ON CLOSURE OF CONFIGURATIONS IN FREELY GENERATED PROJECTIVE PLANES

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ABSTRACT. Let $F$ be an arbitrary freely generated projective plane. Based on Shirshov's combinatorial method, we introduce the notion of a reduced configuration in $F$. We prove that for every subplane $P$ generated in $F$ by some configuration $B$, there is a reduced configuration $B'$ such that $P$ is freely generated by $B'$.

Keywords: projective plane, configuration, incidence, freely generated projective plane, nonassociative word, regular word.

1. INTRODUCTION

In [1], it was proved that if a subplane $P$ of a free projective plane is generated by a finite configuration $B$, then $P$ is also free. The proof of this statement is, in particular, based on the following reasoning. If we represent the process of generating of a projective plane $P$ in the form of a sequence of configurations

$$B = B_0 \subseteq B_1 \subseteq \ldots \subseteq B_n \subseteq \ldots,$$

where the extension $B_n \subseteq B_{n+1}$ is a full and single-step one for every $n \in \omega$, then for every $n$ the rank of the configuration $B_{n+1}$ cannot exceed the rank of the configuration $B_n$, which means that as $n$ increases, the rank of $B_n$ gets stabilized, therefore, there exists $m$ such that $B_m$ freely generates $P$. If we refuse the condition of finiteness of the configuration $B$, then the mentioned reasoning on ranks becomes
incorrect, and there arises a necessity to find other ways to find and describe the configuration $B'$, freely generating the plane $\mathcal{P}$.

In [2, 3, 4, 5, 6, 7], based on the approach by A. I. Shirshov, an algebraic theory of free and freely generated projective planes was developed, which provided a possibility to solve a number of traditional problems in algebra in the mentioned classes of projective planes.

In this article, using the constructions and combinatorial methods of A. I. Shirshov, we introduce the notion of a reduced configuration in an arbitrary freely generated projective plane $\mathcal{F}$ and propose a method for transforming an arbitrary configuration $B$, generating a subplane $\mathcal{P}$ in the projective plane $\mathcal{F}$, into a reduced configuration $B'$, which freely generates $\mathcal{P}$. In the case of finitely generated planes, the proposed method for transforming the configuration $B$ into the reduced configuration $B'$ turns out to be effective, which allows us to state the solvability of the problem of inclusion into the subplane $\mathcal{P}$.

In Section 2 of this article, the necessary definitions related to the theory of projective planes are provided. In Section 3, we introduce the definition of a reduced configuration and prove the main results of this work.

2. Preliminary information

We will now provide the necessary information from the theory of projective planes [2, 8].

We will consider projective planes as algebraic systems of a signature $\sigma = \langle A^0, 0A, I \rangle$, where $A^0$ and $0A$ are unary predicate symbols, and $I$ is a binary predicate symbol. The domain $A$ of each of such systems will be broken into two subsets $A^0 \cup 0A = A$, $A^0 \cap 0A = \emptyset$. We refer to the elements $a, b \in A$ as the ones of the same kind with respect to the given partition, if $a, b \in A^0$ or $a, b \in 0A$. The elements $A^0$ are called points, and the elements $0A$ are called lines.

A configuration is an algebraic system $\mathcal{A} = \langle A, A^0, 0A, I \rangle$ with a partition of the domain $A$ into two subsets $A^0 \cup 0A = A$, $A^0 \cap 0A = \emptyset$, and a symmetric binary relation $I \subseteq A^2$, which is called a incidence relation and satisfies the following conditions:

(P1) If $\langle a, b \rangle \in I$, then the elements $a$ and $b$ are of distinct kinds;

(P2) If $\langle a, c \rangle, \langle b, c \rangle \in I$, $\langle a, d \rangle, \langle b, d \rangle \in I$, then $a = b$ or $c = d$.

If the configuration $\mathcal{A}$ is finite, then we will call the number $2\cdot|A| - |I|/2$ a rank of $\mathcal{A}$. If the configuration $\mathcal{A}$ is infinite, then its rank is the cardinality of the set $A$.

On every configuration $\mathcal{A}$, we additionally define a partial binary commutative operation $\cdot$ (product) in the following way:

(P3) The product $a \cdot b$ is defined and $a \cdot b = c$ if and only if $a, b$ are distinct elements of $A$ of the same kind, such that $\langle a, c \rangle \in I$ and $\langle b, c \rangle \in I$.

A configuration $\mathcal{A}$ is called closed, if for every pair of distinct elements $a, b \in A$ of the same kind in $\mathcal{A}$ the product $a \cdot b$ is defined.

A closed configuration $\mathcal{A}$ is called a projective plane, if it satisfies the following condition of nondegeneracy:

(P4) There exist pairwise distinct $a, b, c, d \in A$ such that in $\mathcal{A}$ there are products $a \cdot b, b \cdot c, c \cdot d, d \cdot a$ that are defined and pairwise distinct.
A configuration \( \mathcal{A} = \langle A, A^0, 0A, I^A \rangle \) is a subconfiguration of the configuration \( \mathcal{B} = \langle B, B^0, 0B, I^B \rangle \), if \( A^0 \subseteq B^0 \), \( 0A \subseteq 0B \) and \( I^A = I^B \cap A^2 \). If \( \mathcal{A} \) is a subconfiguration of \( \mathcal{B} \), then we will also say that \( \mathcal{B} \) is an extension of \( \mathcal{A} \) and write \( \mathcal{A} \subseteq \mathcal{B} \).

A configuration \( \mathcal{A} \) is called degenerate, if it can be embedded into a closed configuration which does not satisfy condition (P4). Otherwise, \( \mathcal{A} \) is called nondegenerate.

An extension \( \mathcal{A} \subseteq \mathcal{B} \) is called a single-step one, if for every \( c \in B \setminus A \) there exist \( a, b \in A \) such that \( a \cdot b = c \). The single-step extension \( \mathcal{A} \subseteq \mathcal{B} \) is called full, if for every distinct \( a, b \in A \) of the same kind there exists \( c \in B \) such that \( a \cdot b = c \). The single-step extension \( \mathcal{A} \subseteq \mathcal{B} \) is called free, if for every \( c \in B \setminus A \) there exist exactly two elements \( a, b \in A \) such that \( a \cdot b = c \).

If \( \mathcal{A} \) is a subconfiguration of a closed configuration \( \mathcal{P} \), then by \( \langle \mathcal{A} \rangle_\mathcal{P} \), we will designate the intersection of all closed configurations of \( \mathcal{P} \), containing \( \mathcal{A} \). Moreover, \( \langle \mathcal{A} \rangle_\mathcal{P} \) is called a closure of the configuration \( \mathcal{A} \) in \( \mathcal{P} \), and it is said that \( \mathcal{A} \) generates the configuration \( \langle \mathcal{A} \rangle_\mathcal{P} \) in \( \mathcal{P} \).

It is well known [8, ch. XI] that if \( \mathcal{A} = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_i \subseteq \cdots \) is a sequence of subconfigurations of a closed configuration \( \mathcal{P} \), such that the extension \( \mathcal{A}_i \subseteq \mathcal{A}_{i+1} \) is a full single-step one for every \( i \in \omega \), then \( \bigcup_{i \in \omega} \mathcal{A}_i = \langle \mathcal{A} \rangle_\mathcal{P} \). In this case, if the extension \( \mathcal{A}_i \subseteq \mathcal{A}_{i+1} \) is free for every \( i \in \omega \), then the closure \( \langle \mathcal{A} \rangle_\mathcal{P} \) is called free, and it is written that \( \mathcal{F}(\mathcal{A}) = \langle \mathcal{A} \rangle_\mathcal{P} \) and said that \( \mathcal{A} \) freely generates the configuration \( \mathcal{F}(\mathcal{A}) \).

Note that every configuration \( \mathcal{A} \) can be embedded in such closed configuration \( \mathcal{P} \), where the closure \( \langle \mathcal{A} \rangle_\mathcal{P} \) is free and defined with respect to \( \mathcal{A} \) in a unique way up to an isomorphism, and does not depend on the choice of \( \mathcal{P} \).

In [2, 4], a construction of a free closure of a configuration, which is usually referred to as Shirshov’s construction, is proposed. In this construction, every element of the free closure has its unambiguous notation in the form of an irreducible nonassociative word over the initial configuration. We will remind the main definitions of the mentioned construction.

A set \( W(A) \) of nonassociative words over the alphabet \( A \) is defined by induction:

(a) if \( a \in A \), then \( a \in W(A) \);
(b) if \( u, v \in W(A) \), then \( uv \in W(A) \).

We will omit the outer brackets in nonassociative words.

For arbitrary nonassociative words \( u, v \in W(A) \), we will write \( u \sqsubseteq v \), if \( v = uv \) or \( v = vu \) for some word \( w \), that is, \( u \) is a maximal proper subword of \( v \). We will write \( u \sqsubseteq^* v \), if there exist words \( u_0, \ldots, u_k \) such that \( u = u_0 \sqsubseteq \cdots \sqsubseteq u_k \sqsubseteq v \). The notation \( u \sqsubseteq^* v \) means that \( u = v \) or \( u \sqsubseteq^* v \).

**Construction A.** Let \( \mathcal{A} = \langle A, A^0, 0A, I^A \rangle \) be a nondegenerate nonclosed configuration, on whose elements a strict complete order \( \prec \) such that \( a \prec b \) for every \( a \in A^0 \) and \( b \in 0A \) is defined.

We will consider the elements of the set \( A \) as alphabetic symbols. We refer as a \( \mathcal{A} \)-length of a nonassociative word \( w \in W(A) \) to a number \( |w| \) of occurrences of elements of \( A \) in the word \( w \). We will call an \( \mathcal{A} \)-weight of a nonassociative word \( w \in W(A) \) a number \( \|w\| = n_1 + 2n_2 \), where \( n_1 \) and \( n_2 \) are the numbers of occurrences in the notation of the word \( w \) of the symbols from \( A^0 \) and \( 0A \), respectively.
We continue the order $\prec$ to the set $W(A)$. For every $w_1 \neq w_2$ from $W(A)$, we put $w_1 \prec w_2$ if and only if either $||w_1|| < ||w_2||$, or $||w_1|| = ||w_2||$, and one of the following conditions is satisfied:

(a) $|w_1| < |w_2|$, or
(b) $|w_1| = |w_2| = 1$ and $w_1 \prec w_2$, or
(c) $|w_1| = |w_2| > 1$, $w_1 = u_1w_2$, $w_2 = u_3u_4$, and $u_1 \prec u_3$, or
(d) $|w_1| = |w_2| > 1$, $w_1 = u_1w_2$, $w_2 = u_3u_4$, $u_1 = u_3$, and $w_2 \prec u_4$.

By induction on $A$-length of the nonassociative word $w \in W(A)$, we define the set of $A$-regular words in the following way:

If $w \in A^0$ ($w \in A^1$), then $w$ is called an $A$-regular word of the first kind (the second kind).

If $w = w_1w_2$, then $w$ is called an $A$-regular word of the first kind (the second kind) if and only if five following conditions are fulfilled:

(A1) $w_1 \supset w_2$ and $w_1, w_2$ are $A$-regular words of the second kind (first kind);

(A2) There does not exist any $w_3$ such that $\langle w_1, w_3 \rangle \in I^A$ and $\langle w_2, w_3 \rangle \in I^A$;

(A3) If $w_1 = w_3w_4$, then $\langle w_3, w_2 \rangle \notin I^A$ and $\langle w_4, w_2 \rangle \notin I^A$;

(A4) If $w_1 = w_3w_4$ and $w_2 = w_5w_6$, then $\{w_3, w_4\} \cap \{w_5, w_6\} = \emptyset$;

(A5) If $w_1 = (w_3w_4)w_5$ or $w_1 = w_5(w_3w_4)$, then $w_2 \notin \{w_3, w_4\}$.

We designate by $F^0$ and $0F$ the sets of all $A$-regular words of the first and the second kinds respectively.

Note that for $w_1, w_2 \in W(A)$, the words $w_1w_2$ and $w_2w_1$ cannot be $A$-regular simultaneously. If one of the words $w_1w_2$ or $w_2w_1$ is $A$-regular, then we will designate that $A$-regular word by $\overline{w_1w_2}$. Use of the notation $\overline{w_1w_2}$ will allow us to simplify the presentation of the proofs of the article’s statements in the future.

The incidence relation $I^F$ on the set $F = F^0 \cup 0F$ of all $A$-regular words is defined with the help of the following equivalence:

$\langle w_1, w_2 \rangle \in I^F \iff \langle w_1, w_2 \rangle \in I^A \lor w_1 \sqsupset w_2 \lor w_2 \sqsupset w_1$.

We consider the configuration $F = \langle F, F^0, 0F, I^F \rangle$ to be the result of the construction $A$. In [2, 4], it was proved that the result of the construction $A$ is up to isomorphism a projective plane $\mathcal{F}(A)$, freely generated by $A$.

3. Reduced configurations and their properties

Let $A = \langle A, A^0, A^1, I^A \rangle$ be a nondegenerate nonclosed configuration. We fix the projective plane $\mathcal{F} = \mathcal{F}(A)$, freely generated by the configuration $A$. Further we will identify $\mathcal{F}(A)$ to the result $\langle F, F^0, 0F, I^F \rangle$ of the construction $A$, considering every element of $\mathcal{F}(A)$ to be uniquely representable in the form of an $A$-regular word.

Construction B. Let $B = \langle B, B^0, B^1, I^B \rangle$ be a nondegenerate nonclosed subconfiguration in the projective plane $\mathcal{F}$. In particular, the elements of $B$ are $A$-regular words.

We will consider the elements of the set $B$ as alphabetic symbols. Note that in this case the elements of $B$ can have $A$-length exceeding one and can be subwords in other elements of $B$.

For an arbitrary nonassociative word $w \in W(B)$, we define its $B$-depth $d_B(w)$:

(a) If $w \in B$, then $d_B(w) = 1$;

(b) If $w \notin B$ and $w = w_1w_2$, then $d_B(w) = \max\{d_B(w_1), d_B(w_2)\} + 1$.
Note that \( W(B) \subseteq W(A) \) and \( d_B(w) \leq d_A(w) \) for every \( w \in W(B) \). (The \( A \)-depth of nonassociative words over \( A \) is defined in a similar way.) Since \( W(B) \subseteq W(A) \), on the elements \( W(B) \), the order \( \prec \), introduced in the construction \( A \), is defined.

We define the set of \( B \)-regular words by induction on \( B \)-depth of the nonassociative word \( w \in W(B) \) in the following way:

If \( d_B(w) = 1 \), then \( w \in B \). Then we will call the word \( w \) \( B \)-regular of the first kind (the second kind), if \( w \in B^0 \) (\( w \in B^0 \)).

If \( d_B(w) \geq 2 \), then \( w = w_1w_2 \) for some \( w_1, w_2 \in W(B) \). Then we will call the word \( w \) \( B \)-regular of the first kind (second kind), if and only if six following conditions are fulfilled:

- \((B1)\) \( w_1 \succ w_2 \) and \( w_1, w_2 \) are \( B \)-regular words of the second kind (first kind);
- \((B2)\) There does not exist any \( w_3 \) such that \( \langle w_1, w_3 \rangle \in I^B \) and \( \langle w_2, w_3 \rangle \in I^B \);
- \((B3)\) If \( d_B(w_1) \geq 2 \) and \( w_1 = w_3w_4 \), then \( \langle w_3, w_2 \rangle \notin I^B \) and \( \langle w_4, w_2 \rangle \notin I^B \);
- \((B4)\) If \( d_B(w_2) \geq 2 \) and \( w_2 = w_3w_4 \), then \( \langle w_3, w_1 \rangle \notin I^B \) and \( \langle w_4, w_1 \rangle \notin I^B \);
- \((B5)\) If \( d_B(w_1) \geq 2 \), \( d_B(w_2) \geq 2 \), \( w_1 = w_3w_4 \), and \( w_2 = w_5w_6 \), then \( \{w_3, w_4\} \cap \{w_5, w_6\} = \emptyset \);
- \((B6)\) If \( d_B(w_1) \geq 3 \), \( w_1 = (w_3w_4)w_5 \) or \( w_1 = w_5(w_3w_4) \), and \( d_B(w_3w_4) \geq 2 \), then \( w_2 \notin \{w_3, w_4\} \).

We consider the set of all \( B \)-regular words to be the result of the construction \( B \).

Note that if \( w \in A \) and \( w \) is an \( B \)-regular word, then \( w \in B \).

**Example.** Suppose that the projective plane \( F \) is freely generated by the construction \( A = \langle A, A^0, 0_A, I^A \rangle \), where \( A = \{a_0 > a_1 > \cdots > a_5\}, A^0 = \{a_1, \ldots, a_5\}, 0_A = \{a_0\}, I^A = \langle a_0, a_i, \{a_i, a_0\} \mid 3 \leq i \leq 5 \rangle \).

If \( B \) is a subconfiguration in \( F \), presented on Figure 1, then the word \( ((a_1a_2)a_0)a_3 \) is \( B \)-regular, but is not \( A \)-regular.

If \( b = (((a_1a_4)(a_2a_5))(a_1a_5)(a_2a_4))a_0 \) and \( B \) is a subconfiguration in \( F \), presented on Figure 2, then every \( B \)-regular word is \( A \)-regular.

So, generally speaking, not every \( B \)-regular word is \( A \)-regular. We will define the configurations of a special form, with respect to which the regular words will at the same time be \( A \)-regular.

**Definition.** We will call the nondegenerate nonclosed configuration \( B \subseteq F \) a reduced one, if for every \( w_1, w_2, w_3 \in F \), the following conditions are satisfied:

- \((R1)\) If \( w_1, w_2 \in B \), \( w_1 \neq w_2 \), \( \langle w_1, w_3 \rangle \in I^A \), and \( \langle w_2, w_3 \rangle \in I^A \), then \( w_3 \in B \).
- \((R2)\) If \( \overline{w_1w_3} \in B \), \( w_3 \in B \), and \( \langle w_1, w_3 \rangle \in I^A \), then \( w_1 \in B \).
- \((R3)\) If \( \overline{w_1w_2} \in B \), \( \overline{w_1w_3} \in B \), and \( \overline{w_1w_2} \neq \overline{w_1w_3} \), then \( w_1 \in B \).
- \((R4)\) If \( \overline{w_1w_2}w_3 \in B \) and the word \( w_1 \) is \( B \)-regular, then \( \overline{w_1w_2} \in B \).
Proposition 1. Let $B$ be a reduced subconfiguration in $F$. Then every $B$-regular word is $A$-regular.

Proof. Note that from condition (R4) if follows that for every $w_1, w_2 \in F$, the following property is valid:

(R5) If $w_1w_2 \in B$ and the word $w_1$ is $B$-regular, then $w_1 \in B$.

Indeed, if $d_B(w_1) = 1$, then $w_1 \in B$. But if $d_B(w_1) \geq 2$, then $w_1 = w_3w_4$ for some $B$-regular words $w_3$ and $w_4$. Then $w_1w_2 = (w_3w_4)w_3$, which allows us to conclude that due to condition (R4) $w_1 = w_3w_4 \in B$.

Now let $w$ be an arbitrary $B$-regular word. We will prove by induction on $B$-depth of the word $w$, that $w$ is $A$-regular.

If $d_B(w) = 1$, then $w \in B$, and it is trivial that the word $w$ is $A$-regular.

But if $d_B(w) \geq 2$, then $w = w_1w_2$ for some $B$-regular words $w_1$ and $w_2$ with the condition $w_1 \succ w_2$. Due to the inductional assumption, $w_1$ and $w_2$ are $A$-regular. Therefore, the word $w$ satisfies condition (A1). We will prove that $w$ satisfies conditions (A2)-(A5).

Assume that $w_1, w_2 \in A$, $\langle w_1, w_3 \rangle \in I^A$ and $\langle w_2, w_3 \rangle \in I^A$ for some $w_3 \in A$. Then $w_1, w_2 \in B$ and, therefore, using condition (R1), we obtain that $w_3 \in B$. Hence, $\langle w_1, w_3 \rangle \in I^B$ and $\langle w_2, w_3 \rangle \in I^B$, which contradicts the condition (B2) for the word $w$.

Assume that $w_1 = w_3w_4$ and $\langle w_3, w_2 \rangle \in I^A$. Since $w_2 \in A$, we conclude that $w_2 \in B$. If $d_B(w_1) \geq 2$, then the words $w_3$ and $w_4$ are $B$-regular. Then $w_3 \in B$ and $\langle w_3, w_2 \rangle \in I^B$, which contradicts the condition (B3) for the word $w$. But if $d_B(w_1) = 1$, then $w_1 \in B$, and from condition (R2) it follows that $w_3 \in B$. Since $w_3 \sqsubset w_1$, we conclude that $\langle w_3, w_1 \rangle \in I^B$. Moreover, $\langle w_3, w_2 \rangle \in I^B$. The latter contradicts the condition (B2) for the word $w$.

Assume that $w_1 = w_3w_4$ and $\langle w_3, w_2 \rangle \in I^B$. If $d_B(w_1) \geq 2$ and $d_B(w_2) \geq 2$, then we obtain a contradiction to the condition (B5) for the word $w$. If $d_B(w_1) \geq 2$ and $d_B(w_2) = 1$, then the word $w_3$ is $B$-regular and $w_3w_4 \in F$. Therefore, due to property (R5), we conclude that $w_3 \in B$. Then, since $w_3 \sqsubset w_2$, we obtain that $\langle w_3, w_2 \rangle \in I^B$, which contradicts the condition (B3). If $d_B(w_1) = 1$ and $d_B(w_2) \geq 2$, then similar reasonings lead to a contradiction to the condition (B4).

If $d_B(w_1) = d_B(w_2) = 1$, then $w_3w_4 \in B$ and $w_3w_5 \in B$, and therefore, due to condition (R3), $w_3 \in B$ is valid. Then $\langle w_1, w_3 \rangle \in I^B$ and $\langle w_2, w_3 \rangle \in I^B$, which contradicts the condition (B2).

Assume that $w_1 = (w_2w_3)w_4$. If $d_B(w_1) = 1$, then $(w_2w_3)w_4 \in B$ and the word $w$ is $B$-regular. Then from condition (R4) it follows that $w_2w_3 \in B$, and from property (R5) we obtain that $w_2 \in B$. Since $w_2 \sqsubset w_2w_3 \sqsubset (w_2w_3)w_4$, we obtain that $\langle w_1, w_2w_3 \rangle \in I^B$ and $\langle w_2, w_2w_3 \rangle \in I^B$, which contradicts the condition (B2). If $d_B(w_1) = 2$, then $w_2w_3 \in B$. Therefore, using property (R5), we conclude that $w_2 \in B$. Then $\langle w_2w_3, w_2 \rangle \in I^B$, which contradicts the condition (B3). Thus, $d_B(w_1) \geq 3$. In this case, if $d_B(w_2w_3) = 1$ and $d_B(w_4) \geq 2$, then using the similar reasoning as for the previous case, we obtain a contradiction to (B3). Hence, it follows that $d_B(w_2w_3) \geq 2$ and $d_B(w_4) \geq 1$, which contradicts the condition (B6). 

\Box

Proposition 2. Suppose that $B$ and $C$ are subconfigurations in $F$, such that $B$ is reduced, and $C$ is a single-step extension of $B$. Then the following statements hold:

(a) The extension $B \subseteq C$ is free;

(b) For every $w \in F$, if $w$ is a $C$-regular word, then $w$ is $B$-regular;
(c) The configuration \( \mathcal{C} \) is reduced.

**Proof.** We will prove that the extension \( \mathcal{B} \subseteq \mathcal{C} \) is free. Suppose that \( w \in \mathcal{C} \setminus \mathcal{B} \). Therefore, there exist distinct \( w_1, w_2 \in \mathcal{B} \) such that \( \langle w_1, w \rangle \in \mathcal{I}^\mathcal{B} \) and \( \langle w_2, w \rangle \in \mathcal{I}^\mathcal{B} \). We can assume that \( w_1 \succ w_2 \). According to the definition of the incidence relation \( \mathcal{I}^\mathcal{B} \), the following cases are possible.

If \( \langle w_1, w \rangle \in \mathcal{I}^\mathcal{A} \) and \( \langle w_2, w \rangle \in \mathcal{I}^\mathcal{A} \), then due to condition (R1), we conclude that \( w \in \mathcal{B} \), which is impossible. If \( w_1 = \overline{w_3w_4w} \) and \( \langle w_2, w \rangle \in \mathcal{I}^\mathcal{A} \), then due to condition (R2), we conclude that \( w \in \mathcal{B} \), which is impossible. If \( \langle w_1, w \rangle = \overline{w_3w_4w} \) and \( w_2 = \overline{w_3w_4w} \), then from condition (R3) it follows again that \( w \in \mathcal{B} \). Therefore, there exist distinct \( w_1, w_2 \in \mathcal{B} \) to the fact that the configuration \( \mathcal{A} \) is not \( \mathcal{A} \)-regular possible.

Assume that there exists \( w_3 \in \mathcal{B} \) such that \( w_3 \neq w_1, w_3 \neq w_2 \) and \( \langle w_3, w \rangle \in \mathcal{I}^\mathcal{B} \). Since the word \( w_3 \) is not a letter from \( \mathcal{A} \) and \( \mathcal{B} \), which is impossible. If \( \langle w_1, w \rangle \in \mathcal{I}^\mathcal{A} \) for some \( \mathcal{A} \)-regular words \( w_1 \) and \( w_2 \), due to the inductive assumption, \( w_1 \) and \( w_2 \) are \( \mathcal{B} \)-regular. We will show that \( w \) satisfies the conditions (B1)–(B6) from the definition of a \( \mathcal{B} \)-regular word.

Suppose that \( w \in \mathcal{F} \) and \( w \) is \( \mathcal{C} \)-regular. We will prove by induction on \( \mathcal{C} \)-depth \( d_C(w) \) of the word \( w \) that \( w \) is \( \mathcal{B} \)-regular. If \( d_C(w) = 1 \), then due what was proved above, \( w \) is \( \mathcal{B} \)-regular. If \( d_C(w) \geq 2 \), then \( w = w_1w_2 \) for some \( \mathcal{C} \)-regular words \( w_1 \) and \( w_2 \). Due to the inductive assumption, \( w_1 \) and \( w_2 \) are \( \mathcal{B} \)-regular. We will show that \( w \) satisfies the conditions (B1)–(B6) from the definition of a \( \mathcal{B} \)-regular word.

Since \( w \in \mathcal{F} \), we conclude that \( w_1 \succ w_2 \), and therefore, condition (B1) is satisfied. From the fact that \( \mathcal{B} \subseteq \mathcal{C} \) and \( w \) is \( \mathcal{C} \)-regular, it follows that the condition (B2) holds. Assume that \( d_B(w_1) \geq 2 \), \( w_1 = \overline{w_3w_4w} \) and \( \langle w_2, w_3 \rangle \in \mathcal{I}^\mathcal{B} \). If \( \langle w_3, w_2 \rangle \in \mathcal{I}^\mathcal{A} \), then \( w_3, w_2 \in \mathcal{A} \), which contradicts the \( \mathcal{A} \)-regularity of the word \( w \). If \( w_3 \preceq w_2 \), then \( w_3 = \overline{w_3w_5} \) for some \( w_5 \in \mathcal{F} \), and therefore, the word \( w = \overline{w_3w_5} \) \( w_3w_5 \) is not \( \mathcal{A} \)-regular. But if \( w_3 \preceq w_3 \), then \( w_3 = \overline{w_3w_5} \) for some \( w_5 \in \mathcal{F} \), and therefore \( w = \overline{w_3w_5}w_4 \) is not \( \mathcal{A} \)-regular. Thus, the condition (B3) holds. In a similar way it can be proved that the condition (B4) is satisfied. (B5) and (B6) hold due to the fact that \( w \) is \( \mathcal{A} \)-regular.

Now we will prove that the configuration \( \mathcal{C} \) is reduced by checking whether the conditions (R1)–(R4) are met.

Let \( w_1, w_2 \in \mathcal{C} \), \( w_1 \neq w_2 \), \( \langle w_1, w_3 \rangle \in \mathcal{I}^\mathcal{A} \) and \( \langle w_2, w_3 \rangle \in \mathcal{I}^\mathcal{A} \) for some \( w_3 \in \mathcal{F} \). Since the words \( w_1 \) and \( w_2 \) are the letters from \( \mathcal{A} \), each of them cannot be represented in the form of an \( \mathcal{A} \)-regular word \( w'w'' \), where \( w', w'' \in \mathcal{B} \). Then it follows that \( w_1 \notin \mathcal{C} \setminus \mathcal{B} \) and \( w_2 \notin \mathcal{C} \setminus \mathcal{B} \). Then \( w_1, w_2 \in \mathcal{B} \), and therefore, due to the fact that the configuration \( \mathcal{B} \) is reduced, we conclude that \( w_3 \in \mathcal{B} \). Hence, \( w_3 \in \mathcal{C} \).
Let \( \overline{w_1w_2} \in C \), \( w_3 \in C \) and \( \langle w_1, w_3 \rangle \in I^A \). Similarly to what we have seen above, from the condition \( w_3 \in A \) it follows that \( w_3 \notin C \setminus B \). Therefore, \( w_3 \notin B \).

If \( \overline{w_1w_2} \in B \), then from the fact that the configuration \( B \) is reduced it follows that \( w_1 \in B \). But if \( \overline{w_1w_2} \in C \setminus B \), then as it has been noted earlier, \( w_1, w_2 \in B \) holds.

In any case, we obtain that \( w_1 \in C \).

Let \( \overline{w_1w_3}, \overline{w_1w_2} \in C \) and \( \overline{w_1w_2} \neq \overline{w_1w_3} \). If \( \overline{w_1w_2} \in B \) and \( \overline{w_1w_3} \in B \), then due to the fact that \( B \) is reduced, we conclude that \( w_1 \in B \), which means that \( w_1 \in C \).

If \( \overline{w_1w_2} \in C \setminus B \) or \( \overline{w_1w_3} \in C \setminus B \), then in any case we obtain that \( w_1 \in B \), and therefore, again we have that \( w_1 \in C \).

Suppose that \( \overline{w_1w_2}w_3 \in C \) and the word \( w_1 \) is \( C \)-regular. Since \( w_1 \in F \) and \( w_1 \) is \( C \)-regular, then \( w_1 \) is \( B \)-regular. Then if \( \overline{w_1w_2}w_3 \in B \), then due to the fact that \( B \) is reduced, it follows that \( \overline{w_1w_2} \in B \), and therefore, \( \overline{w_1w_2} \in C \). But if \( \overline{w_1w_2}w_3 \in C \setminus B \), then as it has been noted earlier, \( \overline{w_1w_2} \in B \) and \( w_3 \in B \) hold, from which we obtain again that \( \overline{w_1w_2} \in C \).

We will show that when closing the reduced configuration \( B \) in the plane \( F \), the elements \( B \) behave as free generators.

**Proposition 3.** Let \( B \) be a reduced subconfiguration in \( F \). Then the closure \( \langle B \rangle_F \) is free and consists exactly of all \( B \)-regular words.

**Proof.** Consider in the projective plane \( F \) a countable sequence of its subconfigurations

\[
B = B_0 \subseteq B_1 \subseteq \ldots \subseteq B_i \subseteq 
\]

such that for every \( i \in \omega \), the extension \( B_i \subseteq B_{i+1} \) is a full single-step one. Then \( \bigcup_{i \in \omega} B_i = \langle B \rangle_F \).

Using items (a) and (c) from Proposition 2, by induction we conclude that every extension \( B_i \subseteq B_{i+1} \) is free. Therefore, the closure \( \langle B \rangle_F \) is free, and \( B \) freely generates the plane \( \langle B \rangle_F \).

Using item (b) from Proposition 2 and induction, we conclude that for every \( w \in F \) and \( i \in \omega \), if the word \( w \) is \( B \)-regular, then \( w \) is \( B \)-regular. In particular, if \( w \in B_i \), then \( w \) is \( B \)-regular. Then it follows that every element of the closure \( \langle B \rangle_F \) is a \( B \)-regular word.

On the other hand, if the word \( w \) is \( B \)-regular, then due to Proposition 1, it is \( A \)-regular. Therefore, \( w \in F \), and hence, \( w \in \langle B \rangle_F \). \( \square \)

**Proposition 4.** Let \( B \) be a nondegenerate nonclosed subconfiguration in \( F \). Then there exists a reduced configuration \( B' \subseteq F \) such that \( B \subseteq B' \) and \( \langle B \rangle_F = \langle B' \rangle_F \).

**Proof.** Consider the following subset in the plane \( F \)

\[
X = A \cup \{ w \in F \mid w \leq^* w' \text{ for some } w' \in B \}.
\]

We designate by \( \aleph_0 \) the cardinality of the set \( X \). We will choose the smallest infinite cardinal number \( \kappa \) such that \( \aleph_0 < \kappa \).

We define the terms over the elements of \( B \) by induction. Every element \( b \in B \) will be called a term over \( b \). If \( t_1 = t_1(b_1, \ldots, b_s) \) is a term over \( b_1, \ldots, b_s \) and \( t_2 = t_2(b_{s+1}, \ldots, b_t) \) is a term over \( b_{s+1}, \ldots, b_t \), then their product \( t_1 \cdot t_2 \) in the plane \( F \) will be called a term over \( b_1, \ldots, b_t \).

We will define by a transfinite induction on \( \alpha \in \kappa \) the sequence \( \{ B_\alpha \}_{\alpha \in \kappa} \) of subsets of \( F \) such that for every \( \alpha \in \kappa \), the following properties are valid:
(a) $B_\beta \subseteq B_\alpha$ for every $\beta \in \alpha$;
(b) $B_\alpha \subseteq X$;
(c) for every $w \in B_\alpha$, there exists a term $t(b_1, \ldots, b_s)$ over some elements $b_1, \ldots, b_s \in B$ such that $w = t(b_1, \ldots, b_s)$.

We put $B_0 = B$. It is clear that $B_0$ satisfies conditions (a)-(c).

Suppose that $\alpha \in \kappa$ and $B_\beta$ is already defined and satisfies the conditions (a)-(c) for every $\beta \in \alpha$.

If $\alpha$ is a limit ordinal, we put that $B_\alpha = \bigcup_{\beta \in \alpha} B_\beta$. It is clear that then $B_\alpha$ satisfies the conditions (a)-(c).

If $\alpha$ is a nonlimit ordinal, then $\alpha = \beta + 1$. The set $B_\beta \subseteq F$ uniquely defines the subconfiguration $B_\beta$ in the plane $F$, such that the domain of $B_\beta$ is $B_\beta$. Since $B \subseteq B_\beta$, the configuration $B_\beta$ is nondegenerate. If the configuration $B_\beta$ satisfies the conditions (R1)-(R4), then $B_\beta$ is reduced, and then we put that $B_\alpha = B_\beta$. But if $B_\beta$ does not satisfy the conditions (R1)-(R4), then at least one of the following sets $Y_1, Y_2, Y_3, Y_4$ is not empty:

$Y_1 = \{ w_3 \in F \mid \exists w_1, w_2 \in B_\beta $ such that $w_1 \neq w_2, (w_1, w_3) \in I_1, (w_2, w_3) \in I_1, $ and $w_3 \notin B_\beta \}$,

$Y_2 = \{ w_1 \in F \mid \exists \overline{w_1w_2} \in B_\beta $ and $w_3 \in B_\beta $ such that $\langle w_1, w_3 \rangle \in I_1 $ and $w_1 \notin B_\beta \}$,

$Y_3 = \{ w_1 \in F \mid \exists \overline{w_1w_2} \in B_\beta $ and $\overline{w_1w_3} \in B_\beta $ such that $\overline{w_1w_2} \neq \overline{w_1w_3} $ and $w_1 \notin B_\beta \}$,

$Y_4 = \{ \overline{w_1w_2} \in F \mid \exists \overline{w_1w_2w_3} \in B_\beta $ such that $w_1$ is $B_\beta$-regular and $\overline{w_1w_2} \notin B_\beta \}$.

If $w_3 \in Y_1$, then $w_3 \in A \subseteq X$ and $w_3 = w_1 \cdot w_2$ for some $w_1, w_2 \in B_\beta$. Since $B_\beta$ satisfies the condition (c), $w_1$ and $w_2$ can be represented in the form of terms over the elements of $B$. Therefore, $w_3$ can also be represented in the form of a term over elements of $B$. Using similar reasoning, it can be proved that $Y_2 \subseteq A$ and every element of $w_1 \in Y_2$ can be represented in the form of a term over the elements of $B$.

If $w_1 \in Y_3$, then $w_1 = \overline{w_1w_2} \cdot \overline{w_1w_3}$, where $\overline{w_1w_2} \in B_\beta$ and $\overline{w_1w_3} \in B_\beta$. Since $B_\beta$ satisfies condition (b), we conclude that $\overline{w_1w_2} \in X \setminus A$, which means that from the property $w_1 \subseteq \overline{w_1w_2}$ it follows that $w_1 \in X$. Moreover, $\overline{w_1w_2}$ and $\overline{w_1w_3}$ can be represented in the form of terms over the elements of $B$. Therefore, $w_1$ can also be represented in the form of a term over the elements of $B$.

If the set $Y_4$ is not empty, then the following set $Y_4'$ is also not empty.

$Y_4' = \{ w \in F \mid w $ is $B_\beta$-regular and exists $(\overline{w_1w_2})w_3 \in B_\beta $ such that $w_1$ is $B_\beta$-regular, $\overline{w_1w_2} \notin B_\beta, w \sqsubset w_1 $ and $d_{B_\beta}(w) \leq d_{B_\beta}(w_1) \}$.

Suppose that $w \in Y_4'$. We will prove by induction on $B_\beta$-depth $d_{B_\beta}(w)$ that $w \in X$ and $w$ can be represented in the form of a term over the elements of $B$. If $d_{B_\beta}(w) = 1$, then $w \in B_\beta$, and the statement follows from the fact that $B_\beta$
satisfies the conditions (b)–(c). If \( d_{B_\beta}(w) \geq 2 \), then \( w = uv \) for some \( B_\beta \)-regular words \( u \) and \( v \). Then \( u, v \in Y'_4 \) and due to the inductive assumption, \( u \) and \( v \) can be represented in the form of terms over the elements of \( B \). Therefore, \( w \) can also be represented in the form of a term over the elements of \( B \). Moreover, for some \( \overline{w_1w_2}w_3 \in B_\beta \) we have that \( w \subseteq^* w_1 \sqcup^* \overline{w_1w_2}w_3 \in X \setminus A \), hence, we conclude that \( w \in X \).

Now consider an arbitrary element \( \overline{w_1w_2} \in Y_4 \). Then there exists \( \overline{w_1w_2}w_3 \in B_\beta \) such that \( w_1 \) is \( B_\beta \)-regular. Since \( \overline{w_1w_2}w_3 \in X \setminus A \) and \( \overline{w_1w_2}w_3 \sqcup \overline{w_1w_2}w_3 \), we conclude that \( \overline{w_1w_2} \in X \). Note that \( w_1 \in Y'_4 \). Therefore, due to what was proved above, \( w_1 \) can be represented in the form of a term over the elements of \( B \). Moreover, the element \( \overline{w_1w_2}w_3 \) can also be represented in the form of a term over the elements of \( B \). Then from the identity \( \overline{w_1w_2} = \overline{w_1w_2}w_3 \cdot w_4 \) we obtain that \( \overline{w_1w_2} \) can be represented in the form of a term over the elements of \( B \).

We put \( B_\alpha = B_\beta \cup Y_1 \cup Y_2 \cup Y_3 \cup Y_4 \cup Y'_4 \). Then \( B_\beta \subseteq B_\alpha \) and \( B_\beta \neq B_\alpha \). Due to what was proved above, \( B_\alpha \) satisfies the conditions (b) and (c).

So, the required sequence \( \{B_\alpha\}_{\alpha \in \omega} \) is constructed. Due to the choice of the cardinal number \( \omega \) and the fact that the property (b) holds, there definitely exists \( \alpha \in \omega \) such that \( B_\beta = B_{\alpha+1} \). Therefore, by construction, the configuration \( B'_\beta = B_\alpha \) is reduced. From the inclusion \( B \subseteq B'_\beta \) it follows that \( \langle B \rangle_\beta \subseteq \langle B'_\beta \rangle_\beta \). On the other hand, every element of \( B'_\beta \) can be represented in the form of a term over the elements of the configuration \( B \). Therefore, \( \langle B'_\beta \rangle_\beta \subseteq \langle B \rangle_\beta \). Thus, \( \langle B' \rangle_\beta = \langle B \rangle_\beta \).

**Theorem 5**. Let \( B \) be a nondegenerate nonclosed subconfiguration in \( F \). Then there exists a configuration \( B'_\beta \subseteq F \) such that \( B'_\beta \) freely generates the projective plane \( \langle B \rangle \).

**Proof**. Due to Proposition 4, there exists a reduced configuration \( B'_\beta \subseteq F \) such that \( \langle B \rangle \subseteq \langle B'_\beta \rangle \). By Proposition 3, the closure \( \langle B'_\beta \rangle_\beta \) is free. Hence, \( B'_\beta \) freely generates the projective plane \( \langle B \rangle \).

**Corollary 6**. Let \( B \) be a reduced subconfiguration in \( F \). Then \( A \cap \langle B \rangle_\beta = A \cap B \).

**Proof**. The inclusion \( A \cap B \subseteq A \cap \langle B \rangle_\beta \) is trivial. Suppose that \( w \in A \cap \langle B \rangle_\beta \). By Proposition 3, the word \( w \) is \( B \)-regular, and moreover, it coincides with a letter from \( A \). Since \( d_B(w) \leq d_A(w) = 1 \), we conclude that \( d_B(w) = 1 \), that is, \( w \in A \cap B \).

In [4], it was established that in finitely generated freely generated projective planes, the problem of inclusion of a word into a finitely generated subplane is algorithmically solvable. The method for transforming a configuration \( B \) into a reduced configuration \( B'_\beta \), which has been proposed above, provides a possibility to obtain another proof for this fact.

**Corollary 7** (A.A. Nikitin [4]). Suppose that \( F = F(A) \) is a projective plane, freely generated by a finite nondegenerate nonclosed configuration \( A \), and \( B \) is a finite nondegenerate nonclosed subconfiguration in \( F \). Then there exists an effective algorithm, which, given any word \( w \in F \), defines whether it is true that \( w \in \langle B \rangle_\beta \).

**Proof**. Note that if the configurations \( A \) and \( B \) are finite, then the set \( X \) in the proof of Proposition 4 is also finite, and therefore, the cardinal number \( \omega \) coincides with \( \omega \). Then it follows that the sequence \( \{B_\alpha\}_{\alpha \in \omega} \) from the proof of Proposition 4 consists of finite sets and gets stabilized after a finite number of steps, that is, there exists \( \alpha \in \omega \) such that the configuration \( B_\alpha \) is reduced. From the effectiveness of the construction \( A \) it follows that the checking of the conditions (R1)–(R4) for
the finite configuration \( B_i \subseteq \mathcal{F} \), \( i \in \omega \), is conducted effectively. Therefore, in the case of finite \( A \) and \( B \), all constructions in the proof of Proposition 4 are effective. Thus, the reduced configuration \( B' \subseteq \mathcal{F} \) with the condition \( \langle B \rangle_{\mathcal{F}} = \langle B' \rangle_{\mathcal{F}} \) can be found effectively.

We will describe an algorithm that defines for any \( w \in \mathcal{F} \) whether the statement \( w \in \langle B \rangle_{\mathcal{F}} \) holds. Let \( w \) be an arbitrary \( A \)-regular word. We will calculate its \( A \)-depth \( d_A(w) = n \) and define a finite set of words

\[
U_n = \{ u \in \mathcal{F} \mid \text{word } u \text{ is } B' \text{-regular and } d_{B'}(u) \leq n \}.
\]

We will prove that \( w \in \langle B \rangle_{\mathcal{F}} \) if and only if \( w \in U_n \). If \( w \in U_n \), then the word \( w \) is \( B' \)-regular, and therefore, due to Proposition 3, we conclude that \( w \in \langle B \rangle_{\mathcal{F}} \). But if \( w \in \langle B \rangle_{\mathcal{F}} \), then \( w \in \langle B' \rangle_{\mathcal{F}} \). By Proposition 3, the word \( w \) is \( B' \)-regular. Since \( d_{B'}(w) \leq d_A(w) = n \), we obtain that \( w \in U_n \).

Thus, due to the finiteness of the set \( U_n \), the checking of the condition \( w \in \langle B \rangle_{\mathcal{F}} \) is conducted effectively. \( \square \)

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