsymbol{x} \in \Omega, \tag{49}$$

the heat equation

$$-\nabla_x \cdot \big(\mathbb{L}^* \nabla_x \theta^*\big) = g, \quad \boldsymbol{x} \in \Omega, \tag{50}$$

and the homogeneous boundary conditions

$$u^*(\boldsymbol{x}) = 0, \quad \theta^*(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \partial\Omega. \tag{51}$$

Proof. We divide the proof of Theorem 4 into three stages. As has already been noted, Problem B_ε naturally splits into two subproblems. At first, we find θ^ε from the variational equation (33). Then we determine u^ε from the variational equation (32), with θ^ε given as the solution of the former subproblem. Correspondingly, the homogenization procedure as $\varepsilon \to 0+$ consists of the two stages. At the first stage, we homogenize the equation (33) and thus derive the equation (48). At the second stage, we homogenize the equation (32) and derive the equation (47). Finally, at the third stage, we prove the uniqueness assertion for the solution of Problem H.

Stage 1. *Derivation of the variational equation* (48). In (33) take the test function of the form

$$\vartheta(\boldsymbol{x}) = \vartheta_\varepsilon(\boldsymbol{x}) = \phi(\boldsymbol{x}) + \varepsilon\phi^1\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\Big),$$

where $\phi \in C_0^\infty(\Omega)$ and $\phi^1 \in C_0^\infty(\Omega; C_\sharp^1(\Xi))$ are arbitrary functions. Thus we get the integral equality

$$\begin{aligned}\int\limits_\Omega \lambda \nabla_x \theta^\varepsilon(\boldsymbol{x}) \cdot \widetilde{\nabla}_x\Big(\phi(\boldsymbol{x}) + \varepsilon\phi^1\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\Big)\Big)\, d\boldsymbol{x} \\ + \lambda_m \varepsilon \int\limits_{\gamma^\varepsilon} \partial_{x_1}\theta^\varepsilon(\boldsymbol{x})\, \widetilde{\partial}_{x_1}\Big(\phi(\boldsymbol{x}) + \varepsilon\phi^1\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\Big)\Big)\, d\sigma^\varepsilon(\boldsymbol{x}) \\ = \int\limits_\Omega g(\boldsymbol{x})\Big(\phi(\boldsymbol{x}) + \varepsilon\phi^1\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\Big)\Big)\, d\boldsymbol{x}.\end{aligned} \tag{52}$$

Notation 5. Here and further by $\widetilde{\nabla}_x$ we denote the full gradient with respect to $\boldsymbol{x}$, i.e., for all sufficiently smooth functions $\Phi = \Phi(\boldsymbol{x}, \boldsymbol{\xi})$ we have

$$\widetilde{\nabla}_x \Phi\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\Big) = \Big(\nabla_x \Phi(\boldsymbol{x}, \boldsymbol{\xi}) + \frac{1}{\varepsilon}\nabla_\xi \Phi(\boldsymbol{x}, \boldsymbol{\xi})\Big)\Big|_{\boldsymbol{\xi} = \boldsymbol{x}/\varepsilon}.$$

At the same time, ∇_x and ∇_ξ are the gradient operators consisting of the partial derivatives, i.e., for all sufficiently smooth functions $\Phi = \Phi(\boldsymbol{x}, \boldsymbol{\xi})$ we have

$$\nabla_x \Phi\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\Big) = \Big(\nabla_x \Phi(\boldsymbol{x}, \boldsymbol{\xi})\Big)\Big|_{\boldsymbol{\xi} = \boldsymbol{x}/\varepsilon}, \quad \nabla_\xi \Phi\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\Big) = \Big(\nabla_\xi \Phi(\boldsymbol{x}, \boldsymbol{\xi})\Big)\Big|_{\boldsymbol{\xi} = \boldsymbol{x}/\varepsilon}.$$

In accord with this, we have that $\widetilde{\nabla}_x = \nabla_x + \dfrac{1}{\varepsilon}\nabla_\xi$.

Quite similarly, we define the full derivative $\widetilde{\partial}_{x_1}$ and the partial derivatives ∂_{x_1} and ∂_{ξ_1}. In particular, we have $\widetilde{\partial}_{x_1} = \partial_{x_1} + \dfrac{1}{\varepsilon}\,\partial_{\xi_1}$.

Let us study each of the integrals in (52) separately and, using Lemmas 1 and 2, pass to the limit in (52) as $\varepsilon \to 0+$.

Remark 6. In order not to repeat ourselves each time, note that each of the following limiting relations at stages 1 and 2 of this proof is valid for some chosen subsequence. As a matter of fact, at stages 1 and 2, we prove that there exists some subsequence $(u^\varepsilon, \theta^\varepsilon)$ of solutions to Problem B_ε that converges to a solution of Problem H weakly in $H_0^1(\Omega) \times H_0^1(\Omega)$.

For the first integral in the left hand side, we have

$$\int_\Omega \lambda \nabla_x \theta^\varepsilon(\boldsymbol{x}) \cdot \widetilde{\nabla}_x \Big(\phi(\boldsymbol{x}) + \varepsilon \phi^1\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\Big)\Big)\, d\boldsymbol{x}$$
$$= \int_\Omega \lambda \nabla_x \theta^\varepsilon(\boldsymbol{x}) \cdot \nabla_x \phi(\boldsymbol{x})\, d\boldsymbol{x} + \varepsilon \int_\Omega \lambda \nabla_x \theta^\varepsilon(\boldsymbol{x}) \cdot \nabla_x \phi^1\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\Big)\, d\boldsymbol{x}$$
$$+ \int_\Omega \lambda \nabla_x \theta^\varepsilon(\boldsymbol{x}) \cdot \nabla_\xi \phi^1\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\Big)\, d\boldsymbol{x} \underset{\varepsilon\to 0+}{\longrightarrow} \dots$$

we pass to the limit using the relations (39) and (46):

$$\dots \underset{\varepsilon\to 0+}{\longrightarrow} \int_\Omega \int_\Xi \lambda(\nabla_x \theta^*(\boldsymbol{x}) + \nabla_x \theta^1(\boldsymbol{x}, \xi)) \cdot (\nabla_x \phi(\boldsymbol{x}) + \nabla_\xi \phi^1(\boldsymbol{x}, \boldsymbol{\xi}))\, d\boldsymbol{\xi}\, d\boldsymbol{x}$$
$$= \int_\Omega \lambda \nabla_x \theta^*(\boldsymbol{x}) \cdot \nabla_x \phi(\boldsymbol{x})\, d\boldsymbol{x} + \int_\Omega \int_\Xi \lambda \nabla_\xi \theta^1(\boldsymbol{x}, \boldsymbol{\xi}) \cdot \nabla_\xi \phi^1(\boldsymbol{x}, \boldsymbol{\xi})\, d\boldsymbol{\xi}\, d\boldsymbol{x}.$$

Here, the last equality holds true due to the identities

$$\int_\Xi d\boldsymbol{\xi} = 1, \quad \int_\Xi \nabla_\xi \theta^1(\boldsymbol{x}, \boldsymbol{\xi}) \cdot \nabla_x \phi(\boldsymbol{x})\, d\boldsymbol{\xi} = 0, \quad \int_\Xi \nabla_x \theta^*(\boldsymbol{x}) \cdot \nabla_\xi \phi^1(\boldsymbol{x}, \boldsymbol{\xi})\, d\boldsymbol{\xi} = 0, \tag{53}$$

which, in turn, are valid due to the definition of the periodicity cell Ξ and due to Ξ-periodicity of functions θ^1 and ϕ^1.

Thus we conclude that the limiting relation

$$\int_\Omega \lambda \nabla_x \theta^\varepsilon(\boldsymbol{x}) \cdot \widetilde{\nabla}_x \Big(\phi(\boldsymbol{x}) + \varepsilon \phi^1\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\Big)\Big)\, d\boldsymbol{x} \underset{\varepsilon\to 0+}{\longrightarrow} \tag{54}$$
$$\int_\Omega \lambda \nabla_x \theta^*(\boldsymbol{x}) \cdot \nabla_x \phi(\boldsymbol{x})\, d\boldsymbol{x} + \int_\Omega \int_\Xi \lambda \nabla_\xi \theta^1(\boldsymbol{x}, \boldsymbol{\xi}) \cdot \nabla_\xi \phi^1(\boldsymbol{x}, \boldsymbol{\xi})\, d\boldsymbol{\xi}\, d\boldsymbol{x}$$

holds true.

Now consider the second integral in the left hand side of (52). We have

$$\lambda_m \varepsilon \int_{\gamma^\varepsilon} \partial_{x_1} \theta^\varepsilon(\boldsymbol{x})\, \widetilde{\partial}_{x_1} \Big(\phi(\boldsymbol{x}) + \varepsilon \phi^1\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\Big)\Big)\, d\sigma^\varepsilon(\boldsymbol{x}) \tag{55}$$
$$= \lambda_m \varepsilon \int_{\gamma^\varepsilon} \partial_{x_1} \theta^\varepsilon(\boldsymbol{x})\, \partial_{x_1} \phi(\boldsymbol{x})\, d\sigma^\varepsilon(\boldsymbol{x}) + \lambda_m \varepsilon^2 \int_{\gamma^\varepsilon} \partial_{x_1} \theta^\varepsilon(\boldsymbol{x})\, \partial_{x_1} \phi^1\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\Big)\, d\sigma^\varepsilon(\boldsymbol{x})$$
$$+ \lambda_m \varepsilon \int_{\gamma^\varepsilon} \partial_{x_1} \theta^\varepsilon(\boldsymbol{x})\, \partial_{\xi_1} \phi^1\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\Big)\, d\sigma^\varepsilon(\boldsymbol{x})$$
$$\overset{(31)}{=} -\lambda_m \varepsilon \int_{\gamma^\varepsilon} \theta^\varepsilon(\boldsymbol{x})\, \partial^2_{x_1} \phi(\boldsymbol{x})\, d\sigma^\varepsilon(\boldsymbol{x}) + \lambda_m \varepsilon^2 \int_{\gamma^\varepsilon} \partial_{x_1} \theta^\varepsilon(\boldsymbol{x})\, \partial_{x_1} \phi^1\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\Big)\, d\sigma^\varepsilon(\boldsymbol{x})$$
$$+ \lambda_m \varepsilon \int_{\gamma^\varepsilon} \partial_{x_1} \theta^\varepsilon(\boldsymbol{x})\, \partial_{\xi_1} \phi^1\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\Big)\, d\sigma^\varepsilon(\boldsymbol{x}).$$

Due to the estimate $(36)_2$ (with $p = 1$), on the strength of Proposition 5, there exist a subsequence in $\{\varepsilon \to 0+\}$ (still labeled by ε) and a limiting function $\mathcal{Q}_1 \in L^2(\Omega \times \gamma_*)$ such that

$$\partial_{x_1}\theta^{\varepsilon} \underset{\varepsilon\to 0+}{\longrightarrow} \mathcal{Q}_1 \quad \text{two-scale in the sense of Definition 4.} \tag{56}$$

Passing to the limit in (55) due to (41), (42), and (56) we get the limiting relation

$$\begin{aligned}(57)\quad & \lambda_m\varepsilon \int\limits_{\gamma^\varepsilon} \partial_{x_1}\theta^{\varepsilon}(\boldsymbol{x})\,\widetilde{\partial}_{x_1}\Big(\phi(\boldsymbol{x}) + \varepsilon\phi^1\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\Big)\Big)\,d\sigma^{\varepsilon}(\boldsymbol{x}) \underset{\varepsilon\to 0+}{\longrightarrow} \\ & -\lambda_m \int\limits_{\Omega}\int\limits_{\gamma_*} \theta^*(\boldsymbol{x})\,\partial^2_{x_1}\phi(\boldsymbol{x})\,d\xi_1\,d\boldsymbol{x} + 0 + \lambda_m \int\limits_{\Omega}\int\limits_{\gamma_*} \mathcal{Q}_1(\boldsymbol{x}, \xi_1, \xi_2^*)\,\partial_{\xi_1}\phi^1(\boldsymbol{x}, \xi_1, \xi_2^*)\,d\xi_1 d\boldsymbol{x}.\end{aligned}$$

Since $g \in L^2(\Omega)$, for the right hand side of (52) we have

$$\int\limits_{\Omega} g(\boldsymbol{x})\Big(\phi(\boldsymbol{x}) + \varepsilon\phi^1\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\Big)\Big)\,d\boldsymbol{x} \underset{\varepsilon\to 0+}{\longrightarrow} \int\limits_{\Omega} g(\boldsymbol{x})\,\phi(\boldsymbol{x})\,d\boldsymbol{x}. \tag{58}$$

As the result, using the limiting relations (54), (57), and (58), integrating by parts in x_1 in the first limiting integral in (57), and taking into account the identity $\int_{\gamma_*} d\xi_1 = 1$, from (52) we derive the integral equality

$$\begin{aligned}(59)\quad & \int\limits_{\Omega} \lambda\nabla_x\theta^*(\boldsymbol{x})\cdot\nabla_x\phi(\boldsymbol{x})\,d\boldsymbol{x} + \int\limits_{\Omega}\int\limits_{\Xi} \lambda\nabla_{\xi}\theta^1(\boldsymbol{x}, \boldsymbol{\xi})\cdot\nabla_{\xi}\phi^1(\boldsymbol{x}, \boldsymbol{\xi})\,d\boldsymbol{\xi}\,d\boldsymbol{x} \\ & + \lambda_m \int\limits_{\Omega} \partial_{x_1}\theta^*(\boldsymbol{x})\,\partial_{x_1}\phi(\boldsymbol{x})\,d\boldsymbol{x} + \lambda_m \int\limits_{\Omega}\int\limits_{\gamma_*} \mathcal{Q}_1(\boldsymbol{x}, \xi_1, \xi_2^*)\,\partial_{\xi_1}\phi^1(\boldsymbol{x}, \xi_1, \xi_2^*)\,d\xi_1\,d\boldsymbol{x} \\ & = \int\limits_{\Omega} g(\boldsymbol{x})\phi(\boldsymbol{x})\,d\boldsymbol{x}.\end{aligned}$$

Note that, since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, in (59) we can take any function belonging to $H_0^1(\Omega)$ as a test function ϕ.

Also note that the macroscopic and microscopic scales are separated in (59). More precisely, in the integrals containing the test function ϕ, only the sought function θ^* takes place and the functions θ^1 and $\mathcal{Q}_1$ and the fast (microscopic) variables ξ_1 and ξ_2 are absent. In turn, the sought function θ^* is absent in the integrals containing the test function ϕ^1.

Inserting the test function $\phi_1 \equiv 0$ into (59), we arrive exactly at the variational equation (48).

Stage 2. *Derivation of the variational equation* (47). In (32) (with $p = 1$) let us take the test function

$$v(\boldsymbol{x}) = v_{\varepsilon}(\boldsymbol{x}) = \psi(\boldsymbol{x}) + \varepsilon\psi^1\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\Big),$$

where $\psi \in C_0^\infty(\Omega)$ and $\psi^1 \in C_0^\infty(\Omega; C^1_\sharp(\Xi))$ are arbitrary. Thus we get the integral equality

$$\int\limits_{\Omega} a\nabla_x u^{\varepsilon}(\boldsymbol{x})\cdot\widetilde{\nabla}_x\Big(\psi(\boldsymbol{x}) + \varepsilon\psi^1\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\Big)\Big)\,d\boldsymbol{x} \tag{60}$$

$$
\begin{aligned}
&- \int\limits_{\Omega} \big(a\beta\theta^{\varepsilon}(\boldsymbol{x})\big)\,\mathbf{1}\cdot\widetilde{\nabla}_x\Big(\psi(\boldsymbol{x})+\varepsilon\psi^1\Big(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\Big)\Big)\,d\boldsymbol{x}\\
&+ a_m\varepsilon\int\limits_{\gamma^{\varepsilon}} \partial_{x_1}u^{\varepsilon}(\boldsymbol{x})\,\widetilde{\partial}_{x_1}\Big(\psi(\boldsymbol{x})+\varepsilon\psi^1\Big(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\Big)\Big)\,d\sigma^{\varepsilon}(\boldsymbol{x})\\
&- a_m\beta_m\varepsilon\int\limits_{\gamma^{\varepsilon}} \theta^{\varepsilon}(\boldsymbol{x})\,\widetilde{\partial}_{x_1}\Big(\psi(\boldsymbol{x})+\varepsilon\psi^1\Big(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\Big)\Big)\,d\sigma^{\varepsilon}(\boldsymbol{x})\\
= &\int\limits_{\Omega} f(\boldsymbol{x})\,\Big(\psi(\boldsymbol{x})+\varepsilon\psi^1\Big(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\Big)\Big)\,d\boldsymbol{x}.
\end{aligned}
$$

Let us study each of the integrals in (60) separately, and, using Lemmas 1 and 2, pass to the limit in (60) as $\varepsilon \to 0+$.

Quite similarly to the limiting relation (54), on the strength of (39) and Ξ-periodicity of the functions u^1 and ψ^1, for the first integral in the left hand side of (60) we deduce

$$
\begin{aligned}
(61)\qquad &\int\limits_{\Omega} a\nabla_x u^{\varepsilon}(\boldsymbol{x})\cdot\widetilde{\nabla}_x\Big(\psi(\boldsymbol{x})+\varepsilon\psi^1\Big(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\Big)\Big)\,d\boldsymbol{x} \underset{\varepsilon\to 0+}{\longrightarrow}\\
&\int\limits_{\Omega} a\nabla_x u^{*}(\boldsymbol{x})\cdot\nabla_x\psi(\boldsymbol{x})\,d\boldsymbol{x} + \int\limits_{\Omega}\int\limits_{\Xi} a\nabla_{\xi}u^1(\boldsymbol{x},\boldsymbol{\xi})\cdot\nabla_{\xi}\psi^1(\boldsymbol{x},\boldsymbol{\xi})\,d\boldsymbol{\xi}\,d\boldsymbol{x}.
\end{aligned}
$$

On the strength of the relation (46), i.e., due to the fact that $\theta^{\varepsilon} \underset{\varepsilon\to 0+}{\longrightarrow} \theta^*$ two-scale in the sense of Definition 3, taking into account identity $(53)_1$ and Ξ-periodicity of ψ^1, for the second integral in the left hand side of (60) we obtain

$$
\begin{aligned}
(62)\qquad &- \int\limits_{\Omega} \big(a\beta\theta^{\varepsilon}(\boldsymbol{x})\big)\,\mathbf{1}\cdot\widetilde{\nabla}_x\Big(\psi(\boldsymbol{x})+\varepsilon\psi^1\Big(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\Big)\Big)\,d\boldsymbol{x}\\
&= - \int\limits_{\Omega} \big(a\beta\theta^{\varepsilon}(\boldsymbol{x})\big)\,\mathbf{1}\cdot\nabla_x\psi(x)\,d\boldsymbol{x} - \varepsilon\int\limits_{\Omega} \big(a\beta\theta^{\varepsilon}(\boldsymbol{x})\big)\,\mathbf{1}\cdot\nabla_x\psi^1\Big(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\Big)\,d\boldsymbol{x}\\
&\quad - \int\limits_{\Omega} \big(a\beta\theta^{\varepsilon}(\boldsymbol{x})\big)\,\mathbf{1}\cdot\nabla_{\xi}\psi^1\Big(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\Big)\,d\boldsymbol{x} \underset{\varepsilon\to 0+}{\longrightarrow}\\
&\quad - \int\limits_{\Omega}\int\limits_{\Xi} \big(a\beta\theta^{*}(\boldsymbol{x})\big)\,\mathbf{1}\cdot\big(\nabla_x\psi(\boldsymbol{x})+\nabla_{\xi}\psi^1(\boldsymbol{x},\boldsymbol{\xi})\big)\,d\boldsymbol{\xi}\,d\boldsymbol{x}\\
&= - \int\limits_{\Omega} \big(a\beta\theta^{*}(\boldsymbol{x})\big)\,\mathbf{1}\cdot\nabla_x\psi(\boldsymbol{x})\,d\boldsymbol{x}.
\end{aligned}
$$

Now we turn to consideration of the third integral in the left hand side of (60). At first note that, on the strength of Proposition 5, due to the bound $(36)_1$ (with $p = 1$) there exist a subsequence from $\{\varepsilon \to 0+\}$ (still labeled by ε) and a limiting function $\mathcal{P}_1 \in L^2(\Omega\times\gamma_*)$ such that

$$
(63)\qquad \partial_{x_1}u^{\varepsilon} \underset{\varepsilon\to 0+}{\longrightarrow} \mathcal{P}_1 \quad \text{two-scale in the sense of Definition 4.}
$$

Now we fulfill the limiting transition in the third integral in the left hand side of (60) on the basis of (31), (40), (38), and (63) quite similarly to the derivation of the limiting relation (57). Thus we get

$$(64)\quad a_m\varepsilon\int_{\gamma^\varepsilon}\partial_{x_1}u^\varepsilon(\boldsymbol{x})\,\widetilde{\partial}_{x_1}\Big(\psi(\boldsymbol{x})+\varepsilon\psi^1\Big(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\Big)\Big)\,d\sigma^\varepsilon(\boldsymbol{x})\underset{\varepsilon\to 0+}{\longrightarrow}$$

$$-\,a_m\int_\Omega\int_{\gamma_*}u^*(\boldsymbol{x})\,\partial^2_{x_1}\psi(\boldsymbol{x})\,d\xi_1\,d\boldsymbol{x}$$

$$+\,a_m\int_\Omega\int_{\gamma_*}\mathcal{P}_1(\boldsymbol{x},\xi_1,\xi_2^*)\,\partial_{\xi_1}\psi^1(\boldsymbol{x},\xi_1,\xi_2^*)\,d\xi_1\,d\boldsymbol{x}$$

$$=a_m\int_\Omega\partial_{x_1}u^*(\boldsymbol{x})\,\partial_{x_1}\psi(\boldsymbol{x})\,d\boldsymbol{x}+a_m\int_\Omega\int_{\gamma_*}\mathcal{P}_1(\boldsymbol{x},\xi_1,\xi_2^*)\,\partial_{\xi_1}\psi^1(\boldsymbol{x},\xi_1,\xi_2^*)\,d\xi_1\,d\boldsymbol{x}.$$

On the strength of (41), (42), and Ξ-periodicity of ψ^1, for the last integral in the left hand side of (60) we deduce

(65)

$$-\,a_m\beta_m\varepsilon\int_{\gamma^\varepsilon}\theta^\varepsilon(\boldsymbol{x})\,\widetilde{\partial}_{x_1}\Big(\psi(\boldsymbol{x})+\varepsilon\psi^1\Big(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\Big)\Big)\,d\sigma^\varepsilon(\boldsymbol{x})$$

$$=-a_m\beta_m\varepsilon\int_{\gamma^\varepsilon}\theta^\varepsilon(\boldsymbol{x})\,\partial_{x_1}\psi(\boldsymbol{x})\,d\sigma^\varepsilon(\boldsymbol{x})$$

$$-\,a_m\beta_m\varepsilon^2\int_{\gamma^\varepsilon}\theta^\varepsilon(\boldsymbol{x})\,\partial_{x_1}\psi^1\Big(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\Big)\,d\sigma^\varepsilon(\boldsymbol{x})$$

$$-\,a_m\beta_m\varepsilon\int_{\gamma^\varepsilon}\theta^\varepsilon(\boldsymbol{x})\,\partial_{\xi_1}\psi^1\Big(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\Big)\,d\sigma^\varepsilon(\boldsymbol{x})\underset{\varepsilon\to 0+}{\longrightarrow}$$

$$-\,a_m\beta_m\int_\Omega\int_{\gamma_*}\theta^*(\boldsymbol{x})\,\partial_{x_1}\psi(\boldsymbol{x})\,d\xi_1\,d\boldsymbol{x}$$

$$-\,a_m\beta_m\int_\Omega\int_{\gamma_*}\theta^*(\boldsymbol{x})\,\partial_{\xi_1}\psi^1(\boldsymbol{x},\xi_1,\xi_2^*)\,d\xi_1\,d\boldsymbol{x}=-a_m\beta_m\int_\Omega\theta^*(\boldsymbol{x})\,\partial_{x_1}\psi(\boldsymbol{x})\,d\boldsymbol{x}.$$

Finally, for the right hand side of (60) the limiting relation

$$(66)\qquad\int_\Omega f(\boldsymbol{x})\Big(\psi(\boldsymbol{x})+\varepsilon\psi^1\Big(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\Big)\Big)\,d\boldsymbol{x}\underset{\varepsilon\to 0+}{\longrightarrow}\int_\Omega f(\boldsymbol{x})\,\psi(\boldsymbol{x})\,d\boldsymbol{x}$$

holds true.

As the result, using the limiting relations (61), (62), (64), (65), and (66), as $\varepsilon\to 0+$ from (60) we derive the integral equality

(67)

$$\int_\Omega a\nabla_x u^*(\boldsymbol{x})\cdot\nabla_x\psi(\boldsymbol{x})\,d\boldsymbol{x}+\int_\Omega\int_\Xi a\nabla_\xi u^1(\boldsymbol{x},\boldsymbol{\xi})\cdot\nabla_\xi\psi^1(\boldsymbol{x},\boldsymbol{\xi})\,d\boldsymbol{\xi}\,d\boldsymbol{x}$$

$$-\int_\Omega\big(a\beta\theta^*(\boldsymbol{x})\big)\,\mathbf{1}\cdot\nabla_x\psi(\boldsymbol{x})\,d\boldsymbol{x}+a_m\int_\Omega\partial_{x_1}u^*(\boldsymbol{x})\,\partial_{x_1}\psi(\boldsymbol{x})\,d\boldsymbol{x}$$

$$+ a_m \int\limits_{\Omega} \int\limits_{\gamma_*} \mathcal{P}_1(\boldsymbol{x}, \xi_1, \xi_2^*)\, \partial_{\xi_1} \psi^1(\boldsymbol{x}, \xi_1, \xi_2^*)\, d\xi_1\, d\boldsymbol{x} - a_m \beta_m \int\limits_{\Omega} \theta^*(\boldsymbol{x}) \partial_{x_1} \psi(\boldsymbol{x})\, d\boldsymbol{x}$$
$$= \int\limits_{\Omega} f(\boldsymbol{x})\, \psi(\boldsymbol{x})\, d\boldsymbol{x}.$$

Notice that, since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, it is admissible to take a $\psi \in H_0^1(\Omega)$ as a test function in (67).

Further note that the macroscopic and microscopic scales are separated in (67), similarly to how in the integral equality (59). More precisely, in the integrals containing the test function ψ, only the sought macroscopic functions u^* and θ^* take place and the functions u^1 and $\mathcal{P}_1$ and the fast (microscopic) variables ξ_1 and ξ_2 are absent. In turn, the sought functions u^* and θ^* are absent in the integrals containing the test function ψ^1.

Inserting the test function $\psi_1 \equiv 0$ into (67), we arrive exactly at the variational equation (47).

Thus, at the stages 1 and 2 we established that there exists a subsequence $\{(u^\varepsilon, \theta^\varepsilon)\}_{\varepsilon \to 0+}$ of solutions to Problem B_ε convergent weakly in $H_0^1(\Omega) \times H_0^1(\Omega)$ to the solution of Problem H. Clearly, the existence assertion for solution of Problem H follows immediately from the limiting passage as $\varepsilon \to 0+$ (along the chosen subsequence).

Stage 3. *Uniqueness of solution to Problem H.* The same as for the uniqueness assertions in Theorems 1 and 3, justification of uniqueness of solution to Problem H is rather simple and relies on the Lax-Milgram theorem. By the way, the use of the Lax-Milgram theorem also provides the proof of the assertion on existence of solutions to Problem H independently of the above constructed limiting passage as $\varepsilon \to 0+$.

Finally, on the strength of uniqueness of the solution (u^*, θ^*), we conclude that ***the whole*** family $\{(u^\varepsilon, \theta^\varepsilon)\}_{\varepsilon>0}$ of solutions to Problem B_ε converges to (u^*, θ^*) as $\varepsilon \to 0+$ and therefore there is no need to choose a subsequence following Remark 6. This observation completes the proof of Theorem 4. □

3. The results of numerical analysis

In this section, we present the results of the series of numerical experiments for the established limiting models. The main goal of these experiments is to show numerically that the family of solutions of the original model Problem A converges to the solution of Problem B as $\delta \to 0+$ and the family of solutions of Problem B_ε converges to the solution of Problem H as $\varepsilon \to 0+$.

3.1. **The problem with one thin inclusion.** At first, let us focus on Problem B, which considers only one elastic inclusion. The input data are taken as follows:

$$\begin{gathered} a_\pm = 1, \quad a_m = 2, \quad \lambda_\pm = 1, \quad \lambda_m = 2, \quad \beta_\pm = 1, \quad \beta_m = 2, \\ f(\boldsymbol{x}) = f(x_1, x_2) = x_1 + 0.25, \quad g(\boldsymbol{x}) = g(x_1, x_2) = 100 \sin x_1. \end{gathered} \tag{68}$$

The rectangle $\{-0.5 < x_1 < 0.5\} \times \{-1 < x_2 < 1\}$ with width $L_{x_1} = 1$ and height $L_{x_2} = 2$ is taken as a computational domain Ω. An elastic inclusion occupies the segment $\{-0.5 < x_1 < 0.5\} \times \{x_2 = 0\}$. The homogeneous Dirichlet conditions $u = 0$ and $\theta = 0$ are imposed on the entire boundary $\partial\Omega$.

Let us introduce a uniform mesh in Ω. We fix the number of triangle elements in the mesh to 10×1000. Characteristic element size along the Ox_2 axis is equal to 2e-2. To approximate the displacement and temperature functions, we use second-order Lagrangian finite elements $\mathbb{P}_2$.

The graphs of the distributions of temperature and displacements in the presence of one elastic inclusion are shown in Figs. 5 and 6.

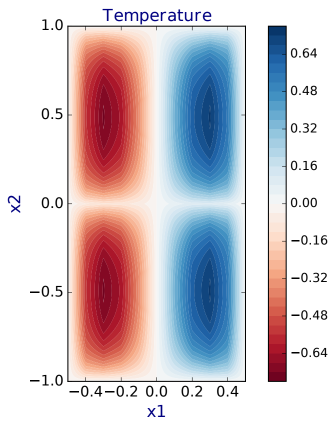

FIG. 5. Temperature

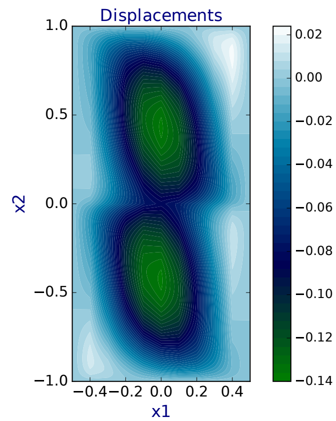

FIG. 6. Displacements

Now, let us consider the initial formulation — Problem A — with various values of parameter δ. Recall that δ is the dimensionless thickness of the inclusion and that δ vanishes in the limit. Let us remind that the coefficients characterizing the intermediate layer (i.e., the bulk inclusion) are defined by the formulas $(11)_2$.

The parameters of the mesh and the types of the finite elements are taken the same as for Problem B. The graphs of the distributions of temperature and displacements for various values of δ are shown in Figs. 7-12. We observe that for sufficiently small value of δ, namely, for δ =5e-4, the distributions of temperature and displacements calculated by Problem A (see Figs. 9 and 12) are close to the solution of Problem B (see Figs. 5 and 6).

Comparison of the results obtained for Problem B with the results obtained for Problem A in the relative L^2-norm is presented in Table 1. The corresponding

TABLE 1. Convergence to the solution of Problem B

δ	$E_{L^2}(\theta)$	$E_{L^2}(u)$
5e-2	0.2789	0.3874
5e-3	0.1078	0.0725
5e-4	0.0173	0.0078
25e-5	0.0555	0.0204

relative errors for displacements and temperature are defined by the formulas

$$E_{L^2}(\theta) = \frac{\|\theta - \theta_\delta\|_{L^2}}{\|\theta\|_{L^2}}, \quad E_{L^2}(u) = \frac{\|u - u_\delta\|_{L^2}}{\|u\|_{L^2}}, \tag{69}$$

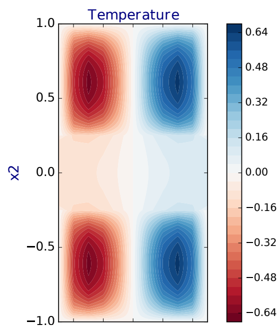

FIG. 7. Temperature, δ =5e-2

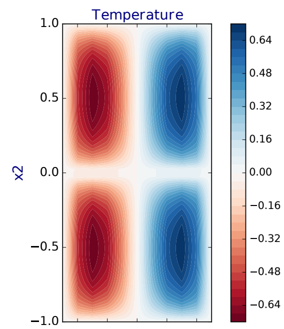

FIG. 8. Temperature, δ =5e-3

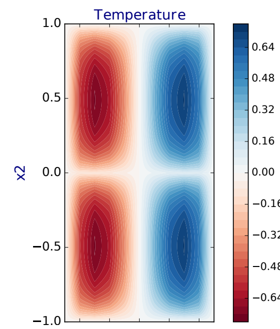

FIG. 9. Temperature, δ =5e-4

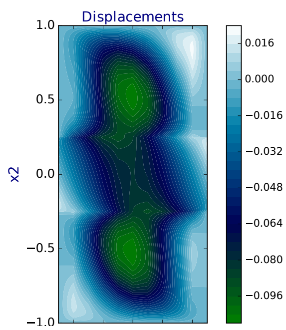

FIG. 10. Displacements, δ =5e-2

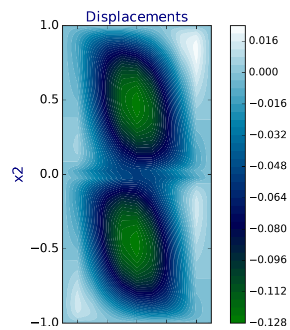

FIG. 11. Displacements, δ =5e-3

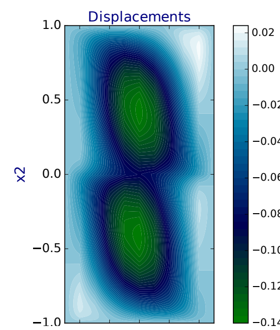

FIG. 12. Displacements, δ =5e-4

where u and θ are the distributions of displacements and temperature calculated by Problem A, and u_δ and θ_δ are the distributions of displacements and temperature calculated by Problem B. The relative error for the distribution of displacement in the case δ =5e-2 is large (38.74 %), as should have been expected. At the same time, in the case δ =5e-4 the relative error is essentially smaller (0.78 %).

Note that, for δ =25e-5 the relative error starts to increase, which is explained by a too small value of the parameter δ for this given size of the mesh. Indeed, in this case, the characteristic size of one element 2e-4 is comparable to the value of δ, which leads to inaccurate approximation of the behavior of the intermediate layer (i.e., of the bulk inclusion).

At the same time, the solution of Problem B does not depend on δ, and therefore the size of the mesh for Problem B can be chosen independently of δ, which makes it possible to take a larger mesh and thereby significantly save computational resources.

3.2. **The problem with multiple inclusions.** In this section we present the results of numerical experiments corresponding to the limiting transition $\varepsilon \to 0+$.

Recall that the parameter ε is the dimensionless distance between two neighboring inclusions. It tends to zero as the number of inclusions infinitely increases. Analogously to the numerical experiments for the case of just one inclusion, we use the uniform mesh consisting of 10×1000 triangular elements and the second-order Lagrangian finite elements. Furthermore, the input data are taken the same as in (68). At first, we consider Problem H. Its numerical solution is shown in Figs. 13 and 14.

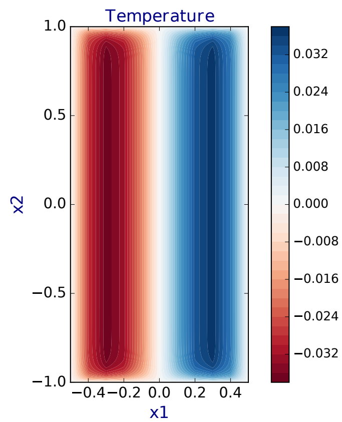

FIG. 13. Temperature

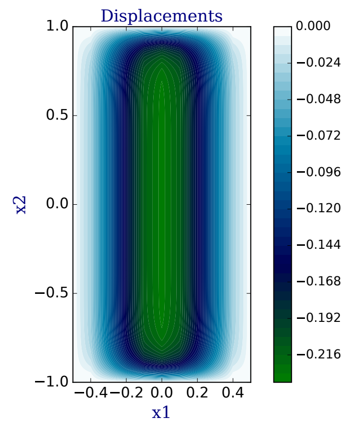

FIG. 14. Displacements

Next, we turn to the case with a finite number of inclusions that was described with Problem B_ε. We thus incorporate a finite number of elastic inclusions equidistant from each other into the computational domain Ω so that the boundaries of the inclusions fall on the boundaries of the mesh. The distributions of temperature and displacements for different values of ε is shown in Figs. 15-20.

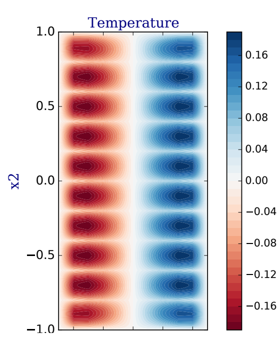

FIG. 15. Temperature, ε=2/10

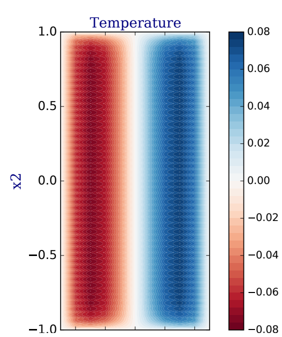

FIG. 16. Temperature, ε=2/50

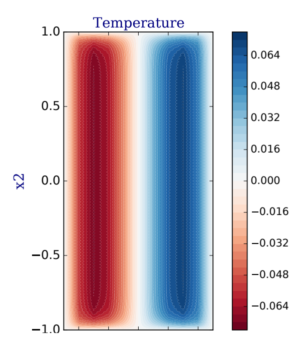

FIG. 17. Temperature, ε=2/100

As ε decreases, the relative errors between the solutions of Problems H and B_ε are defined by the formulas

$$\widetilde{E}_{L^2}(\theta) = \frac{\|\theta^* - \theta_\varepsilon\|_{L^2}}{\|\theta^*\|_{L^2}}, \quad \widetilde{E}_{L^2}(u) = \frac{\|u^* - u_\varepsilon\|_{L^2}}{\|u^*\|_{L^2}}, \tag{70}$$

where u^* and θ^* are the distributions of displacements and temperature calculated via Problem H, and u_ε and θ_ε are the distributions of displacements and temperature calculated via Problem B_ε. Based on the data in Table 2, we conclude that the solution of Problem B_ε tends to the solution of Problem H as ε decreases.

TABLE 2. Passage to the solution of Problem H

ε	$\widetilde{E}_{L_2}(\theta)$	$\widetilde{E}_{L_2}(u)$
2e-1	0.2093	2.580
4e-2	0.0106	0.1286
2e-2	0.00267	0.0325

We underline that, when solving Problem H, there is no need to refine the mesh sharply, while for Problem B_ε it is necessary that the finite element mesh passes through the boundaries of elastic inclusions. This leads to a very fine mesh if a number of inclusions is large. Thus, when implementing Problem H, there is a significant saving in computing resources, as compared to Problem B_ε.

By this remark, we finish observation of numerical experiments in the article.

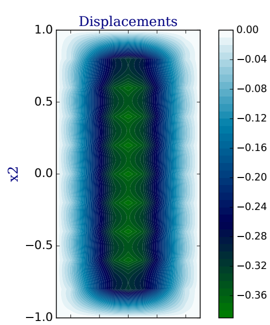

FIG. 18. Displacements, ε=2/10

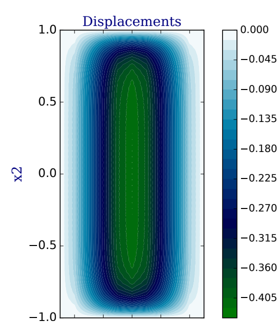

FIG. 19. Displacements, ε=2/50

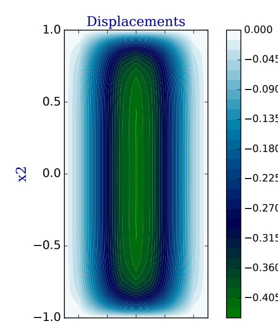

FIG. 20. Displacements, ε=2/100

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Appendix. Nomenclature

In this appendix, we put the fairly complete list of notations used in the article.

Roman Symbols		
Notation	Description	Introduced in
a, a_δ, $a_\pm$	moduli of elasticity	Secs. 1.2-1.4
$\mathbb{A}^*$	matrix of effective elasticity coefficients	formula (47)
$\boldsymbol{a}^*_\beta$	vector of effective coefficients of thermal expansion	formula (47)
b_1^δ, b_2^δ, b_3^δ	bilinear forms	Sec. 1.2
$b_1(\boldsymbol{u},\boldsymbol{v})$, $b_1(\delta;\boldsymbol{u},\boldsymbol{v})$, $b_2(\boldsymbol{\theta},\boldsymbol{v})$, $b_2(\delta;\boldsymbol{\theta},\boldsymbol{v})$, $b_3(\boldsymbol{\theta},\boldsymbol{\vartheta})$, $b_3(\delta;\boldsymbol{\theta},\boldsymbol{\vartheta})$	bilinear forms in the refined formulation of Problem A	Sec. 1.3
$C_\sharp(\Xi)$, $C^1_\sharp(\Xi)$ $C^\infty_\sharp(\Xi)$	spaces of periodic functions	Definition 2
$d\sigma^\varepsilon$	one-dimensional Lebesgue measure on γ^ε	Sec. 2.1
$\boldsymbol{e}_1$, $\boldsymbol{e}_2$	Cartesian basis in $\mathbb{R}^2$	Sec. 2.2
$E_{L^2}(u)$	the relative error for the distributions of displacements in Problems A and B	formula (69)
$E_{L^2}(\theta)$	the relative error for the distributions of temperature in Problems A and B	formula (69)
$\widetilde{E}_{L^2}(u)$	the relative error for the distributions of displacements in Problems B_ε and H	formula (70)
$\widetilde{E}_{L^2}(\theta)$	the relative error for the distributions of temperature in Problems B_ε and H	formula (70)
f, $\tilde{f}$	distributed mass force	Secs. 1.2, 1.3
g, $\tilde{g}$	distributed heat source	Secs. 1.2, 1.3
$H^1_0(\gamma)$	the Sobolev space defined on inclusion γ	Notation 3
$H^1_0(\gamma^\varepsilon)$	the Sobolev space defined on inclusion γ^ε	Sec. 2.1
$H^1_\Gamma(\Omega^\delta_\pm)$, H^δ	Sobolev spaces	Sec. 1.2, Notat. 1
$H^1_\sharp(\Xi)$	Sobolev space of periodic functions	Definition 2
$\mathbf{1}$	$\mathbf{1}=(1,1)^T\in\mathbb{R}^2$	Sec. 1.2
l_1^δ, l_2^δ	linear functions	Sec. 1.2
l_1, l_2	linear functions in the refined	Sec. 1.3

	formulation of Problem A	
$\mathbb{L}^*$	matrix of effective coefficients of heat conductions	formula (48)
L_{x_1}	width of the computational domain	Sec. 3.1
L_{x_2}	height of the computational domain	Sec. 3.1
$\boldsymbol{n}^\delta$	unit outward normal to $\partial\Omega^\delta$	formulas (8), (9)
$\boldsymbol{n}_\pm$	unit outward normals to $\partial\Omega_\pm$	f-las (28), (29)
p	exponent of 'contrast'	formula (33), Remark 4
$\mathcal{P}_1$	limiting two-scale deformation on $\Omega \times \gamma_*$	formula (63)
$\mathcal{Q}_1$	limiting two-scale gradient of temperature on $\Omega \times \gamma_*$	formula (56)
u, u_δ	displacements in Problem A	Secs. 1.2, 1.3
$\boldsymbol{u}$	$\boldsymbol{u} = (u_+, u_-, u_m)$	Notation 1
u^1	limiting two-scale displacement	formula (61)
u_m, u_+, u_-	displacements on $\Omega^\delta_m, \Omega^\delta_+, \Omega^\delta_-$, resp.	Notation 1
u^*	homogenized field of displacements	Theorem 4
V, V_1	Sobolev spaces	Sec. 1.4
V_0	Sobolev space	Sec. 1.3
V^δ	Sobolev space	Sec. 1.2
V^ε	Sobolev space	Sec. 2.1
x_1, x_2	Cartesian coord. in Problems B, B_ε, H	formula (12)
y_1, y_2	Cartesian coord. in Problem P$\hbar$	Sec. 1.1
y_1^*	abscissa of the right endpoint of the inclusion in Problem A	Sec. 1.1
z_1, z_2	Cartesian coordinates in the proof of Theorem 1	formula (12)

Greek Symbols

Notation	Descripition	Introduced in
β, β_δ, $\beta_\pm$	coefficients of thermal expansion	Secs. 1.2-1.4
γ	symmetry axis of the set Ω^δ_m, the thin inclusion	Secs. 1.1, 1.2
$\gamma_{\rm utd}$	union of thin inclusions in $\mathbb{R}^2_\xi$	Sec. 2.2
γ_+	shift of γ to 1/2 upwards in Oy_1y_2-system	Sec. 1.3
γ_-	shift of γ to 1/2 downwards in Oy_1y_2-system	Sec. 1.1
Γ^δ_+, Γ^δ_-	parts of the exterior boundary of Ω^δ	Sec. 1.1
γ^δ_+	shift of γ to $\delta/2$ upwards	Sec. 1.1
γ^δ_-	shift of γ to $\delta/2$ downwards	Sec. 1.1
γ^ε	set of thin inclusions (fibers)	Sec. 1.5
γ_*	inclusion in periodic cell Ξ	Sec. 2.2, Fig. 4
δ	thickness of bulk inclusion, small parameter	Sec. 1.1
ε	distance between neighboring inclusions, small parameter	Sec. 1.5

θ, θ_δ	temperature in Problem A	Secs. 1.2, 1.3
$\boldsymbol{\theta}$	$\boldsymbol{\theta} = (\theta_+, \theta_-, \theta_m)$	Notation 1
θ^1	limiting two-scale temperature	formula (54)
θ_m, θ_+, θ_-	temperature on $\Omega^\delta_m, \Omega^\delta_+, \Omega^\delta_-$, resp.	Notation 1
θ^*	homogenized temperature field	Theorem 4
λ, λ_δ, $\lambda_\pm$	coefficients of heat conduction	Secs. 1.2-1.4
Ξ	periodicity cell	Notation 4
$\boldsymbol{\xi}$, ξ_1, ξ_2	fast (microscopic) variables	Notation 4
ξ_2^*	ordinate of thin inclusion γ_* on Ξ	Sec. 2.2
$\Omega \subset \mathbb{R}^2_x$	domain of composite in Problems $\mathrm{B}, \mathrm{B}_\varepsilon, \mathrm{H}$	Secs. 1.1, 1.5, 2.1
$\Omega \subset \mathbb{R}^2_y$	pre-image of Ω^δ in Problem A	Sec. 1.1
Ω^δ	domain of composite in Problem A	Sec. 1.1
Ω^δ_m	bulk inclusion	Sec. 1.1
Ω_+	pre-image of Ω^δ_+ when shifting on $+\delta/2$	Sec. 1.1
Ω_-	pre-image of Ω^δ_- when shifting on $-\delta/2$	Sec. 1.1
Ω^δ_+, Ω^δ_-	components of the thermoelastic body in Problem A	Sec. 1.1

Some operators

Notation	Description	Introduced in
$\widetilde{\partial}_{x_k}$	full derivative w.r.t. x_k	Notation 5
∇_x, ∇_y	gradient operators	Sec. 1.2
$\widetilde{\nabla}_x$	full gradient in $\boldsymbol{x}$	Notation 5

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