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COMPUTABLE METRICS ABOVE THE STANDARD REAL METRIC

R. KORNEV

Abstract. We construct a sequence of computable real metrics pairwise incomparable under weak reducibility $\leq_{ch}$ and located above the standard real metric w. r. t. computable reducibility $\leq_c$. Iterating the construction, we obtain that the ordering $(P(\omega), \subseteq)$ of subsets of ω is embeddable into the ordering of ch-degrees of real metrics above the standard metric. It is also proved that the countable atomless Boolean algebra is embeddable with preservation of joins and meets into the ordering of c-degrees of computable real metrics.

Keywords: computable metric space, representation of real numbers, Cauchy representation, reducibility of representations, computable analysis.

We continue the study of two notions of computable reducibility on real metrics started in [1]: a *computable* reducibility, or c-reducibility for short, and a weaker version of it, ch-reducibility. In [1] it was shown that there exists a countable sequence of computable metrics on $\mathbf{R}$ pairwise ch-incomparable to each other and that the functional tree $\omega^{<\omega}$ is embeddable as a poset into the ordering of c-degrees of computable metrics. An inherent property of the construction was that all constructed metrics were c-reducible to the standard real metric, $\rho_R(x, y) = |x - y|$. In the present paper, in contrast to that, we show the existence of metrics above ρ_R.

The main result of the paper is the following theorem.

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Theorem 1. *There exists an infinite sequence of computable metrics* $\rho_i >_c \rho_R$, $i \in \omega$, *such that* $\rho_i \,|_{ch}\, \rho_j$ *for all* $i \neq j$.

Section 2 consists of the proof of Theorem 1. In Section 3 we analyze the proof and obtain several stronger results. By combining metrics ρ_i with each other in a certain way, we embed the ordering $(P(\omega), \subseteq)$ of subsets of ω into the ordering of ch-degrees of real metrics. From this we conclude that there are continuum many different ch-degrees of metrics on $\mathbf{R}$. Moreover, we are able to embed any countable partial ordering into the ordering of ch-degrees of computable metrics above ρ_R. Finally, we turn to the ordering of c-degrees of computable metrics and show that the countable atomless Boolean algebra is embeddable into this ordering with preservation of joins and meets.

1. Preliminaries and notations

Main definitions and notations we use can be found in [1]. Basic definitions and classic results in computability theory can be found in [2] and [3]. For the background in computable analysis we refer the reader to [4].

The *Baire space* is the set ω^ω of all countable sequences of natural numbers endowed with the product topology of countably many copies of ω with discrete topology. For $f = (f(0), f(1), \ldots) \in \omega^\omega$, $f \downharpoonright n = (f(n), f(n+1), \ldots)$ denotes the nth tail of f. Also, for $n, m, k \in \omega$, we will use the following notations:

$$\bar{n} = (n, n, \ldots) \in \omega^\omega, \;\; m^k\bar{n} = (\underbrace{m, \ldots, m}_{k}, n, n, \ldots) \in \omega^\omega.$$

Following [5], we denote partial computable functions and Turing functionals by uppercase letter Φ and corresponding use functions by lowercase letter φ. For an oracle $f \in \omega^\omega$ and a natural number n, $\Phi_{e,s}(f)(n)$ is the result of computation of $\Phi_e(f)(n)$ in s steps, and $\varphi_{e,s}(f)(n)$ is the use of this computation.

Let X be a set of at most continuum cardinality. A *representation* of X is a partial surjection $\delta\colon \omega^\omega \to X$. The notion of a representation was introduced in [6] and is a generalization of the notion of a numbering of a countable set; for the background on the theory of numberings, see [7]. Let $\delta_X\colon \omega^\omega \to X$, $\delta_Y\colon \omega^\omega \to Y$ be representations of sets X and Y. Partial mapping $\Phi\colon \omega^\omega \to \omega^\omega$ is called a (δ_X, δ_Y)-*realization* of a partial function $F\colon X \to Y$ if

$$F \circ \delta_X(f) = \delta_Y \circ \Phi(f) \text{ for } f \in \mathrm{dom}(F \circ \delta_X).$$

Function F is called (δ_X, δ_Y)-*computable* if it has a computable (δ_X, δ_Y)-realization, i. e., is realized by a Turing functional. This definition is equivalent to the Weihrauch's definition [4] via Type-2 machines, cf. also [8, 9].

Let δ, δ' be representations of a set X. Representation δ is *computably reducible* to δ' (written $\delta \leq_c \delta'$ [6]) if there exists a Turing functional Φ_z such that

$$\delta(f) = \delta' \circ \Phi_z(f) \text{ for } f \in \mathrm{dom}(\delta),$$

or, equivalently, if the identity mapping id_X is (δ, δ')-computable. Binary relation $\leq_c$ is a preordering on the set of all representations of X. One can define the equivalence relation $\equiv_c$ in the usual way and consider equivalence classes of representations under this relation, which we will call *c-degrees*. The c-degree of a representation δ will be denoted by $\deg_c(\delta)$. It is well-known [4] that it is possible to define the *join* $\vee$ and *meet* $\wedge$ operations on representations of X that respect $\equiv_c$, making the ordering of c-degrees of representations of X a lattice.

Let $\mathbf{X} = (X, \rho, W, \nu)$ be a complete separable metric space with a distinguished countable dense subset W and a numbering $\nu\colon \omega \to W$. Denote $w_n = \nu n$. *Cauchy representation* $\rho\colon \omega^\omega \to X$, defined as

$$\rho(f) = x \text{ if } w_{f(n)} \underset{n\to\infty}{\longrightarrow} x \text{ and } \rho(w_{f(n)}, w_{f(m)}) \leqslant 2^{-n} \text{ for } m > n,$$

will be denoted by the same character as the underlying metric; any element f with the above property is called a *Cauchy name* for x. Space $\mathbf{X}$ is called a *computable metric space* if the distance $\rho(w_n, w_m)$ is a computable real number uniformly in n and m.

Remark 1. If $\rho(f) = x$, then $\rho(w_{f(n)}, x) \leqslant 2^{-n}$ for all n.

Suppose that the countable dense set W and its numbering ν are fixed. Consider another metric ρ' on X such that (X, ρ') is a complete metric space with the same topology as on (X, ρ). We say that ρ is *computably reducible* to ρ', $\rho \leq_c \rho'$, if the reducibility holds for respective Cauchy representations. This means that there is an effective procedure that, for any $x \in X$, given a ρ-name for x, outputs a ρ'-name for x. As we have already noted, this is equivalent to the fact that the identity homeomorphism id_X is (ρ, ρ')-computable. Generalizing this, we say that ρ is *weakly reducible* to ρ', $\rho \leq_{ch} \rho'$, if there exists at least one (ρ, ρ')-computable autohomeomorphism of X. It is easy to see that $\leq_{ch}$ is a preordering. Obviously, $\rho \leq_c \rho'$ implies $\rho \leq_{ch} \rho'$. We can define equivalence relations $\equiv_c$ and $\equiv_{ch}$ and speak about c- and ch-degrees of metrics.

Since $\leq_{ch}$ is weaker than $\leq_c$, any c-degree is contained in a ch-degree. Also, by definition, the ordering of c-degrees of metrics canonically embeds into the ordering of c-degrees of representations: degree of a metric can be put into correspondence with the degree of its Cauchy representation.

Of course, we could give the definition of $\leq_{ch}$ for representations in general, not restricting ourselves to metrics inducing one given topology, but currently that is outside the scope of our investigation.

Let (X, ρ_X) and (Y, ρ_Y) be metric spaces, A be a subset of X. A *modulus of uniform continuity* of a function $F\colon A \to Y$ is a function $\mathrm{mod}\colon \omega \to \omega$ such that for all $x, y \in A$ and $n \in \omega$

$$\rho_X(x, y) \leqslant 2^{-\mathrm{mod}(n)} \;\Rightarrow\; \rho_Y(F(x), F(y)) \leqslant 2^{-n}.$$

Throughout the paper we also assume that $\mathrm{mod}(n+1) \geqslant \mathrm{mod}(n) \geqslant n$. We will also omit the word “uniform”. Obviously, function F has a modulus of uniform continuity if and only if F is uniformly continuous. See [10] and [11] for more information on computability-theoretic aspects of moduli of continuity.

Henceforth, unless specified otherwise, we will be working with the space $\mathbf{R}$ of real numbers with the standard topology and with the set of rational numbers $\mathbf{Q}$ taken as the dense subset. Fix a Gödel numbering of the rationals, $\mathbf{Q} = (q_i)_{i\in\omega}$. By a *metric* we will mean a complete real metric inducing the standard topology.

2. Proof of Theorem 1

2.1. **Requirements.** To prove Theorem 1, we will construct a countable family $(\rho_i)_{i\in\omega}$ of computable metrics, satisfying the following requirements for $i, e \in \omega$:

$\mathcal{R}_{ie}$: $\rho_i \not\leq_{ch} \rho_j$ via Φ_e for all $j \neq i$,

$\mathcal{S}$: $\rho_R \leq_c \rho_i$ for all i.

Notice that in the requirement $\mathcal{R}_{ie}$ we diagonalize against all ρ_j, $j \neq i$, simultaneously. Since $\leq_c$ implies $\leq_{ch}$, satisfaction of all requirements $\mathcal{R}_{ie}$ and $\mathcal{S}$ ensures that for all $i \neq j$

$$\deg_{ch}(\rho_R) < \deg_{ch}(\rho_i) \mid \deg_{ch}(\rho_j)$$

and

$$\deg_c(\rho_R) < \deg_c(\rho_i) \mid \deg_c(\rho_j).$$

In order to make metrics ρ_i computable, during the construction we will be building an increasing sequence of finite sets A_s of rational numbers such that the distances between numbers in A_s will not change after stage s. To be precise, sets A_s will have the following properties:

(1) For all i, (A_s, ρ_i) is a finite subspace of metric space $(\mathbf{R}, \rho_i)$,
(2) $A_s \subseteq A_t$ for $s \leqslant t$,
(3) $\bigcup_{i\in\omega} A_s = \mathbf{Q}$.

This gives us a method of computing $\rho_i(q, r)$ uniformly in $q, r \in \mathbf{Q}$ for all i: in order to compute this value, it suffices to wait for stage s at which $q, r \in A_s$.

For the requirement $\mathcal{R}_{ie}$ to be met it suffices to show that, for all $j \neq i$, if Φ_e computes an autohomeomorphism F of the reals w. r. t. ρ_i and ρ_j, then there exist $x \in \mathbf{R}$ and a ρ_i-name f for x such that $\Phi_e(f)$, if defined, is not a ρ_j-name for $F(x)$. Such a name f will be obtained by making certain real numbers close in ρ_i while at the same time keeping Φ_e-"images" of these numbers distant in all other ρ_j.

Let us describe the elementary deformations that will be added to our metrics to meet $\mathcal{R}$-requirements. Fix a cylindrical coordinate system (x, r, θ) in $\mathbf{R}^3$. Take an element $a \in \mathbf{R}$ and a closed real interval J to the right of a. Divide J into 6 even subintervals $J^k = [b_k, b_{k+1}]$, $k = 0, \ldots, 5$. Define a mapping $\Gamma\colon J \to \mathbf{R}^3$ such that

$$\Gamma(b_0) = (b_0, 0, 0), \quad \Gamma(b_3) = (a, h, \tfrac{s}{s+1}\pi), \quad \Gamma(b_6) = (b_6, 0, 0),$$
$$\Gamma(b_1) = (b_0, 2, \tfrac{s}{s+1}\pi), \quad \Gamma(b_4) = (a, 2.5, \tfrac{s}{s+1}\pi),$$
$$\Gamma(b_2) = (a + 0.25, 2, \tfrac{s}{s+1}\pi), \quad \Gamma(b_5) = (b_6, 2.5, \tfrac{s}{s+1}\pi),$$

and Γ is linear on each subinterval J^k. Here, $h < 2$ is a real number and s is the stage at which we introduce Γ in the construction.

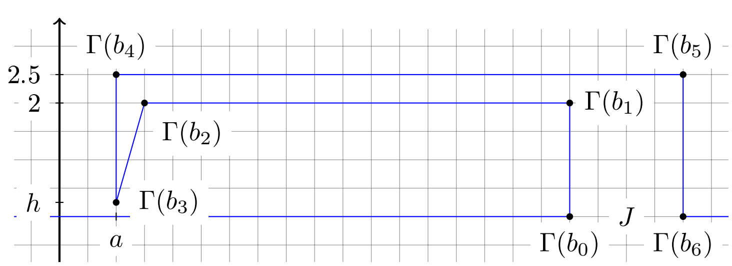

FIG. 1. Mapping $\Gamma_{a,J,h,s}$ within the plane $\theta = \frac{s}{s+1}\pi$.

We can extend Γ to a mapping $\gamma\colon \mathbf{R} \to \mathbf{R}^3$ by putting $\gamma(x) = (x, 0, 0)$ for $x \notin J$. Define a metric ρ on $\mathbf{R}$ as $\rho(x, y) = \|\gamma(x) - \gamma(y)\|$, where $\|\cdot\|$ is the Euclidean norm in $\mathbf{R}^3$. Clearly, ρ is a computable metric, provided that a, h and the endpoints of J are computable real numbers. We have $\rho(a, b_3) = h$ in this metric, thus we can control the distance between these points by choosing an appropriate h. We will use subscripts to indicate the parameters of Γ. For instance, we may want to write $\Gamma_{a,J,h,s}$, meaning that $\Gamma\colon J \to \mathbf{R}^3$ is defined at stage s to make $\rho(a, b_3) = h$ where b_3 is the midpoint of J. Some (or all) of these subscripts may be omitted, where it is convenient.

Fix integer points $\mathbf{a}_i = 10i \in \mathbf{R}$ and intervals $J_i = [10i+4, 10i+6] \subset \mathbf{R}$ (so that J_i is located between $\mathbf{a}_i$ and $\mathbf{a}_{i+1}$). Points $\mathbf{a}_i$ will be used as followers of our requirements, and intervals J_i will serve as domains of mappings Γ defined in the construction.

2.2. **Strategy for $\mathcal{R}_{ie}$ in isolation.** Metric ρ_i will have the form

$$\rho_i(x,y) = \|\gamma_i(x) - \gamma_i(y)\|,$$

where the mapping $\gamma_i\colon \mathbf{R} \to \mathbf{R}^3$ is built in stages. At stage 0 we have $\gamma_{i,0}(x) = (x, 0, 0)$ so $\rho_{i,0} = \rho_R$. Suppose for now that $\rho_j = \rho_R$ for $j \neq i$. Pick a follower $q_a = \mathbf{a}_k$ and wait until $\Phi_e(\bar{a})(0)\downarrow= u$. In order to diagonalize against Φ_e, suppose that Φ_e does realize a surjective homeomorphism $F\colon (\mathbf{R}, \rho_{i,t}) \to (\mathbf{R}, \rho_R)$, where $\rho_{i,t}$ is the current approximation of ρ_i at stage t. We want to modify this approximation by means of an elementary deformation, yielding another approximation $\rho_{i,s}$ at a subsequent stage s, and provide elements $f \in \omega^\omega$ and $x \in \mathbf{R}$ such that $\rho_{i,s}(f) = x$ but $\Phi_e(f)$ is not a ρ_R-name for $F(x)$. Element f will have the form $f = a^p\bar{d}$, where $p = \varphi_e(\bar{a})(0)$ and $x = q_d$ is a midpoint of an interval that we will connect to q_a via said deformation (as noted above, then we can control the $\rho_{i,s}$-distance between q_a and q_d, so, setting $\rho_{i,s}(q_a, q_d) = 2^{-p+1}$, we see that f is indeed a $\rho_{i,s}$-name for q_d). By the choice of p we have $\Phi_e(f)(0) = \Phi_e(\bar{a})(0) = u$. To ensure that $\Phi_e(f)$ is not a ρ_R-name for $F(q_d)$, by Remark 1 it suffices to make sure that $\rho_R(q_u, F(q_d)) > 1$. To achieve this, we would like to work not directly with q_d, but with a "proxy" of it, an auxiliary point that we will call a second follower of $\mathcal{R}_{ie}$. So, when we see $\Phi_e(\bar{a})(0)\downarrow= u$, we pick a second follower $q_b = \mathbf{a}_l > q_a$ and wait until $\Phi_e(\bar{b})(0)\downarrow= v$. Our goal is to check if q_v is far enough from q_u, and in case it is, we will be able to conclude that $F(q_d)$ is far enough from q_u as well, since every surjective homeomorphism of the reals is monotone. We can observe the following two cases.

Case 1. $\rho_R(q_u, q_v) > 2$. This situation is favourable for us. Suppose we are at stage s of the construction. Choose an interval J_m to the right of both followers and define a mapping $\Gamma_{q_a, J_m, 2^{-p+1}, s}$, where $p = \varphi_{e,s}(\bar{a})(0)$. Define

$$\gamma_{i,s}(x) = \gamma_{i,s-1}(x) \uplus \Gamma_s(x) = \begin{cases} \Gamma_s(x), & x \in J_m \\ \gamma_{i,s-1}(x), & x \notin J_m \end{cases}$$

and let $\rho_{i,s}(x,y) = \|\gamma_{i,s}(x) - \gamma_{i,s}(y)\|$ for $x, y \in \mathbf{R}$. See Fig. 4 below for the idea of how a new mapping Γ_s is added to $\gamma_{i,s-1}$.

It is easy to see that Φ_e can't be a $(\rho_{i,s}, \rho_R)$-realization of any autohomeomorphism of $\mathbf{R}$. Indeed, suppose the opposite, that Φ_e realizes a surjective homeomorphism $F\colon \mathbf{R} \to \mathbf{R}$. Then F is strictly monotonic. Let q_d be the midpoint of interval J_m. As noted above, we have $\rho_{i,s}(q_a, q_d) = 2^{-p+1}$ so $a^p\bar{d}$ is a $\rho_{i,s}$-name of q_d such that $\Phi_e(a^p\bar{d})(0) = \Phi_e(\bar{a})(0) = u$. We need to show that $\rho_R(q_u, F(q_d)) > 1$. Since $q_a < q_b < q_d$ by the choice of J_m, we must have

$$F(q_a) < F(q_b) < F(q_d) \text{ or } F(q_a) > F(q_b) > F(q_d).$$

Assume that $F(q_a) < F(q_b) < F(q_d)$, the other case being symmetric. Since $\rho_R(q_u, q_v) > 2$, $\rho_R(q_u, F(q_a)) \leqslant 1$ and $\rho_R(q_v, F(q_b)) \leqslant 1$ by Remark 1, then

$$\rho_R(q_u, F(q_b)) \geqslant |\rho_R(q_u, q_v) - \rho_R(q_v, F(q_b))| > 1,$$

and so $q_u < F(q_b) < F(q_d)$. Hence we have $\rho_R(q_u, F(q_d)) > \rho_R(q_u, F(q_b)) > 1$, which means that $\Phi_e(a^p \bar{d})$ can't be a Cauchy name for $F(q_d)$ (see Fig. 2).

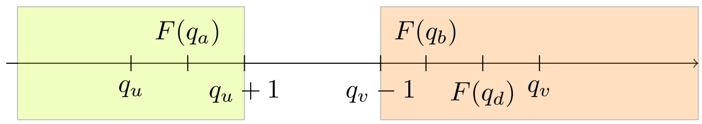

FIG. 2. q_u can't be close enough to $F(q_d)$. Case $F(q_a) < F(q_b) < F(q_d)$.

Case 2. $\rho_R(q_u, q_v) \leqslant 2$. In this case we simply pick a new second follower $\mathbf{a}_{l+1}$ and repeat the process. If F is an autohomeomorphism of the reals, this process will eventually end since the distance between $F(q_a)$ and $F(q_b)$ must grow as the distance between q_a and q_b increases. If the process never ends, then $\mathcal{R}_{ie}$ is automatically satisfied with no effort from our side. Similarly, if $\Phi_e(\bar{a})(0)\uparrow$ or $\Phi_e(\bar{b})(0)\uparrow$, then Φ_e clearly cannot realize a total function, and $\mathcal{R}_{ie}$ is satisfied.

Now we can reveal the role of the second follower q_b. This auxiliary point is needed to provide us with the information that the distance between $F(q_a)$ and $F(q_d)$ is large enough without even knowing q_d, which permits us to diagonalize against Φ_e while always being able to pick fresh intervals J_m from the area not used in the construction yet. This way we will be able to compute moduli of continuity of mappings Γ_{J_m} uniformly in m, which will give us c-reduction of ρ_R to metrics ρ_i, satisfying the requirement $\mathcal{S}$.

2.3. **Interactions between strategies.** Metrics ρ_j will not look like ρ_R all the time and will be defining their own mappings Γ that can decrease the distance $\rho_j(q_u, F(q_d))$, possibly injuring $\mathcal{R}_{ie}$. This happens when q_u is close to a follower z of a requirement of the form $\mathcal{R}_{je'}$ and the strategy for $\mathcal{R}_{je'}$ introduces a mapping $\Gamma_{z,J}$ such that $F(q_d) \in J$. Then $F(q_d)$ becomes close to z and thus to q_u in the metric ρ_j. We eliminate this possibility by means of a priority argument. To protect $\mathcal{R}_{ie}$, we restrict the area used in the construction so far from being changed by weaker priority requirements, which implies that their followers z have to be chosen outside that area, in particular, far from q_u. This way, q_u and $F(q_d)$ will be kept far from each other unless a stronger priority requirement takes action. Fix an effective enumeration $(\mathcal{R}_n)_{n\in\omega}$ of requirements $\mathcal{R}_{ie}$, $i, e \in \omega$. Requirement $\mathcal{R}_n$ has priority over $\mathcal{R}_m$ if $n < m$. When $\mathcal{R}_n$ gets to act at some stage, it initializes all weaker requirements, which means that they cancel their followers and have to pick them anew.

Followers of the requirements are appointed in the following manner. Recall that each requirement wants to choose two followers. If a requirement $\mathcal{R}_n = \mathcal{R}_{ie}$ currently has no followers, we appoint the first follower of $\mathcal{R}_n$ to be a *fresh large* (i. e., greater than all rational numbers seen in the construction so far) number $\mathbf{a}_k = q_a$. Once $\Phi_e(\bar{a})(0)\downarrow= u$, we choose a second follower q_b of $\mathcal{R}_n$, $q_b = \mathbf{a}_l > \mathbf{a}_k$, and restrict the area used in the construction by initializing weaker priority requirements. The restriction is uniform for all metrics ρ_j, $j \neq i$. This will imply that from now on no weaker requirement will be able to make distances $\rho_j(q_u, x)$ too small for all $j \neq i$

and any point x already far from q_u at this moment, killing off the above mentioned possibility of $\mathcal{R}_n$ being injured by weaker requirements.

Now, like in the basic strategy, we gradually increase the value of the second follower until we find one guaranteeing us that, if we choose an interval J_m with midpoint q_d to the right of it, then $\rho_j(q_u, F(q_d)) > 1$ for all $j \neq i$. Recall that we need to exclude the possibility of $F(q_d)$ to belong to the domain of a mapping Γ connecting it to q_u in some metric ρ_j. As noted above, weaker requirements will not be able to introduce such a Γ in the future since they were initialized and their first followers are guaranteed to be far from q_u. Therefore, we only need to make sure that $F(q_d)$ does not belong to a domain of any mapping Γ defined in the past, and here the second follower $q_b = \mathbf{a}_l$ comes into play again. We want $F(q_b)$ to be outside the area used in the construction so far. In order to ensure it, when $\Phi_e(\bar{a})(0) \downarrow = u$, we introduce two numbers C_n and D_n serving as left and right bounds of this area. In particular, the interval $[C_n, D_n]$ contains q_u and the domains of all mappings Γ defined so far. Wait until $\Phi_e(\bar{b})(0) \downarrow = v$ and check whether $q_v \in [C_n, D_n]$. If yes, reappoint the second follower to be $\mathbf{a}_{l+1}$ and repeat the process. Once we see that $q_v \notin [C_n, D_n]$, the strategy finally ends and we can choose a fresh interval J_m and introduce the map Γ_{q_a, J_m} connecting the midpoint q_d of J_m to q_a in the metric ρ_i. Since $q_d > q_b$ and every autohomeomorphism of the reals is monotone, $F(q_d)$ will be outside $[C_n, D_n]$ as well. See the following figure for an illustration. The red area is the entire area used in the construction at the moment when $\Phi_e(\bar{a})(0) \downarrow$, we restrict this whole area from being accessed by q_v. Any mapping Γ defined after this moment by a weaker priority requirement will operate outside this area and thus cannot connect anything to q_u.

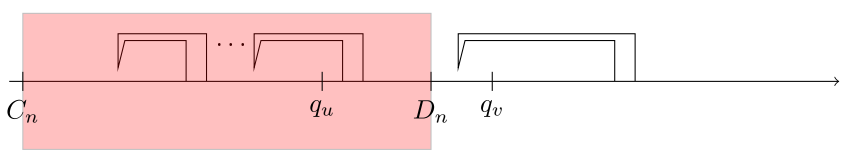

FIG. 3. Restricted area.

Of course, stronger priority requirements are still able to introduce a mapping Γ violating this restriction and injuring $\mathcal{R}_n$, but we will see that this can happen only finitely many times.

In order for ρ_i to be a correctly defined metric, we need γ_i to be injective. Let us explain how we combine different maps Γ into a full picture, preserving the injectivity of γ_i. Suppose that we want to meet a requirement $\mathcal{R}_n = \mathcal{R}_{ie}$. We have seen that the strategy for $\mathcal{R}_n$ consists of two main phases: computation of $\Phi_e(\bar{a})(0)$ and searching for an appropriate second follower q_b. What happens if a strategy for a weaker requirement $\mathcal{R}_m = \mathcal{R}_{ie'}$ meddles between these two phases? In other words, suppose that, while $\mathcal{R}_n$ is finding a suitable second follower, $\mathcal{R}_m$-strategy is started and completed and a mapping Γ_t is defined at some stage t that connects a follower of $\mathcal{R}_m$ with some interval. If after that, at stage $s > t$, $\mathcal{R}_n$ successfully finishes the second phase and defines a map Γ_s, then Γ_s will have to connect its own fresh interval J with the first follower q_a of $\mathcal{R}_n$ all the way through the area where Γ_t is doing its job. In order to prevent the images of Γ_s and Γ_t from intersecting with each other, we place them into different planes in $\mathbf{R}^3$: recall that

$\Gamma_s(x) = (y, r, \theta) \in \mathbf{R}^3$, where the angular coordinate $\theta = \frac{s}{s+1}\pi$ is unique for each Γ_s. This way, images of different mappings Γ do not intersect with each other, and as a result, all mappings $\gamma_i \colon \mathbf{R} \to \mathbf{R}^3$ are injective.

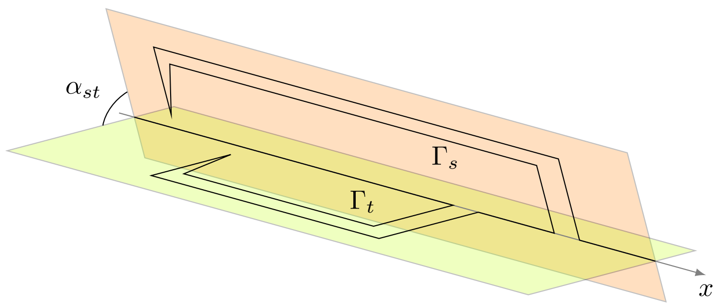

FIG. 4. Gluing the mappings Γ_s and Γ_t. $\alpha_{st} = |\frac{s}{s+1} - \frac{t}{t+1}|\pi$ is the angle between planes containing images of Γ_s and Γ_t.

2.4. **Construction.** *Stage* $s = 0$. Let $A_0 = \varnothing$, $\gamma_{i,0}(x) = (x, 0, 0)$ for all i.

Stage $s+1$. Let $A_{s+1} = A_s \cup \{q_s\}$. Let $n = \langle s+1 \rangle_0$, where $\langle \cdot \rangle_0$ is the left projection of the usual Cantor bijective pairing function. We work with requirement $\mathcal{R}_n = \mathcal{R}_{ie}$ if it hasn't been met. Let $\mathbf{a}_k = q_a$ be the first follower of $\mathcal{R}_n$ (if $\mathcal{R}_n$ has no first follower, appoint it to be a fresh large number $\mathbf{a}_k = 10k$). If $\mathcal{R}_n$ has no second follower, proceed to substage 0. Otherwise, proceed to substage 1.

Substage 0. We have two possibilities:

(1) $\Phi_{e,s+1}(\bar{a})(0) \uparrow$. In this case, end the stage $s+1$.
(2) $\Phi_{e,s+1}(\bar{a})(0) \downarrow = u$. We say that $\mathcal{R}_n$ *acts at substage* 0. Initialize all weaker priority requirements. Define $C_n = \min(-2, q_u - 2)$ and $D_n = \max(q_u + 2, D) + 1$, where D is the right end of the rightmost interval used in the construction so far. Appoint the second follower $\mathbf{a}_l = \mathbf{a}_{k+1}$ and proceed to substage 1.

Substage 1. $\mathcal{R}_n$ has appointed the second follower $\mathbf{a}_l = q_b$. Find the first among the following possibilities that applies:

(1) If $\Phi_{e,s+1}(\bar{b})(0) \uparrow$, end the stage $s+1$.
(2) If $\Phi_{e,s+1}(\bar{b})(0) \downarrow = v$ and ($q_v < C_n$ or $q_v > D_n$), we say that $\mathcal{R}_n$ acts at substage 1. Initialize all weaker priority requirements. Pick a fresh interval J_m and define a map $\Gamma_{q_a, J_m, 2^{-p+1}, s+1}$ where $p = \varphi_{e,s+1}(\bar{a})(0)$. End the stage $s+1$.
(3) Otherwise (i.e. when $\Phi_{e,s+1}(\bar{b})(0) \downarrow$ but the extra condition fails), redefine the second follower to be $\mathbf{a}_{l+1}$ and end the stage $s+1$.

At the end of the stage, if $\mathcal{R}_n$ acted at substage 1, let $\gamma_{i,s+1} = \gamma_{i,s} \uplus \Gamma_{q_a, J_m, 2^{-p+1}, s+1}$, otherwise let $\gamma_{i,s+1} = \gamma_{i,s}$. Let $\gamma_{j,s+1} = \gamma_{j,s}$ for $j \neq i$.

2.5. **Verification.** We need to show that metrics ρ_i are computable and induce the standard topology on $\mathbf{R}$ as well as prove that all requirements are satisfied. For the latter we will use a standard finite priority argument.

First of all, it is easy to see that pointwise limits $\gamma_i(x) = \lim_s \gamma_{i,s}(x)$ exist for all i, since for any $x \in \mathbf{R}$ there is at most one stage s such that $\gamma_{i,s+1}(x) \neq \gamma_{i,s}(x)$. It is also easy to see that metrics ρ_i are computable: in order to compute $\rho_i(q_m, q_n)$

it suffices to wait for the stage $s = \max(m, n) + 1$ at which q_m and q_n will both be contained in the set A_s. Distances between these points will be forbidden to change from now on, so we have

$$\rho_i(q_m, q_n) = \rho_{i,s}(q_m, q_n) = \|\gamma_{i,s}(q_m) - \gamma_{i,s}(q_n)\|,$$

which, as follows from the construction, is a computable real number.

Lemma 1. *For all i, every ρ_i-ball is bounded in $(\mathbf{R}, \rho_R)$. More precisely, for all $x, y, \varepsilon \in \mathbf{R}$, $\rho_i(x, y) \leqslant \varepsilon$ only if $\min(x - \varepsilon, -\varepsilon) \leqslant y < \max(x + 1 + \varepsilon, D)$, where D is the right end of the rightmost interval J such that a mapping $\Gamma_{z,J}$ with $z < x + 1 + \varepsilon$ was defined in the construction of ρ_i (that is, at some stage s we put $\gamma_{i,s+1} = \gamma_{i,s} \uplus \Gamma_{z,J}$).*

Proof. Fix $i \in \omega$ and $x, y \in \mathbf{R}$. We have $\gamma_i(x) = (x', r_0, \theta_0), \gamma_i(y) = (y', r_1, \theta_1) \in \mathbf{R}^3$ and $\rho_i(x, y) = \|\gamma_i(x) - \gamma_i(y)\| \geqslant |x' - y'|$, so $\rho_i(x, y) \leqslant \varepsilon$ only if $|x' - y'| \leqslant \varepsilon$. We break the proof into the following cases.

(1) $x, y < 0$. Then $x' = x$ and $y' = y$ as the set of negative real numbers is not affected by the construction, so $|x' - y'| = |x - y| \leqslant \varepsilon$ if and only if $x - \varepsilon \leqslant y \leqslant x + \varepsilon$.

(2) $x < 0$ and $y \geqslant 0$. We need to take care of two following subcases.
   - (a) $y' \geqslant y$. Then $|x' - y'| = |x - y'| \geqslant |x - y|$ and thus $|x' - y'| \leqslant \varepsilon$ only if $|x - y| \leqslant \varepsilon$.
   - (b) $y' < y$. Note that $y' \neq y$ only when $\gamma_i(y) = \Gamma_{z,J}(y) = (y', r_1, \theta_1)$ for a mapping $\Gamma_{z,J}$ introduced in the construction and $y \in J$. By definition of $\Gamma_{z,J}$, $z \leqslant y'$, so $|x' - y'| = |x - y'| \leqslant \varepsilon$ only if $z \leqslant x + \varepsilon$.

(3) $x \geqslant 0$ and $y < 0$. We proceed in the same way as in the previous case. If $x' \geqslant x$, then $|x' - y'| = |x' - y| \leqslant \varepsilon$ only if $|x - y| \leqslant \varepsilon$. When $x' < x$, note that, because of the choice of followers $\mathbf{a}_k$, x' must be greater than or equal to zero, so in order to $|x' - y| \leqslant \varepsilon$ we must have $y \geqslant -\varepsilon$.

(4) $x, y \geqslant 0$. Again we split the proof into subcases.
   - (a) $x' \leqslant x$ and $y' \geqslant y$. Then $|x' - y'| \leqslant \varepsilon$ only if $y \leqslant y' \leqslant x' + \varepsilon \leqslant x + \varepsilon$.
   - (b) $x' \leqslant x$ and $y' < y$. Then $y \in J$ for an interval J such that a mapping $\Gamma_{z,J}$ was defined in the construction, and $|x' - y'| \leqslant \varepsilon$ only if $z \leqslant x' + \varepsilon \leqslant x + \varepsilon$.
   - (c) $x' > x$. In this case, obviously, x belongs to an interval J_* used in the construction; more precisely, $x \in J_*^4$ or $x \in J_*^5$, where $J_*^0, \ldots, J_*^5$ is the partition of J_* into 6 subintervals, each of length $1/3$. So we have $x' \leqslant x + 2/3 < x + 1$ and, as in two previous subcases, $|x' - y'| < \varepsilon$ only if $y \leqslant x' + \varepsilon < x + 1 + \varepsilon$ or $y \in J$, where a mapping $\Gamma_{z,J}$ was defined in the construction with $z \leqslant x' + \varepsilon < x + 1 + \varepsilon$.

Assembling these estimates together, we obtain the inequality from the statement of the lemma. Since there are only finitely many followers $z < x + 1 + \varepsilon$, the lemma is proved. □

Lemma 2. *All metrics ρ_i are complete and induce the standard topology on $\mathbf{R}$.*

Proof. Fix $i \in \omega$. We have already noted that the map $\gamma_i\colon \mathbf{R} \to \mathbf{R}^3$ is injective; it is easy to see that it is continuous. To show that ρ_i and ρ_R induce the same topology on $\mathbf{R}$, notice first that the identity mapping $\mathrm{id}\colon (\mathbf{R}, \rho_R) \to (\mathbf{R}, \rho_i)$ is continuous because γ_i is continuous, so any set open in $(\mathbf{R}, \rho_i)$ is open in $(\mathbf{R}, \rho_R)$. On the other hand, it is not hard to see that for any $x \in \mathbf{R}$ there is an $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ open ρ_i-ball $B(x, \varepsilon)$ is a real interval. Then any set open in $(\mathbf{R}, \rho_R)$ is open in $(\mathbf{R}, \rho_i)$, so the two topologies coincide.

It is left to show that ρ_i is complete. Let $(x_n)_{n\in\omega}$ be a Cauchy sequence in $(\mathbf{R}, \rho_i)$. Then there exists $\varepsilon > 0$ such that $\rho_i(x_0, x_n) < \varepsilon$ for all n. Since ρ_i-balls are bounded in $(\mathbf{R}, \rho_R)$, $(x_n)_{n\in\omega}$ has a subsequence converging in $(\mathbf{R}, \rho_R)$. By the above, this subsequence converges in $(\mathbf{R}, \rho_i)$, so $(x_n)_{n\in\omega}$ converges in $(\mathbf{R}, \rho_i)$ as well. □

Lemma 3. *$\rho_R \leq_c \rho_i$ for all i.*

Proof. It is clear that every mapping Γ_{J_m} defined in the construction is uniformly continuous. Moreover, modulus of continuity of Γ_{J_m} is computable uniformly in m. That is, there exists a computable function $\mathrm{mod}\colon \omega^2 \to \omega$ such that for each m the unary function $\mathrm{mod}_m(n) = \mathrm{mod}(m,n)$ is a modulus of continuity of Γ_{J_m}, if a map Γ_{J_m} was defined in the construction, and the unary identity function, otherwise.

Indeed, given m, we can effectively decide if interval J_m was used in the construction to define a mapping Γ_{J_m}: it can happen only before stage t at which the set $A_t \cap J_m$ becomes nonempty. If J_m was not used, let $\mathrm{mod}(m,n) = n$ for all n. Otherwise, if Γ_{J_m} has been defined, we can compute a, p and s such that $\Gamma_{J_m} = \Gamma_{q_a, J_m, 2^{-p+1}, s}$. Using the fact that Γ_{J_m} is piecewise linear, we can let

$$\mathrm{mod}(m,n) = \max_{k=0,\dots,5} \mathrm{mod}(m,n,k),$$

where $\mathrm{mod}(m,n,k)$ is a modulus of continuity of the linear function $\Gamma_{J_m} \restriction J_m^k$ that can be computed directly, using q_a, J_m and 2^{-p+1}.

We build a Turing functional Φ reducing ρ_R to all metrics ρ_i at once. Let $f \in \omega^\omega$ be a ρ_R-name for $x \in \mathbf{R}$. We want to translate f into a ρ_i-name $\Phi(f)$ for x. We can have two different situations:

(1) $\rho_R(q_{f(0)}, J_m) < 1$ for an interval J_m used in the construction to define a mapping Γ_{J_m}. Then by definition of a modulus of continuity and convention $\mathrm{mod}_m(n) \geqslant n$ it is easy to see that $f \circ \mathrm{mod}_m$ is a ρ_i-name for x for all $i \in \omega$.

(2) Otherwise. In this case we can be sure that all elements $q_{f(n)}$ belong to an area in $\mathbf{R}$ not deformated by the construction, i. e., $\gamma_i(q_{f(n)}) = (q_{f(n)}, 0, 0)$ for all i. Thus $\forall n, l\ \big(\rho_i(q_{f(n)}, q_{f(l)}) = \rho_R(q_{f(n)}, q_{f(l)})\big)$, so f is also a ρ_i-name for x for all i.

The functional $\Phi\colon \omega^\omega \to \omega^\omega$ can be defined as

$$\Phi(f) = \begin{cases} f \circ \mathrm{mod}_m, & \text{if there is an } m \text{ such that } \rho_R(q_{f(0)}, J_m) < 1, \\ f, & \text{otherwise.} \end{cases}$$

Clearly, Φ is a Turing functional, and if $\rho_R(f) = x$, then $\rho_i(\Phi(f)) = x$ for all i. □

Lemma 4. *Any requirement is initialized only finitely many times.*

Proof. Consider by induction the stage after which requirement $\mathcal{R}_n$ is never initialized. After this stage it can act and initialize $\mathcal{R}_{n+1}$ at most twice. □

Lemma 5. *If $i \neq j$, then $\rho_i \nleq_{ch} \rho_j$.*

Proof. Suppose that $\rho_i \leq_{ch} \rho_j$ via Φ_e. First observe that, for each $a \in \omega$, $\Phi_e(\bar{a})(0)\downarrow$. We know that ρ_j-balls are bounded w. r. t. the standard metric. Since Φ_e computes an autohomeomorphism F of $\mathbf{R}$, we claim that Φ_e is *unbounded*, that is,

$$\forall b_0\, \exists b_1\, \forall b\, \big((q_{b_0} > 0 \,\&\, q_b > q_{b_1}) \Rightarrow\ q_{\Phi_e(\bar{b})(0)} \notin [-q_{b_0}, q_{b_0}]\big).$$

Indeed, fix b_0 and consider the closed real interval $[-q_{b_0}, q_{b_0}]$. Being compact, it is contained in some ρ_j-ball $B(x,\varepsilon)$. Consider the closed ball $B = B[x, 1+\varepsilon]$. By Lemmas 1 and 2 it is compact, so $F^{-1}(B)$ is compact as well. Take an upper bound q_{b_1} for the set $F^{-1}(B)$, then $F(q_b) \notin B$ for all $q_b > q_{b_1}$, so for all $y \in [-q_{b_0}, q_{b_0}]$

$$\rho_j(F(q_b), y) \geqslant |\rho_j(F(q_b), x) - \rho_j(x,y)| > 1 + \varepsilon - \varepsilon = 1,$$
$$\rho_j(y, q_{\Phi_e(\bar{b})(0)}) \geqslant |\rho_j(y, F(q_b)) - \rho_j(F(q_b), q_{\Phi_e(\bar{b})(0)})| > 1 - 1 = 0,$$

which means that $q_{\Phi_e(\bar{b})(0)} \neq y$, and our claim is proved.

By the previous lemma, there is a stage s after which requirement $\mathcal{R}_n = \mathcal{R}_{ie}$ is never initialized and has a permanent first follower q_a. Let $s_0 > s$ be the stage at which $\mathcal{R}_n$ acts at substage 0 and appoints the second follower q_b. $\mathcal{R}_n$-strategy only visits substage 1 after stage s_0, so the values C_n and D_n remain constant, starting from this stage. Because Φ_e is unbounded, at some stage $s_1 \geqslant s_0$ requirement $\mathcal{R}_n$ will have a second follower big enough to act at substage 1, defining a map $\Gamma_{q_a,J}$. Let q_d be the midpoint of J. From the definition of $\Gamma_{q_a,J}$ we have $\rho_i(q_a, q_d) = 2^{-p+1}$ where $p = \varphi_{e,s_1}(\bar{a})(0)$, so $a^p\bar{d}$ is a ρ_i-name for q_d such that $\Phi_e(a^p\bar{d})(0) = \Phi_e(\bar{a})(0) = u$. It is left to show that $\Phi_e(a^p\bar{d})$ is not a ρ_j-name for $F(q_d)$, that is, $\rho_j(q_u, F(q_d)) > 1$.

Observe that after stage s_0 no requirement apart from $\mathcal{R}_{ie}$ will be able to define a mapping Γ_z with $z < D_n$: all stronger priority requirements don't act after stage s and at stage s_0 we initialize all weaker priority requirements, forcing them to choose fresh first followers larger than D_n. From this and Lemma 1 we conclude that for all $j \neq i$

$$\rho_j(x, q_u) \leqslant 1\ \Rightarrow\ C_n + 1 \leqslant x < D_n - 1.$$

At stage s_1 we have $\Phi_{e,s_1}(\bar{b})(0)\downarrow= v$, $q_v \notin [C_n, D_n]$. Since $\rho_j(q_v, F(q_b)) \leqslant 1$ and no requirement can define a mapping Γ_z with $z < D_n$, it is easy to see that $F(q_b) \notin [C_n+1, D_n-1]$. Since $q_a < q_b < q_d$ and F is monotone, we obtain $F(q_d) \notin [C_n+1, D_n-1]$, so $\rho_j(F(q_d), q_u) > 1$. □

3. Generalization and further results

3.1. **Generalized construction.** Fix a set $A \subseteq \omega$. Define a mapping $\gamma_A\colon \mathbf{R} \to \mathbf{R}^3$ in stages. Let $\gamma_{A,0}(x) = (x,0,0)$. At stage $s+1$ of the construction from Theorem 1, if we let $\gamma_{i,s+1} = \gamma_{i,s} \uplus \Gamma$ for some $i \in A$, also let $\gamma_{A,s+1} = \gamma_{A,s} \uplus \Gamma$, otherwise let $\gamma_{A,s+1} = \gamma_{A,s}$.

In other words, γ_A accumulates all the maps Γ defined in the constructions of ρ_i for all $i \in A$:

$$\gamma_A(x) = \begin{cases} \Gamma_{J_m}(x), & \text{if } x \in J_m, \text{ where mapping } \Gamma_{J_m} \text{ was defined in the} \\ & \text{construction of } \rho_i \text{ for some } i \in A, \\ (x,0,0), & \text{if } x \text{ does not belong to any of intervals } J_m \text{ used} \\ & \text{in the construction of } \rho_i \text{ for all } i \in A. \end{cases}$$

Define a metric $\rho_A(x,y) = \|\gamma_A(x) - \gamma_A(y)\|$ on $\mathbf{R}$.

Lemma 6. *Metric ρ_A is computable if and only if the set A is computable.*

Proof. "If" direction is trivial. For the "only if" direction, suppose that ρ_A is a computable metric. We want to show that A is computable. Fix an e such that $\Phi_e = \mathrm{id}_{\omega^\omega}$. In particular, Φ_e is a (ρ_R, ρ_R)-realization of $\mathrm{id}_{\mathbf{R}}$. Using this and the proof of Lemma 5 (Φ_e is unbounded), we see that for each i there is a stage s of the construction from Theorem 1 at which the requirement $\mathcal{R}_{ie}$ acts at substage 1, defining a mapping $\Gamma_{q_a,J}$. In order to find out whether $i \in A$, wait for this stage s. Let q_d be the midpoint of interval J. Now, $i \in A$ if and only if $\rho_A(q_a, q_d) < 2$. □

It is easy to see that Lemmas 1 and 2 are also valid for all metrics ρ_A, $A \subseteq \omega$. If $A = \omega - \{i\}$, for convenience we will write $\rho_{\neq i}$ instead of $\rho_{\omega-\{i\}}$. Note that the proof of Lemma 5 in fact gives us that $\rho_i \not\leq_{ch} \rho_{\neq i}$, as shown below.

Lemma 7. *For all i, $\rho_i \not\leq_{ch} \rho_{\neq i}$.*

Proof. We use notations from Lemma 5. If Φ_e ch-reduces ρ_i to $\rho_{\neq i}$, then Φ_e is unbounded. Then $\mathcal{R}_n$ will act at stages s_0 and s_1 and so will be satisfied. Restraint imposed on interval $[C_n, D_n]$ after stage s_0 is uniform for the entire construction, so the remaining part of the proof does not depend on j and is valid with $\rho_{\neq i}$ in place of ρ_j. Thus, $\rho_{\neq i}(q_u, F(q_d)) > 1$ and the name $a^p\bar{d}$ witnesses the failure of reduction $\rho_i \leq_{ch} \rho_{\neq i}$ by Φ_e. □

Lemma 8. *If $A \subseteq B$, then $\rho_A \leq_c \rho_B$.*

Proof. Fix sets $A \subseteq B$. Let f be a ρ_A-name for $x \in \mathbf{R}$. We will try to predict positions of all elements $q_{f(n)}$ based on the position of the initial element $q_{f(0)}$ and obtain from it a ρ_B-name for x. From the definition of a Cauchy name, choice of followers $\mathbf{a}_k$ and intervals J_m and definition of $\Gamma_{\mathbf{a}_k,J_m}$ (see Fig. 1) it can be seen that one and only one of the following possibilities holds:

(1) $\rho_R(q_{f(0)}, J_m) < 1$ for an interval J_m used in the construction of ρ_A to define a mapping $\Gamma_{\mathbf{a}_k,J_m}$ and:
   - (a) $q_{f(0)} \in J_m$ and there are $n \in \omega$ and $p \neq m$ such that $q_{f(n)} \in J_p$. This situation happens only when J_p was used in the construction of ρ_A to define a mapping Γ_{J_p} and the angle between planes containing images of $\Gamma_{\mathbf{a}_k,J_m}$ and Γ_{J_p} is small enough to let $q_{f(n)}$ "jump away" to J_p (see Fig. 4).
   - (b) $q_{f(0)} \in J_m$ and there is $n \in \omega$ such that $\rho_R(q_{f(n)}, \mathbf{a}_k) < 1.25$, i. e., $q_{f(n)}$ "jumps off" to $\mathbf{a}_k$.
   - (c) Elements $q_{f(n)}$ do not "jump away" from J_m, i. e., $\rho_R(q_{f(n)}, J_m) < 2$ for all n.

   Since $A \subseteq B$ so every mapping Γ defined in the construction of ρ_A is also defined in the construction of ρ_B, in all these subcases we can be sure that $\gamma_A(q_{f(n)}) = \gamma_B(q_{f(n)})$ for all n, so f is also a ρ_B-name for x.

(2) $\rho_R(q_{f(0)}, \mathbf{a}_k) < 1.25$ for a follower $\mathbf{a}_k$ used in the construction of ρ_A to define a mapping $\Gamma_{\mathbf{a}_k,J_m}$. In this case, like above, it is possible that some elements of the sequence $(q_{f(n)})_{n\in\omega}$ can jump to J_m, but in the end we know that $\gamma_A(q_{f(n)}) = \gamma_B(q_{f(n)})$ for all n, so f is also a ρ_B-name for x.

(3) Otherwise, i. e., $\rho_R(q_{f(0)}, \mathbf{a}_k) \geqslant 1.25$ and $\rho_R(q_{f(0)}, J_m) \geqslant 1$ for all followers $\mathbf{a}_k$ and intervals J_m used in the construction of ρ_A. Then all elements $q_{f(n)}$ belong to an area in $\mathbf{R}$ not deformated by the construction of ρ_A, that is,

$\gamma_A(q_{f(n)}) = (q_{f(n)}, 0, 0)$, so f is a ρ_R-name. As in Lemma 3, we can obtain a ρ_B-name for x via the function $\mathrm{mod}(m, n)$.

Now it is not hard to see that the functional Φ from the proof of Lemma 3 reduces ρ_A to ρ_B. Here we again use the convention $\mathrm{mod}_m(n) \geqslant n$ so whenever $\rho_B(f) = x$, we have $\rho_B(f \circ \mathrm{mod}_m) = x$ as well. □

Recall [3] that a sequence of sets $A_i \subseteq \omega$ is called *computably independent* if, for every i, $A_i \not\leq_T \bigoplus_{j \neq i} A_j$. Existence of a computably independent sequence of sets leads to the fact that any countable partial ordering can be embedded into the Turing degrees. By the Friedberg-Muchnik construction one can obtain a computably independent sequence of c. e. sets and embed any countable partial ordering into the c. e. Turing degrees. Metrics ρ_i from Theorem 1 exhibit a property similar to computable independence in the sense that, for all $i \neq j$, $\rho_j \leq_{ch} \rho_{\neq i}$ and $\rho_i \not\leq_{ch} \rho_{\neq i}$. Using this, we will prove that any countable partial ordering can be embedded into ch-degrees of computable metrics, as stated in Theorem 2 below.

Lemma 9. *If $A \not\subseteq B$, then $\rho_A \not\leq_{ch} \rho_B$.*

Proof. Take $i \in A - B$. Suppose that $\rho_A \leq_{ch} \rho_B$. Then $\rho_i \leq_c \rho_A \leq_{ch} \rho_B \leq_c \rho_{\neq i}$, which is a contradiction. □

Theorem 2. *The following hold:*

(1) *The ordering $(P(\omega), \subseteq)$ of subsets of ω is embeddable into the ordering of ch-degrees of real metrics above ρ_R.*
(2) *There are exactly $2^{\aleph_0}$ ch-degrees of metrics.*
(3) *There are exactly $2^{\aleph_0}$ c-degrees of metrics.*
(4) *Any countable partial ordering is embeddable into ch-degrees of computable metrics above ρ_R.*

Proof. (1) follows from Lemmas 8 and 9.

(2): By (1), there are at least $2^{\aleph_0}$ ch-degrees of metrics. On the other hand, there are exactly continuum many continuous real functions of two variables, in particular, there are exactly continuum many real metrics.

(3): Any c-degree is contained in a ch-degree, so there are at least $2^{\aleph_0}$ c-degrees of metrics.

(4): Consider the computable countably-universal partial order $\mathcal{P} = (\omega, \leqslant_P)$ [12]. Since $\mathcal{P}$ is computable, the sets $A_i = \{k \in \omega \mid k \leqslant_P i\}$ are uniformly computable, and the metrics ρ_{A_i} are computable. Then for all $i, j \in \omega$

$$i \leqslant_P j \;\Leftrightarrow\; A_i \subseteq A_j \;\Leftrightarrow\; \rho_{A_i} \leq_{ch} \rho_{A_j}.$$ □

3.2. **Embeddings of lattices into the degree structure.** Another property of our generalized construction is that the mapping $A \mapsto \rho_A$ preserves greatest lower and least upper bounds of A and B in the ordering $(P(\omega), \subseteq)$, for A and B computable sets; that is, there is an isomorphic embedding of the lattice of computable subsets of ω into the ordering of c-degrees of metrics. More formally, the following result holds.

Lemma 10. *If $A, B \subseteq \omega$ are computable, then $\deg_c(\rho_{A\cup B}) = \deg_c(\rho_A) \vee \deg_c(\rho_B)$ and $\deg_c(\rho_{A\cap B}) = \deg_c(\rho_A) \wedge \deg_c(\rho_B)$ in the lattice of c-degrees of representations of $\mathbf{R}$. (Recall that by abuse of notation ρ denotes both the metric and the corresponding Cauchy representation)*

Proof. Obviously, $\rho_{A\cap B} \leq_c \rho_A \leq_c \rho_{A\cup B}$ (and the same for ρ_B).

We will now show that c-degree of $\rho_{A\cup B}$ is the least upper bound of the degrees of ρ_A and ρ_B. For this purpose, suppose that δ is a representation of $\mathbf{R}$ and Φ_e and Φ_z are functionals c-reducing ρ_A and ρ_B to δ, respectively. We want to show that $\rho_{A\cup B} \leq_c \delta$. Suppose that f is a $\rho_{A\cup B}$-name for x. We prove that one can effectively translate f into a ρ_A-name or a ρ_B-name and then, using Φ_e or Φ_z, obtain a δ-name for x.

As in the proof of Lemma 8, one and only one of the following possibilities holds:

(1) $\rho_R(q_{f(0)}, J_m) < 1$ for an interval J_m used in the construction of $\rho_{A\cup B}$ to define a mapping $\Gamma_{\mathbf{a}_k,J_m}$ and:
  (a) $q_{f(0)} \in J_m$ and there are $n \in \omega$ and $p \neq m$ such that $q_{f(n)} \in J_p$.
  (b) $q_{f(0)} \in J_m$ and there is $n \in \omega$ such that $\rho_R(q_{f(n)}, \mathbf{a}_k) < 1.25$.
  (c) $\rho_R(q_{f(n)}, J_m) < 2$ for all n.

(2) $\rho_R(q_{f(0)}, \mathbf{a}_k) < 1.25$ for a follower $\mathbf{a}_k$ used in the construction of $\rho_{A\cup B}$ to define a mapping $\Gamma_{\mathbf{a}_k,J_m}$.

(3) $\rho_R(q_{f(0)}, \mathbf{a}_k) \geqslant 1.25$ and $\rho_R(q_{f(0)}, J_m) \geqslant 1$ for all followers $\mathbf{a}_k$ and intervals J_m used in the construction of $\rho_{A\cup B}$.

Take a closer look at the first case. Since $\rho_{A\cup B}$ induces the standard topology on $\mathbf{R}$ and $(q_{f(n)})_{n\in\omega}$ is a convergent sequence, it is clear that jumps between intervals (possibility (1a)) eventually have to stop. That is,

$$\exists N, p\, \forall n, l \ \big(n \geqslant N \ \&\ l \neq p \Rightarrow q_{f(n)} \notin J_l\big).$$

These N and p can be found effectively as follows. Consider the stage s such that $\Gamma_{\mathbf{a}_k,J_m} = \Gamma_{\mathbf{a}_k,J_m,s}$. Recall that the angle between planes containing images of $\Gamma_{\mathbf{a}_k,J_m,s}$ and $\Gamma_{J_l,t}$ is $\alpha_{st} = |\frac{t}{t+1} - \frac{s}{s+1}|\pi$, and for fixed s its value is minimal when $t = s+1$. Then it is clear that

$$\rho_{A\cup B}(J_m, J_l) = \inf_{x\in J_m, y\in J_l} \|\Gamma_{\mathbf{a}_k,J_m,s}(x) - \Gamma_{J_l,t}(y)\| \geqslant M_s = 2\sin\big(\tfrac{s+1}{s+2} - \tfrac{s}{s+1}\big)\pi.$$

Fix an N_0 such that $2^{-N_0} < M_s$. If $q_{f(N_0)} \in J_m$, then also $q_{f(n)} \in J_m$ for all $n > N_0$ by the above, and our algorithm ends with $N = N_0$ and $p = m$. If $q_{f(N_0)} \in J_l$ with $l \neq m$, there is a t such that a map $\Gamma_{J_l,t}$ was defined in the construction, and we only need to repeat the process for J_l, i.e. find $N_1 > N_0$ such that $2^{-N_1} < M_t$, then for $n > N_1$ the elements $q_{f(n)}$ will not be able to jump away from J_l provided that $q_{f(N_1)} \in J_l$, and so on. This process is finite, and in the end we obtain the needed N and p.

Interval J_p must have been used in the construction to define a map $\Gamma_{\mathbf{a}_{k'},J_p}$. Using the computability of A and B, we can tell whether $\Gamma_{\mathbf{a}_{k'},J_p}$ was defined in the construction of ρ_A or ρ_B. In the first case, for all $n \geqslant N$ we have $\gamma_{A\cup B}(q_{f(n)}) = \gamma_A(q_{f(n)})$, thus $\rho_A(f \restriction N) = x$, and $\delta(\Phi_e(f \restriction N)) = x$. In the second case, $\delta(\Phi_z(f \restriction N)) = x$.

In case (2), similarly, we can tell whether $\Gamma_{\mathbf{a}_k,J_m}$ was defined in the construction of ρ_A or ρ_B. So $\rho_A(f) = x$ or $\rho_B(f) = x$, respectively, and $\delta(\Phi_e(f)) = x$ or $\delta(\Phi_z(f)) = x$, respectively.

In case (3), $\rho_A(f) = \rho_B(f) = x$ so $\delta(\Phi_e(f)) = x$.

We have proved that any $\rho_{A\cup B}$-name can be translated into a δ-name for the same element via an effective procedure, so $\rho_{A\cup B} \leq_c \delta$.

In a similar fashion one can prove that c-degree of $\rho_{A\cap B}$ is the greatest lower bound of c-degrees of ρ_A and ρ_B. Suppose that δ is a representation of $\mathbf{R}$ that is

c-reduced to ρ_A and ρ_B by functionals Φ_e and Φ_z, respectively. Given a δ-name for x, we can use the functionals Φ_e and Φ_z to obtain a ρ_A-name f and a ρ_B-name g for x. We show that from these names one can effectively construct a $\rho_{A\cap B}$-name for x. Consider the following situations:

(1) $\rho_R(q_{f(0)}, J_m) < 1$ where J_m was used in the construction of ρ_A. Then we can wait for the end of jumps described above for some $N, p \in \omega$. Obviously, J_p must have been used in the construction of ρ_A to define a map $\Gamma_{\mathbf{a}_{k'}, J_p}$. There are two possible subcases:
   (a) $\Gamma_{\mathbf{a}_{k'}, J_p}$ was also defined in the construction of ρ_B. Then $f \restriction N$ is a $\rho_{A\cap B}$-name for x.
   (b) $\Gamma_{\mathbf{a}_{k'}, J_p}$ was not defined in the construction of ρ_B. In this case, note that x belongs to an area not used in the construction of ρ_B and g is both a ρ_B-name and a $\rho_{A\cap B}$-name for x.

(2) $\rho_R(q_{f(0)}, \mathbf{a}_k) < 1.25$ where $\mathbf{a}_k$ was used in the construction of ρ_A. There also are two subcases:
   (a) $\mathbf{a}_k$ was used in the construction of ρ_B. Then f is a $\rho_{A\cap B}$-name for x.
   (b) $\mathbf{a}_k$ was not used in the construction of ρ_B. Again we can deduce that x belongs to an area not used in the construction of ρ_B and g is a $\rho_{A\cap B}$-name for x.

(3) $\rho_R(q_{f(0)}, \mathbf{a}_k) \geqslant 1.25$ and $\rho_R(q_{f(0)}, J_m) \geqslant 1$ for all followers $\mathbf{a}_k$ and intervals J_m used in the construction of $\rho_{A\cup B}$. In this case, f is both a ρ_A-name and a $\rho_{A\cap B}$-name for x.

From this we conclude that a $\rho_{A\cap B}$-name for x can be obtained effectively from any δ-name, thus $\delta \leq_c \rho_{A\cap B}$, and the lemma is proved. □

As an immediate corollary, we are able to embed the countable atomless Boolean algebra into c-degrees of computable metrics preserving joins and meets, similarly to Binns and Simpson's result [13] for Muchnik degrees. Recall that the countable atomless Boolean algebra can be represented as the interval algebra $\mathrm{Int}(1+\eta)$, where η is the order type of the rational numbers (see [14]).

Theorem 3. *The following lattices are embeddable with preservation of joins and meets into the ordering of c-degrees of computable metrics above ρ_R:*

(1) *Boolean algebra of computable subsets of ω;*
(2) $\mathrm{Int}(1+\eta)$*, the countable atomless Boolean algebra;*
(3) *Any countable distributive lattice.*

Proof. (1): follows from the previous lemma.

(2): $\mathrm{Int}(1+\eta)$ is isomorphically embeddable into the Boolean algebra of computable subsets of ω (folklore; see also [13, Theorem 4.7]).

(3): any countable distributive lattice is embeddable into $\mathrm{Int}(1+\eta)$, see e. g. [13, Lemma 4.10]. □

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Ruslan Aleksandrovich Kornev
Novosibirsk State University,
1, Pirogova str.,
Novosibirsk, 630090, Russia
Email address: kornevrus@gmail.com