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ON CLOSURE OF CONFIGURATIONS IN FREELY GENERATED PROJECTIVE PLANES

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Abstract. Let $\mathcal{F}$ be an arbitrary freely generated projective plane. Based on Shirshov's combinatorial method, we introduce the notion of a reduced configuration in $\mathcal{F}$. We prove that for every subplane $\mathcal{P}$ generated in $\mathcal{F}$ by some configuration $\mathcal{B}$, there is a reduced configuration $\mathcal{B}'$ such that $\mathcal{P}$ is freely generated by $\mathcal{B}'$.

Keywords: projective plane, configuration, incidence, freely generated projective plane, nonassociative word, regular word.

1. Introduction

In [1], it was proved that if a subplane $\mathcal{P}$ of a free projective plane is generated by a finite configuration $\mathcal{B}$, then $\mathcal{P}$ is also free. The proof of this statement is, in particular, based on the following reasoning. If we represent the process of generating of a projective plane $\mathcal{P}$ in the form of a sequence of configurations

$$\mathcal{B} = \mathcal{B}_0 \subseteq \mathcal{B}_1 \subseteq \ldots \subseteq \mathcal{B}_n \subseteq \ldots,$$

where the extension $\mathcal{B}_n \subseteq \mathcal{B}_{n+1}$ is a full and single-step one for every $n \in \omega$, then for every n the rank of the configuration $\mathcal{B}_{n+1}$ cannot exceed the rank of the configuration $\mathcal{B}_n$, which means that as n increases, the rank of $\mathcal{B}_n$ gets stabilized, therefore, there exists m such that $\mathcal{B}_m$ freely generates $\mathcal{P}$. If we refuse the condition of finiteness of the configuration $\mathcal{B}$, then the mentioned reasoning on ranks becomes

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incorrect, and there arises a necessity to find other ways to find and describe the configuration $\mathcal{B}'$, freely generating the plane $\mathcal{P}$.

In [2, 3, 4, 5, 6, 7], based on the approach by A. I. Shirshov, an algebraic theory of free and freely generated projective planes was developed, which provided a possibility to solve a number of traditional problems in algebra in the mentioned classes of projective planes.

In this article, using the constructions and combinatorial methods of A. I. Shirshov, we introduce the notion of a *reduced* configuration in an arbitrary freely generated projective plane $\mathcal{F}$ and propose a method for transforming an arbitrary configuration into a reduced one. We prove that for every configuration $\mathcal{B}$, generating a subplane $\mathcal{P}$ in the projective plane $\mathcal{F}$, there exists a reduced configuration $\mathcal{B}'$, which freely generates $\mathcal{P}$. In the case of finitely generated planes, the proposed method for transforming the configuration $\mathcal{B}$ into the reduced configuration $\mathcal{B}'$ turns out to be effective, which allows us to state the solvability of the problem of inclusion into the subplane $\mathcal{P}$.

In Section 2 of this article, the necessary definitions related to the theory of projective planes are provided. In Section 3, we introduce the definition of a reduced configuraion and prove the main results of this work.

2. Preliminary information

We will now provide the necessary information from the theory of projective planes [2, 8].

We will consider projective planes as algebraic systems of a signature $\sigma = \langle A^0, {}^0A, I\rangle$, where A^0 and ${}^0A$ are unary predicate symbols, and I is a binary predicate symbol. The domain A of each of such systems will be broken into two subsets $A^0 \cup {}^0A = A$, $A^0 \cap {}^0A = \varnothing$. We refer to the elements $a, b \in A$ as the ones of *the same kind* with respect to the given partition, if $a, b \in A^0$ or $a, b \in {}^0A$. The elements A^0 are called *points*, and the elements ${}^0A$ are called *lines*.

A *configuration* is an algebraic system $\mathcal{A} = \langle A, A^0, {}^0A, I\rangle$ with a partition of the domain A into two subsets $A^0 \cup {}^0A = A$, $A^0 \cap {}^0A = \varnothing$, and a symmetric binary relation $I \subseteq A^2$, which is called a *incidence relation* and satisfies the following conditions:

(P1) If $\langle a, b\rangle \in I$, then the elements a and b are of distinct kinds;

(P2) If $\langle a, c\rangle \in I$, $\langle b, c\rangle \in I$, $\langle a, d\rangle \in I$, $\langle b, d\rangle \in I$, then $a = b$ or $c = d$.

If the configuration $\mathcal{A}$ is finite, then we will call the number $2\cdot|A| - |I|/2$ a *rank* of $\mathcal{A}$. If the configuration $\mathcal{A}$ is infinite, then its *rank* is the cardinality of the set A.

On every configuration $\mathcal{A}$, we additionally define a partial binary commutative operation «$\cdot$» (product) in the following way:

(P3) The product $a \cdot b$ is defined and $a \cdot b = c$ if and only if a, b are distinct elements of A of the same kind, such that $\langle a, c\rangle \in I$ and $\langle b, c\rangle \in I$.

A configuration $\mathcal{A}$ is called *closed*, if for every pair of distinct elements $a, b \in A$ of the same kind in $\mathcal{A}$ the product $a \cdot b$ is defined.

A closed configuration $\mathcal{A}$ is called a *projective plane*, if it satisfies the following *condition of nondegenerancy*:

(P4) There exist pairwise distinct $a, b, c, d \in A$ such that in $\mathcal{A}$ there are products $a \cdot b$, $b \cdot c$, $c \cdot d$, $d \cdot a$ that are defined and pairwise distinct.

A configuration $\mathcal{A} = \langle A, A^0, {}^0A, I^A\rangle$ is a *subconfiguration* of the configuration $\mathcal{B} = \langle B, B^0, {}^0B, I^B\rangle$, if $A^0 \subseteq B^0$, ${}^0A \subseteq {}^0B$ and $I^A = I^B \cap A^2$. If $\mathcal{A}$ is a subconfiguration of $\mathcal{B}$, then we will also say that $\mathcal{B}$ is an *extension* of $\mathcal{A}$ and write $\mathcal{A} \subseteq \mathcal{B}$.

A configuration $\mathcal{A}$ is called *degenerate*, if it can be embedded into a closed configuration which does not satisfy condition (P4). Otherwise, $\mathcal{A}$ is called *nondegenerate*.

An extension $\mathcal{A} \subseteq \mathcal{B}$ is called an *single-step* one, if for every $c \in B\backslash A$ there exist $a, b \in A$ such that $a \cdot b = c$. The single-step extension $\mathcal{A} \subseteq \mathcal{B}$ is called *full*, if for every distinct $a, b \in A$ of the same kind there exists $c \in B$ such that $a \cdot b = c$. The single-step extension $\mathcal{A} \subseteq \mathcal{B}$ is called *free*, if for every $c \in B\backslash A$ there exist exactly two elements $a, b \in A$ such that $a \cdot b = c$.

If $\mathcal{A}$ is a subconfiguration of a closed configuration $\mathcal{P}$, then by $\langle\mathcal{A}\rangle_{\mathcal{P}}$ we will designate the intersection of all closed configurations of $\mathcal{P}$, containing $\mathcal{A}$. Moreover, $\langle\mathcal{A}\rangle_{\mathcal{P}}$ is called a *closure* of the configuration $\mathcal{A}$ in $\mathcal{P}$, and it is said that $\mathcal{A}$ *generates* the configuration $\langle\mathcal{A}\rangle_{\mathcal{P}}$ in $\mathcal{P}$.

It is well known [8, ch. XI] that if $\mathcal{A} = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \ldots \subseteq \mathcal{A}_i \subseteq \ldots$ is a sequence of subconfigurations of a closed configuration $\mathcal{P}$, such that the extension $\mathcal{A}_i \subseteq \mathcal{A}_{i+1}$ is a full single-step one for every $i \in \omega$, then $\bigcup_{i\in\omega} \mathcal{A}_i = \langle\mathcal{A}\rangle_{\mathcal{P}}$. In this case, if the extension $\mathcal{A}_i \subseteq \mathcal{A}_{i+1}$ is free for every $i \in \omega$, then the closure $\langle\mathcal{A}\rangle_{\mathcal{P}}$ is called *free*, and it is written that $\mathcal{F}(\mathcal{A}) = \langle\mathcal{A}\rangle_{\mathcal{P}}$ and said that $\mathcal{A}$ *freely generates* the configuration $\mathcal{F}(\mathcal{A})$.

Note that every configuration $\mathcal{A}$ can be embedded in such closed configuration $\mathcal{P}$, where the closure $\langle\mathcal{A}\rangle_{\mathcal{P}}$ is free and defined with respect to $\mathcal{A}$ in a unique way up to an isomorphism, and does not depend on the choice of $\mathcal{P}$.

In [2, 4], a construction of a free closure of a configuration, which is usually referred to as *Shirshov's construction*, is proposed. In this construction, every element of the free closure has its unambiguous notation in the form of an irreducible nonassociative word over the initial configuration. We will remind the main definitions of the mentioned construction.

A set $W(A)$ of *nonassociative words over the alphabet* A is defined by induction:

(a) if $a \in A$, then $a \in W(A)$;

(b) if $u, v \in W(A)$, then $(uv) \in W(A)$.

We will omit the outer brackets in nonassociative words.

For arbitrary nonassociative words $u, v \in W(A)$, we will write $u \sqsubset v$, if $v = uw$ or $v = wu$ for some word w, that is, u is a maximal proper subword of v. We will write $u \sqsubset^* v$, if there exist words $u_0, \ldots, u_k$ such that $u = u_0 \sqsubset \cdots \sqsubset u_k \sqsubset v$. The notaion $u \sqsubseteq^* v$ means that $u = v$ or $u \sqsubset^* v$.

Construction A. Let $\mathcal{A} = \langle A, A^0, {}^0A, I^A\rangle$ be a nondegenerate nonclosed configuration, on whose elements a strict complete order $\prec$ such that $a \prec b$ for every $a \in A^0$ and $b \in {}^0A$ is defined.

We will consider the elements of the set A as alphabetic symbols. We refer as a *A-length* of a nonassociative word $w \in W(A)$ to a number $|w|$ of occurrences of elements of A in the word w. We will call an *A-weight* of a nonassociative word $w \in W(A)$ a number $\|w\| = n_1 + 2n_2$, where n_1 and n_2 are the numbers of occurrences in the notation of the word w of the symbols from A^0 and ${}^0A$, respectively.

We continue the order $\prec$ to the set $W(A)$. For every $w_1 \neq w_2$ from $W(A)$, we put $w_1 \prec w_2$ if and only if either $\|w_1\| < \|w_2\|$, or $\|w_1\| = \|w_2\|$, and one of the following conditions is satisfied:

(a) $|w_1| < |w_2|$, or

(b) $|w_1| = |w_2| = 1$ and $w_1 \prec w_2$, or

(c) $|w_1| = |w_2| > 1$, $w_1 = u_1u_2$, $w_2 = u_3u_4$, and $u_1 \prec u_3$, or

(d) $|w_1| = |w_2| > 1$, $w_1 = u_1u_2$, $w_2 = u_3u_4$, $u_1 = u_3$, and $u_2 \prec u_4$.

By induction on A-length of the nonassociative word $w \in W(A)$, we define the set of *$\mathcal{A}$-regular words* in the following way:

If $w \in A^0$ ($w \in {}^0A$), then w is called an *$\mathcal{A}$-regular word of the first kind* (*the second kind*).

If $w = w_1w_2$, then w is called an *$\mathcal{A}$-regular word of the first kind* (*the second kind*) if and only if five following conditions are fulfilled:

(A1) $w_1 \succ w_2$ and w_1, w_2 are $\mathcal{A}$-regular words of the second kind (first kind);

(A2) There does not exist any w_3 such that $\langle w_1, w_3\rangle \in I^A$ and $\langle w_2, w_3\rangle \in I^A$;

(A3) If $w_1 = w_3w_4$, then $\langle w_3, w_2\rangle \notin I^A$ and $\langle w_4, w_2\rangle \notin I^A$;

(A4) If $w_1 = w_3w_4$ and $w_2 = w_5w_6$, then $\{w_3, w_4\} \cap \{w_5, w_6\} = \varnothing$;

(A5) If $w_1 = (w_3w_4)w_5$ or $w_1 = w_5(w_3w_4)$, then $w_2 \notin \{w_3, w_4\}$.

We designate by F^0 and ${}^0F$ the sets of all $\mathcal{A}$-regular words of the first and the second kinds respectively.

Note that for $w_1, w_2 \in W(A)$, the words w_1w_2 and w_2w_1 cannot be $\mathcal{A}$-regular simultaneously. If one of the words w_1w_2 or w_2w_1 is $\mathcal{A}$-regular, then we will designate that $\mathcal{A}$-regular word by $\overline{w_1w_2}$. Use of the notation $\overline{w_1w_2}$ will allow us to simplify the presentation of the proofs of the article's statements in the future.

The incidence relation I^F on the set $F = F^0 \cup {}^0F$ of all $\mathcal{A}$-regular words is defined with the help of the following equivalence:

$$\langle w_1, w_2\rangle \in I^F \iff \langle w_1, w_2\rangle \in I^A \vee w_1 \sqsubset w_2 \vee w_2 \sqsubset w_1.$$

We consider the configuration $\mathcal{F} = \langle F, F^0, {}^0F, I^F\rangle$ to be the result of the construction A. In [2, 4], it was proved that the result of the construction A is up to isomorphism a projective plane $\mathcal{F}(\mathcal{A})$, freely generated by $\mathcal{A}$.

3. Reduced configurations and their properties

Let $\mathcal{A} = \langle A, A^0, {}^0A, I^A\rangle$ be a nondegenerate nonclosed configuration. We fix the projective plane $\mathcal{F} = \mathcal{F}(\mathcal{A})$, freely generated by the configuration $\mathcal{A}$. Further we will identify $\mathcal{F}(\mathcal{A})$ to the result $\langle F, F^0, {}^0F, I^F\rangle$ of the construction A, considering every element of $\mathcal{F}(\mathcal{A})$ to be uniquely representable in the form of an $\mathcal{A}$-regular word.

Construction B. Let $\mathcal{B} = \langle B, B^0, {}^0B, I^B\rangle$ be a nondegenerate nonclosed subconfiguration in the projective plane $\mathcal{F}$. In particular, the elements of $\mathcal{B}$ are $\mathcal{A}$-regular words.

We will consider the elements of the set B as alphabetic symbols. Note that in this case the elements of B can have A-length exceeding one and can be subwords in other elements of B.

For an arbitrary nonassosiative word $w \in W(B)$, we define its *B-depth* $d_B(w)$:

(a) If $w \in B$, then $d_B(w) = 1$;

(b) If $w \notin B$ and $w = w_1w_2$, then $d_B(w) = \max\{d_B(w_1), d_B(w_2)\} + 1$;

Note that $W(B) \subseteq W(A)$ and $d_B(w) \leqslant d_A(w)$ for every $w \in W(B)$. (The A-depth of nonassociative words over A is defined in a similar way.) Since $W(B) \subseteq W(A)$, on the elements $W(B)$, the order $\prec$, introduced in the construction A, is defined.

We define the set of *$\mathcal{B}$-regular words* by induction on B-depth of the nonassociative word $w \in W(B)$ in the following way:

If $d_B(w) = 1$, then $w \in B$. Then we will call the word w *$\mathcal{B}$-regular of the first kind* (*the second kind*), if $w \in B^0$ ($w \in {}^0B$).

If $d_B(w) \geqslant 2$, then $w = w_1w_2$ for some $w_1, w_2 \in W(B)$. Then we will call the word w *$\mathcal{B}$-regular of the first kind* (*second kind*), if and only if six following conditions are fulfilled:

(B1) $w_1 \succ w_2$ and w_1, w_2 are $\mathcal{B}$-regular words of the second kind (first kind);

(B2) There does not exist any w_3 such that $\langle w_1, w_3\rangle \in I^B$ and $\langle w_2, w_3\rangle \in I^B$;

(B3) If $d_B(w_1) \geqslant 2$ and $w_1 = w_3w_4$, then $\langle w_3, w_2\rangle \notin I^B$ and $\langle w_4, w_2\rangle \notin I^B$;

(B4) If $d_B(w_2) \geqslant 2$ and $w_2 = w_3w_4$, then $\langle w_3, w_1\rangle \notin I^B$ and $\langle w_4, w_1\rangle \notin I^B$;

(B5) If $d_B(w_1) \geqslant 2$, $d_B(w_2) \geqslant 2$, $w_1 = w_3w_4$, and $w_2 = w_5w_6$, then $\{w_3, w_4\} \cap \{w_5, w_6\} = \varnothing$;

(B6) If $d_B(w_1) \geqslant 3$, $w_1 = (w_3w_4)w_5$ or $w_1 = w_5(w_3w_4)$, and $d_B(w_3w_4) \geqslant 2$, then $w_2 \notin \{w_3, w_4\}$.

We consider the set of all $\mathcal{B}$-regular words to be the result of the construction B.

Note that if $w \in A$ and w is an $\mathcal{B}$-regular word, then $w \in B$.

Example. Suppose that the projective plane $\mathcal{F}$ is freely generated by the construction $\mathcal{A} = \langle A, A^0, {}^0A, I^A\rangle$, where $A = \{a_0 \succ a_1 \succ \cdots \succ a_5\}$, $A^0 = \{a_1, \ldots, a_5\}$, ${}^0A = \{a_0\}$, $I^A = \{\langle a_0, a_i\rangle, \langle a_i, a_0\rangle \mid 3 \leqslant i \leqslant 5\}$.

If $\mathcal{B}$ is a subconfiguration in $\mathcal{F}$, presented on Figure 1, then the word $((a_1a_2)a_0)a_3$ is $\mathcal{B}$-regular, but is not $\mathcal{A}$-regular.

If $b = [((a_1a_4)(a_2a_5))((a_1a_5)(a_2a_4))]a_0$ and $\mathcal{B}$ is a subconfiguration in $\mathcal{F}$, presented on Figure 2, then every $\mathcal{B}$-regular word is $\mathcal{A}$-regular.

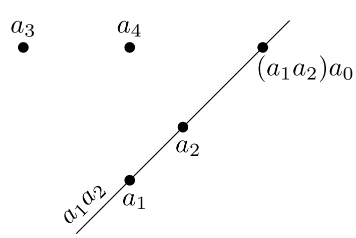

Fig. 1.

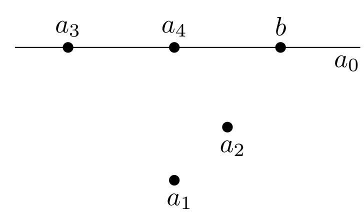

Fig. 2.

So, generally speaking, not every $\mathcal{B}$-regular word is $\mathcal{A}$-regular. We will define the configurations of a special form, with respect to which the regular words will at the same time be $\mathcal{A}$-regular.

Definition. We will call the nondegenerate nonclosed configuration $\mathcal{B} \subseteq \mathcal{F}$ a *reduced* one, if for every $w_1, w_2, w_3 \in F$, the following conditions are satisfied:

(R1) If $w_1, w_2 \in B$, $w_1 \neq w_2$, $\langle w_1, w_3\rangle \in I^A$, and $\langle w_2, w_3\rangle \in I^A$, then $w_3 \in B$.

(R2) If $\overline{w_1w_2} \in B$, $w_3 \in B$, and $\langle w_1, w_3\rangle \in I^A$, then $w_1 \in B$.

(R3) If $\overline{w_1w_2} \in B$, $\overline{w_1w_3} \in B$, and $\overline{w_1w_2} \neq \overline{w_1w_3}$, then $w_1 \in B$.

(R4) If $\overline{(\overline{w_1w_2})w_3} \in B$ and the word w_1 is $\mathcal{B}$-regular, then $\overline{w_1w_2} \in B$.

Proposition 1. *Let $\mathcal{B}$ be a reduced subconfiguration in $\mathcal{F}$. Then every $\mathcal{B}$-regular word is $\mathcal{A}$-regular.*

Proof. Note that from condition (R4) if follows that for every $w_1, w_2 \in F$, the following property is valid:

(R5) If $\overline{w_1 w_2} \in B$ and the word w_1 is $\mathcal{B}$-regular, then $w_1 \in B$.

Indeed, if $d_B(w_1) = 1$, then $w_1 \in B$. But if $d_B(w_1) \geqslant 2$, then $w_1 = w_3 w_4$ for some $\mathcal{B}$-regular words w_3 and w_4. Then $\overline{w_1 w_2} = \overline{(w_3 w_4) w_3}$, which allows us to conclude that due to condition (R4) $w_1 = w_3 w_4 \in B$.

Now let w be an arbitrary $\mathcal{B}$-regular word. We will prove by induction on B-depth of the word w, that w is $\mathcal{A}$-regular.

If $d_B(w) = 1$, then $w \in B$, and it is trivial that the word w is $\mathcal{A}$-regular.

But if $d_B(w) \geqslant 2$, then $w = w_1 w_2$ for some $\mathcal{B}$-regular words w_1 and w_2 with the condition $w_1 \succ w_2$. Due to the inductional assumption, w_1 and w_2 are $\mathcal{A}$-regular. Therefore, the word w satisfies condition (A1). We will prove that w satisfies conditions (A2)–(A5).

Assume that $w_1, w_2 \in A$, $\langle w_1, w_3 \rangle \in I^A$ and $\langle w_2, w_3 \rangle \in I^A$ for some $w_3 \in A$. Then $w_1, w_2 \in B$ and, therefore, using condition (R1), we obtain that $w_3 \in B$. Hence, $\langle w_1, w_3 \rangle \in I^B$ and $\langle w_2, w_3 \rangle \in I^B$, which contradicts the condition (B2) for the word w.

Assume that $w_1 = \overline{w_3 w_4}$ and $\langle w_3, w_2 \rangle \in I^A$. Since $w_2 \in A$, we conclude that $w_2 \in B$. If $d_B(w_1) \geqslant 2$, then the words w_3 and w_4 are $\mathcal{B}$-regular. Then $w_3 \in B$ and $\langle w_3, w_2 \rangle \in I^B$, which contradicts the condition (B3) for the word w. But if $d_B(w_1) = 1$, then $w_1 \in B$, and from condition (R2) it follows that $w_3 \in B$. Since $w_3 \sqsubset w_1$, we conclude that $\langle w_3, w_1 \rangle \in I^B$. Moreover, $\langle w_3, w_2 \rangle \in I^B$. The latter contradicts the condition (B2) for the word w.

Assume that $w_1 = \overline{w_3 w_4}$ and $w_2 = \overline{w_3 w_5}$. If $d_B(w_1) \geqslant 2$ and $d_B(w_2) \geqslant 2$, then we obtain a contradiction to the condition (B5) for the word w. If $d_B(w_1) \geqslant 2$ and $d_B(w_2) = 1$, then the word w_3 is $\mathcal{B}$-regular and $\overline{w_3 w_5} \in B$. Therefore, due to property (R5), we conclude that $w_3 \in B$. Then, since $w_3 \sqsubset w_2$, we obtain that $\langle w_3, w_2 \rangle \in I^B$, which contradicts the condition (B3). If $d_B(w_1) = 1$ and $d_B(w_2) \geqslant 2$, then similar reasonings lead to a contradiction to the condition (B4). If $d_B(w_1) = d_B(w_2) = 1$, then $\overline{w_3 w_4} \in B$ and $\overline{w_3 w_5} \in B$, and therefore, due to condition (R3), $w_3 \in B$ is valid. Then $\langle w_1, w_3 \rangle \in I^B$ and $\langle w_2, w_3 \rangle \in I^B$, which contradicts the condition (B2).

Assume that $w_1 = \overline{(\overline{w_2 w_3}) w_4}$. If $d_B(w_1) = 1$, then $\overline{(\overline{w_2 w_3}) w_4} \in B$ and the word w_2 is $\mathcal{B}$-regular. Then from condition (R4) it follows that $\overline{w_2 w_3} \in B$, and from property (R5) we obtain that $w_2 \in B$. Since $w_2 \sqsubset \overline{w_2 w_3} \sqsubset \overline{(\overline{w_2 w_3}) w_4}$, we obtain that $\langle w_1, \overline{w_2 w_3} \rangle \in I^B$ and $\langle w_2, \overline{w_2 w_3} \rangle \in I^B$, which contradicts the condition (B2). If $d_B(w_1) = 2$, then $\overline{w_2 w_3} \in B$. Therefore, using property (R5), we conclude that $w_2 \in B$. Then $\langle \overline{w_2 w_3}, w_2 \rangle \in I^B$, which contradicts the condition (B3). Thus, $d_B(w_1) \geqslant 3$. In this case, if $d_B(\overline{w_2 w_3}) = 1$ and $d_B(w_4) \geqslant 2$, then using the similar reasoning as for the previous case, we obtain a contradiction to (B3). Hence, it follows that $d_B(\overline{w_2 w_3}) \geqslant 2$ and $d_B(w_4) \geqslant 1$, which contradicts the condition (B6). □

Proposition 2. *Suppose that $\mathcal{B}$ and $\mathcal{C}$ are subconfigurations in $\mathcal{F}$, such that $\mathcal{B}$ is reduced, and $\mathcal{C}$ is a single-step extension of $\mathcal{B}$. Then the following statements hold:*

(a) The extension $\mathcal{B} \subseteq \mathcal{C}$ is free;

(b) For every $w \in F$, if w is a $\mathcal{C}$-regular word, then w is $\mathcal{B}$-regular;

(c) The configuration $\mathcal{C}$ is reduced.

Proof. We will prove that the extension $\mathcal{B} \subseteq \mathcal{C}$ is free. Suppose that $w \in C \setminus B$. Therefore, there exist distinct $w_1, w_2 \in B$ such that $\langle w_1, w\rangle \in I^F$ and $\langle w_2, w\rangle \in I^F$. We can assume that $w_1 \succ w_2$. According to the definition of the incidence relation I^F, the following cases are possible.

If $\langle w_1, w\rangle \in I^A$ and $\langle w_2, w\rangle \in I^A$, then due to condition (R1), we conclude that $w \in B$, which is impossible. If $w_1 = \overline{w_3 w}$ and $\langle w_2, w\rangle \in I^A$, then due to condition (R2), we conclude that $w \in B$, which is impossible. If $w_1 = \overline{w_3 w}$ and $w_2 = \overline{w_4 w}$, then from condition (R3) it follows again that $w \in B$. If $w_1 = \overline{(\overline{w_2 w_3}) w_4}$ and $w = \overline{w_2 w_3}$, then from condition (R4) it follows that $w \in B$.

Thus, only one case where $w = w_1 w_2$ and the word $w_1 w_2$ is $\mathcal{A}$-regular is possible. Assume that there exists $w_3 \in B$ such that $w_3 \neq w_1$, $w_3 \neq w_2$ and $\langle w_3, w\rangle \in I^F$. Since w is not a letter from A and the relations $w_1 \sqsubset w$ and $w_2 \sqsubset w$ hold, we conclude that $w_3 \notin A$ and $w_3 \not\sqsubset w$. Therefore, for some $w_4 \in F$, we have that $w_3 = \overline{w w_4} = \overline{(w_1 w_2) w_4}$. Hence, due to condition (R4), we conclude that $w \in B$. It means that there is no such w_3. Therefore, there exist exactly two elements $w_1, w_2 \in B$ with the condition $w_1 \cdot w_2 = w$, that is, the extension $\mathcal{B} \subseteq \mathcal{C}$ is free.

We will prove that if $w \in C \setminus B$, then the word w is $\mathcal{B}$-regular. Indeed, as it has been established above, there exist $w_1, w_2 \in B$ such that $w_1 \succ w_2$ and $w = w_1 w_2$. It is clear that in this case $d_B(w) = 2$. If there exists $w_3 \in B$ such that $\langle w_1, w_3\rangle \in I^B$ and $\langle w_2, w_3\rangle \in I^B$, then in the plane $\mathcal{F}$, the equality $w_1 \cdot w_2 = w_3$ will hold. On the other hand, the word $w_1 w_2$ is $\mathcal{A}$-regular, which means that in $\mathcal{F}$ we have that $w_1 \cdot w_2 = w_1 w_2$. Therefore, $w = w_3$, which contradicts the condition $w \notin B$. Thus, we have established that w satisfies the conditions (B1) and (B2). The fact that the conditions (B3)–(B6) hold for the word w is trivial.

Suppose that $w \in F$ and w is $\mathcal{C}$-regular. We will prove by induction on C-depth $d_C(w)$ of the word w that w is $\mathcal{B}$-regular. If $d_C(w) = 1$, then due what was proved above, w is $\mathcal{B}$-regular. If $d_C(w) \geqslant 2$, then $w = w_1 w_2$ for some $\mathcal{C}$-regular words w_1 and w_2. Due to the inductional assumption, w_1 and w_2 are $\mathcal{B}$-regular. We will show that w satisfies the conditions (B1)–(B6) from the definition of a $\mathcal{B}$-regular word.

Since $w \in F$, we conclude that $w_1 \succ w_2$, and therefore, condition (B1) is satisfied. From the fact that $\mathcal{B} \subseteq \mathcal{C}$ and w is $\mathcal{C}$-regular, it follows that the condition (B2) holds. Assume that $d_B(w_1) \geqslant 2$, $w_1 = \overline{w_3 w_4}$ and $\langle w_3, w_2\rangle \in I^B$. If $\langle w_3, w_2\rangle \in I^A$, then $w_3, w_2 \in A$, which contradicts the $\mathcal{A}$-regularity of the word w. If $w_3 \sqsubset w_2$, then $w_2 = \overline{w_3 w_5}$ for some $w_5 \in F$, and therefore, the word $w = (\overline{w_3 w_4})(\overline{w_3 w_5})$ is not $\mathcal{A}$-regular. But if $w_2 \sqsubset w_3$, then $w_3 = \overline{w_2 w_5}$ for some $w_5 \in F$, and therefore $w = ((\overline{w_2 w_5}) w_4) w_2$ is not $\mathcal{A}$-regular. Thus, the condition (B3) holds. In a similar way it can be proved that the condition (B4) is satisfied. (B5) and (B6) hold due to the fact that w is $\mathcal{A}$-regular.

Now we will prove that the configuration $\mathcal{C}$ is reduced by checking whether the conditions (R1)–(R4) are met.

Let $w_1, w_2 \in C$, $w_1 \neq w_2$, $\langle w_1, w_3\rangle \in I^A$ and $\langle w_2, w_3\rangle \in I^A$ for some $w_3 \in F$. Since the words w_1 and w_2 are the letters from A, each of them cannot be represented in the form of an $\mathcal{A}$-regular word $w' w''$, where $w', w'' \in B$. Then it follows that $w_1 \notin C \setminus B$ and $w_2 \notin C \setminus B$. Then $w_1, w_2 \in B$, and therefore, due to the fact that the configuration $\mathcal{B}$ is reduced, we conclude that $w_3 \in B$. Hence, $w_3 \in C$.

Let $\overline{w_1w_2} \in C$, $w_3 \in C$ and $\langle w_1, w_3\rangle \in I^A$. Similarly to what we have seen above, from the condition $w_3 \in A$ it follows that $w_3 \notin C \setminus B$. Therefore, $w_3 \in B$. If $\overline{w_1w_2} \in B$, then from the fact that the configuration $\mathcal{B}$ is reduced it follows that $w_1 \in B$. But if $\overline{w_1w_2} \in C \setminus B$, then as it has been noted earlier, $w_1, w_2 \in B$ holds. In any case, we obtain that $w_1 \in C$.

Let $\overline{w_1w_2}, \overline{w_1w_3} \in C$ and $\overline{w_1w_2} \neq \overline{w_1w_3}$. If $\overline{w_1w_2} \in B$ and $\overline{w_1w_3} \in B$, then due to the fact that $\mathcal{B}$ is reduced, we conclude that $w_1 \in B$, which means that $w_1 \in C$. If $\overline{w_1w_2} \in C \setminus B$ or $\overline{w_1w_3} \in C \setminus B$, then in any case we obtain that $w_1 \in B$, and therefore, again we have that $w_1 \in C$.

Suppose that $\overline{(\overline{w_1w_2})w_3} \in C$ and the word w_1 is $\mathcal{C}$-regular. Since $w_1 \in F$ and w_1 is $\mathcal{C}$-regular, then w_1 is $\mathcal{B}$-regular. Then if $\overline{(\overline{w_1w_2})w_3} \in B$, then due to the fact that $\mathcal{B}$ is reduced, it follows that $\overline{w_1w_2} \in B$, and therefore, $\overline{w_1w_2} \in C$. But if $\overline{(\overline{w_1w_2})w_3} \in C \setminus B$, then as it has been noted earlier, $\overline{w_1w_2} \in B$ and $w_3 \in B$ hold, from which we obtain again that $\overline{w_1w_2} \in C$. □

We will show that when closing the reduced configuration $\mathcal{B}$ in the plane $\mathcal{F}$, the elements $\mathcal{B}$ behave as free generators.

Proposition 3. *Let $\mathcal{B}$ be a reduced subconfiguration in $\mathcal{F}$. Then the closure $\langle\mathcal{B}\rangle_{\mathcal{F}}$ is free and consists exactly of all $\mathcal{B}$-regular words.*

Proof. Consider in the projective plane $\mathcal{F}$ a countable sequence of its subconfigurations

$$\mathcal{B} = \mathcal{B}_0 \subseteq \mathcal{B}_1 \subseteq \ldots \subseteq \mathcal{B}_i \subseteq \ldots$$

such that for every $i \in \omega$, the extension $\mathcal{B}_i \subseteq \mathcal{B}_{i+1}$ is a full single-step one. Then $\bigcup_{i\in\omega} \mathcal{B}_i = \langle\mathcal{B}\rangle_{\mathcal{F}}$.

Using items (a) and (c) from Proposition 2, by induction we conclude that every extension $\mathcal{B}_i \subseteq \mathcal{B}_{i+1}$ is free. Therefore, the closure $\langle\mathcal{B}\rangle_{\mathcal{F}}$ is free, and $\mathcal{B}$ freely generates the plane $\langle\mathcal{B}\rangle_{\mathcal{F}}$.

Using item (b) from Proposition 2 and induction, we conclude that for every $w \in F$ and $i \in \omega$, if the word w is $\mathcal{B}_i$-regular, then w is $\mathcal{B}$-regular. In particular, if $w \in \mathcal{B}_i$, then w is $\mathcal{B}$-regular. Then it follows that every element of the closure $\langle\mathcal{B}\rangle_{\mathcal{F}}$ is a $\mathcal{B}$-regular word.

On the other hand, if the word w is $\mathcal{B}$-regular, then due to Proposition 1, it is $\mathcal{A}$-regular. Therefore, $w \in F$, and hence, $w \in \langle\mathcal{B}\rangle_{\mathcal{F}}$. □

Proposition 4. *Let $\mathcal{B}$ be a nondegenerate nonclosed subconfiguration in $\mathcal{F}$. Then there exists a reduced configuration $\mathcal{B}' \subseteq \mathcal{F}$ such that $\mathcal{B} \subseteq \mathcal{B}'$ and $\langle\mathcal{B}\rangle_{\mathcal{F}} = \langle\mathcal{B}'\rangle_{\mathcal{F}}$.*

Proof. Consider the following subset in the plane $\mathcal{F}$

$$X = A \cup \{w \in F \mid w \sqsubseteq^* w' \text{ for some } w' \in B\}.$$

We designate by $\varkappa_0$ the cardinality of the set X. We will choose the smallest infinite cardinal number $\varkappa$ such that $\varkappa_0 < \varkappa$.

We define the *terms over the elements of* B by induction. Every element $b \in B$ will be called a term over b. If $t_1 = t_1(b_1, \ldots, b_s)$ is a term over $b_1, \ldots, b_s \in B$ and $t_2 = t_2(b_{s+1}, \ldots, b_t)$ is a term over $b_{s+1}, \ldots, b_t \in B$, $t_1 \neq t_2$ and t_1, t_2 are of the same kind, then their product $t_1 \cdot t_2$ in the plane $\mathcal{F}$ will be called a term over $b_1, \ldots, b_t$.

We will define by a transfinite induction on $\alpha \in \varkappa$ the sequence $\{B_\alpha\}_{\alpha\in\varkappa}$ of subsets of F such that for every $\alpha \in \varkappa$, the following properties are valid:

(a) $B_\beta \subseteq B_\alpha$ for every $\beta \in \alpha$;

(b) $B_\alpha \subseteq X$;

(c) for every $w \in B_\alpha$, there exists a term $t(b_1, \ldots, b_s)$ over some elements $b_1, \ldots, b_s \in B$ such that $w = t(b_1, \ldots, b_s)$.

We put $B_0 = B$. It is clear that B_0 satisfies conditions (a)–(c).

Suppose that $\alpha \in \varkappa$ and B_β is already defined and satisfies the conditions (a)–(c) for every $\beta \in \alpha$.

If α is a limit ordinal, we put that $B_\alpha = \bigcup_{\beta\in\alpha} B_\beta$. It is clear that then B_α satisfies the conditions (a)–(c).

If α is a nonlimit ordinal, then $\alpha = \beta + 1$. The set $B_\beta \subseteq F$ uniquely defines the subconfiguration $\mathcal{B}_\beta$ in the plane $\mathcal{F}$, such that the domain of $\mathcal{B}_\beta$ is B_β. Since $\mathcal{B} \subseteq \mathcal{B}_\beta$, the configuration $\mathcal{B}_\beta$ is nondegenerate. If the configuration $\mathcal{B}_\beta$ satisfies the conditions (R1)–(R4), then $\mathcal{B}_\beta$ is reduced, and then we put that $B_\alpha = B_\beta$. But if $\mathcal{B}_\beta$ does not satisfy the conditions (R1)–(R4), then at least one of the following sets Y_1, Y_2, Y_3, Y_4 is not empty:

$$Y_1 = \{w_3 \in F \mid \text{exist } w_1, w_2 \in B_\beta \text{ such, that } w_1 \neq w_2, \langle w_1, w_3\rangle \in I^A, \langle w_2, w_3\rangle \in I^A, \text{ and } w_3 \notin B_\beta\},$$

$$Y_2 = \{w_1 \in F \mid \text{exist } \overline{w_1w_2} \in B_\beta \text{ and } w_3 \in B_\beta \text{ such that } \langle w_1, w_3\rangle \in I^A \text{ and } w_1 \notin B_\beta\},$$

$$Y_3 = \{w_1 \in F \mid \text{exist } \overline{w_1w_2} \in B_\beta \text{ and } \overline{w_1w_3} \in B_\beta \text{ such that } \overline{w_1w_2} \neq \overline{w_1w_3} \text{ and } w_1 \notin B_\beta\},$$

$$Y_4 = \{\overline{w_1w_2} \in F \mid \text{exist } \overline{(\overline{w_1w_2})w_3} \in B_\beta \text{ such that } w_1 \text{ is } \mathcal{B}_\beta\text{-regular and } \overline{w_1w_2} \notin B_\beta\}.$$

If $w_3 \in Y_1$, then $w_3 \in A \subseteq X$ and $w_3 = w_1 \cdot w_2$ for some $w_1, w_2 \in B_\beta$. Since B_β satisfies the condition (c), w_1 and w_2 can be represented in the form of terms over the elements of B. Therefore, w_3 can also be represented in the form of a term over elements of B. Using similar reasoning, it can be proved that $Y_2 \subseteq A$ and every element of $w_1 \in Y_2$ can be represented in the form of a term over the elements of B.

If $w_1 \in Y_3$, then $w_1 = \overline{w_1w_2} \cdot \overline{w_1w_3}$, where $\overline{w_1w_2} \in B_\beta$ and $\overline{w_1w_3} \in B_\beta$. Since B_β satisfies condition (b), we conclude that $\overline{w_1w_2} \in X \setminus A$, which means that from the property $w_1 \sqsubset \overline{w_1w_2}$ it follows that $w_1 \in X$. Moreover, $\overline{w_1w_2}$ and $\overline{w_1w_3}$ can be represented in the form of terms over the elements of B. Therefore, w_1 can also be represented in the form of a term over the elements of B.

If the set Y_4 is not empty, then the following set Y_4' is also not empty.

$$Y_4' = \{w \in F \mid w \text{ is } \mathcal{B}_\beta\text{-regular and exists } \overline{(\overline{w_1w_2})w_3} \in B_\beta \text{ such that } w_1 \text{ is } \mathcal{B}_\beta\text{-regular}, \overline{w_1w_2} \notin B_\beta, w \sqsubseteq^* w_1 \text{ and } d_{B_\beta}(w) \leqslant d_{B_\beta}(w_1)\}.$$

Suppose that $w \in Y_4'$. We will prove by induction on B_β-depth $d_{B_\beta}(w)$ that $w \in X$ and w can be represented in the form of a term over the elements of B. If $d_{B_\beta}(w) = 1$, then $w \in B_\beta$, and the statement follows from the fact that B_β

satisfies the conditions (b) Pë (c). If $d_{B_\beta}(w) \geqslant 2$, then $w = uv$ for some $\mathcal{B}_\beta$-regular words u and v. Then $u, v \in Y_4'$ and due to the inductional assumption, u and v can be represented in the form of terms over the elements of B. Therefore, w can also be represented in the form of a term over the elements of B. Moreover, for some $\overline{(\overline{w_1w_2})w_3} \in B_\beta$ we have that $w \sqsubseteq^* w_1 \sqsubset^* \overline{(\overline{w_1w_2})w_3} \in X \setminus A$, hence, we conclude that $w \in X$.

Now consider an arbitrary element $\overline{w_1w_2} \in Y_4$. Then there exists $\overline{(\overline{w_1w_2})w_3} \in B_\beta$ such that w_1 is $\mathcal{B}_\beta$-regular. Since $\overline{(\overline{w_1w_2})w_3} \in X \setminus A$ and $\overline{w_1w_2} \sqsubset \overline{(\overline{w_1w_2})w_3}$, we conclude that $\overline{w_1w_2} \in X$. Note that $w_1 \in Y_4'$. Therefore, due to what was proved above, w_1 can be represented in the form of a term over the elements of B. Moreover, the element $\overline{(\overline{w_1w_2})w_3}$ can also be represented in the form of a term over the elements of B. Then from the identity $\overline{w_1w_2} = \overline{(\overline{w_1w_2})w_3} \cdot w_4$ we obtain that $\overline{w_1w_2}$ can be represented in the form of a term over the elements of B.

We put $B_\alpha = B_\beta \cup Y_1 \cup Y_2 \cup Y_3 \cup Y_4 \cup Y_4'$. Then $B_\beta \subseteq B_\alpha$ and $B_\beta \neq B_\alpha$. Due to what was proved above, B_α satisfies the conditions (b) and (c).

So, the required sequence $\{B_\alpha\}_{\alpha\in\varkappa}$ is constructed. Due to the choice of the cardinal number $\varkappa$ and the fact that the property (b) holds, there definitely exists $\alpha \in \varkappa$ such that $B_\alpha = B_{\alpha+1}$. Therefore, by construction, the configuration $\mathcal{B}' = \mathcal{B}_\alpha$ is reduced. From the inclusion $\mathcal{B} \subseteq \mathcal{B}'$ it follows that $\langle\mathcal{B}\rangle_\mathcal{F} \subseteq \langle\mathcal{B}'\rangle_\mathcal{F}$. On the other hand, every element of $\mathcal{B}'$ can be represented in the form of a term over the elements of the configuration $\mathcal{B}$. Therefore, $\langle\mathcal{B}'\rangle_\mathcal{F} \subseteq \langle\mathcal{B}\rangle_\mathcal{F}$. Thus, $\langle\mathcal{B}\rangle_\mathcal{F} = \langle\mathcal{B}'\rangle_\mathcal{F}$. □

Theorem 5. *Let $\mathcal{B}$ be a nondegenerate nonclosed subconfiguration in $\mathcal{F}$. Then there exists a configuration $\mathcal{B}' \subseteq \mathcal{F}$ such that $\mathcal{B}'$ freely generates the projective plane $\langle\mathcal{B}\rangle_\mathcal{F}$.*

Proof. Due to Proposition 4, there exists a reduced configuration $\mathcal{B}' \subseteq \mathcal{F}$ such that $\langle\mathcal{B}\rangle_\mathcal{F} = \langle\mathcal{B}'\rangle_\mathcal{F}$. By Proposition 3, the closure $\langle\mathcal{B}'\rangle_\mathcal{F}$ is free. Hence, $\mathcal{B}'$ freely generates the projective plane $\langle\mathcal{B}\rangle_\mathcal{F}$. □

Corollary 6. *Let $\mathcal{B}$ be a reduced subconfiguration in $\mathcal{F}$. Then $\mathcal{A} \cap \langle\mathcal{B}\rangle_\mathcal{F} = \mathcal{A} \cap \mathcal{B}$.*

Proof. The inclusion $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A} \cap \langle\mathcal{B}\rangle_\mathcal{F}$ is trivial. Suppose that $w \in \mathcal{A} \cap \langle\mathcal{B}\rangle_\mathcal{F}$. By Proposition 3, the word w is $\mathcal{B}$-regular, and moreover, it coincides with a letter from A. Since $d_B(w) \leqslant d_A(w) = 1$, we conclude that $d_B(w) = 1$, that is, $w \in \mathcal{A} \cap \mathcal{B}$. □

In [4], it was established that in finitely generated freely generated projective planes, the problem of inclusion of a word into a finitely generated subplane is algorithmically solvable. The method for transforming a configuration $\mathcal{B}$ into a reduced configuration $\mathcal{B}'$, which has been proposed above, provides a possibility to obtain another proof for this fact.

Corollary 7 (A.A. Nikitin [4]). *Suppose that $\mathcal{F} = \mathcal{F}(\mathcal{A})$ is a projective plane, freely generated by a finite nondegenerate nonclosed configuration $\mathcal{A}$, and $\mathcal{B}$ is a finite nondegenerate nonclosed subconfiguration in $\mathcal{F}$. Then there exists an effective algorithm, which, given any word $w \in \mathcal{F}$, defines whether it is true that $w \in \langle\mathcal{B}\rangle_\mathcal{F}$.*

Proof. Note that if the configurations $\mathcal{A}$ and $\mathcal{B}$ are finite, then the set X in the proof of Proposition 4 is also finite, and therefore, the cardinal number $\varkappa$ coincides with ω. Then it follows that the sequence $\{B_\alpha\}_{\alpha\in\varkappa}$ from the proof of Proposition 4 consists of finite sets and gets stabilized after a finite number of steps, that is, there exists $\alpha \in \omega$ such that the configuration $\mathcal{B}_\alpha$ is reduced. From the effectiveness of the construction A it follows that the checking of the conditions (R1)–(R4) for the finite configuration $\mathcal{B}_i \subseteq \mathcal{F}$, $i \in \omega$, is conducted effectively. Therefore, in the case of finite $\mathcal{A}$ and $\mathcal{B}$, all constructions in the proof of Proposition 4 are effective. Thus, the reduced configuration $\mathcal{B}' \subseteq \mathcal{F}$ with the condition $\langle \mathcal{B} \rangle_{\mathcal{F}} = \langle \mathcal{B}' \rangle_{\mathcal{F}}$ can be found effectively.

We will describe an algorithm that defines for any $w \in \mathcal{F}$ whether the statement $w \in \langle \mathcal{B} \rangle_{\mathcal{F}}$ holds. Let w be an arbitrary $\mathcal{A}$-regular word. We will calculate its A-depth $d_A(w) = n$ and define a finite set of words

$$U_n = \{u \in F \mid \text{word } u \text{ is } \mathcal{B}'\text{-regular and } d_{B'}(u) \leqslant n\}.$$

We will prove that $w \in \langle \mathcal{B} \rangle_{\mathcal{F}}$ if and only if $w \in U_n$. If $w \in U_n$, then the word w is $\mathcal{B}'$-regular, and therefore, due to Proposition 3, we conclude that $w \in \langle \mathcal{B} \rangle_{\mathcal{F}}$. But if $w \in \langle \mathcal{B} \rangle_{\mathcal{F}}$, then $w \in \langle \mathcal{B}' \rangle_{\mathcal{F}}$. By Proposition 3, the word w is $\mathcal{B}'$-regular. Since $d_{B'}(w) \leqslant d_A(w) = n$, we obtain that $w \in U_n$.

Thus, due to the finiteness of the set U_n, the checking of the condition $w \in \langle \mathcal{B} \rangle_{\mathcal{F}}$ is conducted effectively. □

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