S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 18, №1, стр. 456–463 (2021) УДК 519.172.2
DOI 10.33048/semi.2021.18.031 MSC 05C75

ALL TIGHT DESCRIPTIONS OF MAJOR 3-PATHS IN 3-POLYTOPES WITHOUT 3-VERTICES

TS.CH-D. BATUEVA, O.V. BORODIN, A.O. IVANOVA, D.V. NIKIFOROV

A 3-path uvw is an (i, j, k)-path if $d(u) \leq i$, $d(v) \leq j$, and $d(w) \leq k$, where $d(x)$ is the degree of a vertex x. It is well-known that each 3-polytope has a vertex of degree at most 5, called minor. A description of 3-paths in a 3-polytope is minor or major if the central item of each its triplet is at most 5 or at least 6, respectively. Back in 1922, Franklin proved that each 3-polytope with minimum degree 5 has a $(6, 5, 6)$-path, which description is tight. Recently, Borodin and Ivanova extended Franklin's theorem by producing all the ten tight minor descriptions of 3-paths in the class $\mathbf{P_4}$ of 3-polytopes with minimum degree at least 4. In 2016, Borodin and Ivanova proved that each polytope with minimum degree 5 has a $(5, 6, 6)$-path, and there exists no tight description of 3-paths in this class of 3-polytopes other than $\{(6, 5, 6)\}$ and $\{(5, 6, 6)\}$.

The purpose of this paper is to prove that there exist precisely the following four major tight descriptions of 3-paths in $\mathbf{P_4}$: $\{(4, 9, 4), (4, 7, 5), (5, 6, 6)\}$, $\{(4, 9, 4), (5, 7, 6)\}$, $\{(4, 9, 5), (5, 6, 6)\}$, and $\{(5, 9, 6)\}$.

Keywords: plane graph, 3-polytope, structural properties, 3-path, tight description.

1. Introduction

The *degree* $d(x)$ of a vertex or face x in a plane graph G is the number of its incident edges. A *k-vertex* (*k-face*) is a vertex (face) with degree k, a k^+*-vertex* has degree at least k, etc. The minimum vertex degree of G is $\delta(G)$. We will drop the arguments whenever this does not lead to confusion.

Batueva, Ts.Ch-D., Borodin, O.V., Ivanova, A.O., Nikiforov, D.V., All tight descriptions of major 3-paths in 3-polytopes without 3-vertices.

This research was funded by the Russian Science Foundation (grant 16-11-10054).
Received March, 25, 2021, published April, 22, 2021.

A *k-path* is a path on k vertices. A path uvw is an (i, j, k)*-path* if $d(u) \leq i$, $d(v) \leq j$, and $d(w) \leq k$. The *weight* $w(H)$ of a subgraph H of a graph G is the degree-sum of the vertices of H in G. By $\mathbf{P}_\delta$ denote the class of 3-polytopes with minimum degree δ; in particular, $\mathbf{P_3}$ is the set of all 3-polytopes. It is well-known that each 3-polytope has a vertex of degree at most 5, called *minor*.

In 1904, Wernicke [22] proved that if $P_5 \in \mathbf{P_5}$ then P_5 contains a 5-vertex adjacent to a 6^--vertex. This result was strengthened by Franklin [14] in 1922 by proving the existence of a $(6, 5, 6)$-path in every P_5.

We recall that a description of 3-paths is *tight* if none of its parameters can be strengthened and no term dropped. The tightness of Franklin's description is shown by putting a vertex inside each face of the dodecahedron and joining it to the five boundary vertices.

Franklin's Theorem [14] is fundamental in the structural theory of planar graphs; it has been extended or refined in several directions, see, for example, [1–7, 9, 12, 13, 15, 16, 18–21] and surveys Jendrol'–Voss [17] and Borodin–Ivanova [8].

We now mention only a few easily formulated results on $\mathbf{P_5}$, which are the closest to Franklin's Theorem and whose parameters are all sharp. Borodin [3] proved that there is a 3-face with weight at most 17. Jendrol' and Madaras [16] ensured a 5-vertex that has three neighbors whose weight sums to at most 18 and a 4-path with weight at most 23. For triangulations in $\mathbf{P_5}$, Madaras [19] found a 5-path with weight at most 29, the same upper bound was proved (but never published) in MSc thesis of Z. Micova (formerly P.J. Safarik University) in 2003.

In 2014, Borodin and Ivanova proved [6] that there exist precisely seven tight descriptions of 3-paths in triangle-free 3-polytopes: $\{(5, 3, 6), (4, 3, 7)\}$, $\{(3, 5, 3), (3, 4, 4)\}$, $\{(5, 3, 6), (3, 4, 3)\}$, $\{(3, 5, 3), (4, 3, 4)\}$, $\{(5, 3, 7)\}$, $\{(3, 5, 4)\}$, $\{(5, 4, 6)\}$, which was a result of a new type in the structural theory of plane graphs.

A description of 3-paths in a 3-polytope is *minor* or *major* if the central item of each its triplet is at most 5 or at least 6, respectively.

Recently, Borodin and Ivanova [10] extended Franklin' Theorem to P_4 by proving that there exist precisely the following ten tight minor descriptions of 3-paths:
$\{(6, 5, 6), (4, 4, 9), (6, 4, 8), (7, 4, 7)\}$, $\{(6, 5, 6), (4, 4, 9), (7, 4, 8)\}$,
$\{(6, 5, 6), (6, 4, 9), (7, 4, 7)\}$, $\{(6, 5, 6), (7, 4, 9)\}$, $\{(6, 5, 8), (4, 4, 9), (7, 4, 7)\}$,
$\{(6, 5, 9), (7, 4, 7)\}$, $\{(7, 5, 7), (4, 4, 9), (6, 4, 8)\}$,
$\{(7, 5, 7), (6, 4, 9)\}$, $\{(7, 5, 8), (4, 4, 9)\}$, and $\{(7, 5, 9)\}$.

Back in 1996, Jendrol' [15] gave the following description of 3-paths in $\mathbf{P_3}$: $\{(10, 3, 10), (7, 4, 7), (6, 5, 6), (3, 4, 15), (3, 6, 11), (3, 8, 5), (3, 10, 3), (4, 4, 11), (4, 5, 7), (4, 7, 5)\}$.

The first tight description of 3-paths in $\mathbf{P_3}$ was obtained in 2013 by Borodin et al. [12]: $\{(3, 4, 11), (3, 7, 5), (3, 10, 4), (3, 15, 3), (4, 4, 9), (6, 4, 8), (7, 4, 7), (6, 5, 6)\}$.

Another tight description was given by Borodin, Ivanova and Kostochka [13]: $\{(3, 15, 3), (3, 10, 4), (3, 8, 5), (4, 7, 4), (5, 5, 7), (6, 5, 6), (3, 4, 11), (4, 4, 9), (6, 4, 7)\}$. Also, it is shown in [13] that there exist precisely three one-term descriptions: $\{(5, 15, 6)\}$, $\{(5, 10, 15)\}$, and $(10, 5, 10)$, two of which are major. Recently, the third major tight description for $\mathbf{P_3}$ was found in Borodin-Ivanova [11]:
$\{(3, 18, 3), (3, 11, 4), (3, 8, 5), (3, 7, 6), (4, 9, 4), (4, 7, 5), (5, 6, 6)\}$.

In 2016, it was proved in Borodin-Ivanova [7] that $\mathbf{P_5}$ allows precisely one tight description of 3-paths in addition to Franklin's Theorem, namely $\{(5, 6, 6)\}$.

The purpose of this paper is to extend theresults in [7] from $\mathbf{P_5}$ to $\mathbf{P_4}$ as follows.

Theorem 1. *There exist precisely the following four tight major descriptions of 3-paths in* $\mathbf{P_4}$*:*
$(td1)$: $\{(4,9,4),(4,7,5),(5,6,6)\}$,
$(td2)$: $\{(4,9,4),(5,7,6)\}$,
$(td3)$: $\{(4,9,5),(5,6,6)\}$, *and*
$(td4)$: $\{(5,9,6)\}$.

The problem posed in [13] of describing alltight descriptions of 3-paths in $\mathbf{P_3}$ is widely open. Even for $\mathbf{P_4}$, we have found already almost 40 tight descriptions of 3-paths with no limitations on the central items of triplets, but this list still seems to be far from being complete.

2. Proving Theorem 1

Figures 1–3 show 3-polytopes H_1–H_3 important for the proof; here, H_1 and H_3 are derived from the well-known Platonic and Archimedean solids. Concerning Fig. 2 from [11], it should be noted that the actual 3-polytope H_2 is formed by two identical copies of the depicted configuration by shifted gluing on their outer 6-faces (in the way that six 8-vertices are formed).

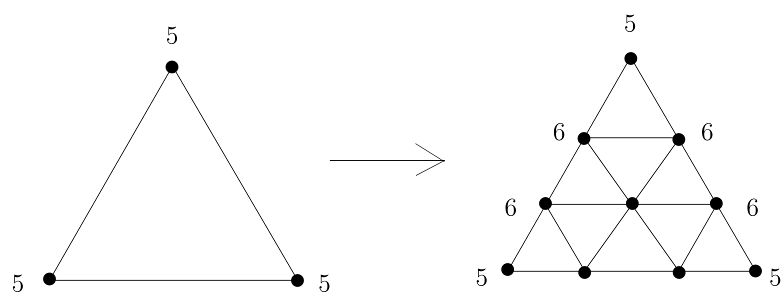

FIG. 1. 3-polytope H_1 having only $(5,6,6)$-paths.

2.1. **Proving that** $(td1)$**:** $\{(4,9,4),(4,7,5),(5,6,6)\}$ **is a description of** 3**-paths in** $\mathbf{P_4}$**.** Suppose that P' is a counterexample, so it does not obey $(td1)$.

2.1.1. *Constructing a triangular counterexample to description* $(td1)$. Let P be a counterexample on $V(P')$ with the maximum number of edges. From now on, by $d(v)$ of $v \in V(P)$ we mean the degree of v in P.

We abbreviate the clause "since P does not contain a path xyz such that $d(x) \le i$, $d(y) \le j$ and $d(z) \le k$" to "by non-(i,j,k)!".

We now prove that P is a triangulation. Suppose there is a 4^+-face $f = abc\dots$ in P. Let a be a vertex of maximum degree among all vertices incident with f. It suffices to prove that $P + ac$ is also a counterexample to Theorem 1 (recall that a, c are not adjacent outside, for otherwise $\{a, c\}$ would be a 2-cut in P).

First observe that $d(a) \ge 6$, for otherwise P would contain a $(5,5,5)$-path abc contrary to the absence of $(5,6,6)$-paths by assumption.

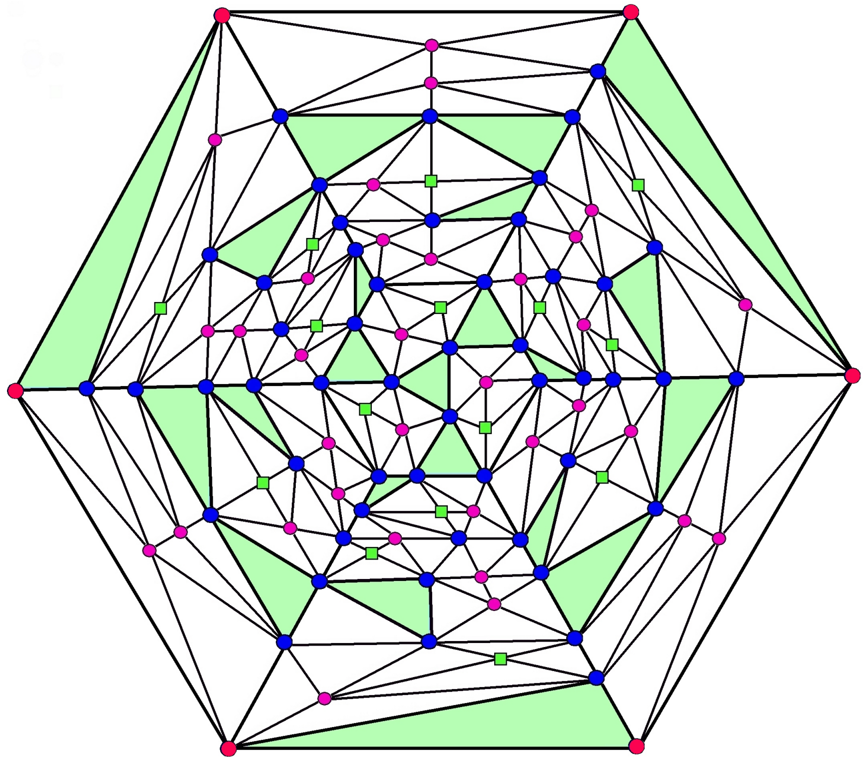

FIG. 2. 3-polytope H_2 with only $(4, 7, 5)$-paths [11].

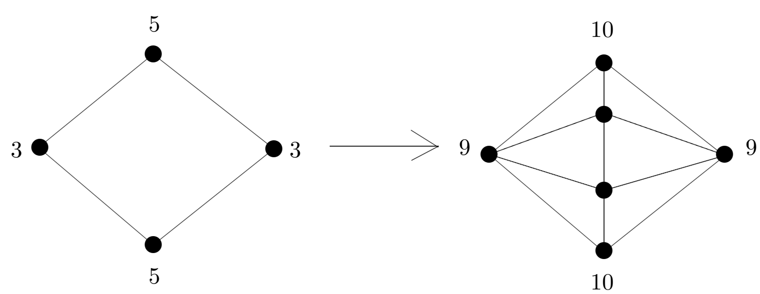

FIG. 3. 3-polytope H_3 having only $(4, 9, 4)$-paths

Now note that $P + ac$ cannot have a forbidden path avoiding the edge ac since the degree of each vertex in $P + ac$ is not smaller than in the counterexample P.

So suppose $P+ac$ has a 3-path acz or yac forbidden by $(td1)$. Since a and c have degrees 7^+ and 5^+ in $P + ac$, respectively, the only danger is to create a $(4, 7, 5)$-path in $P + ac$. However, this could happen only if $d(a) = 6$ and $d(c) = 4$, and then we have a $(5, 6, 6)$-path abc already in P since $d(b) \leq d(a)$ by assumption, a contradiction.

2.1.2. *Discharging.* Euler's formula $|V|-|E|+|F|=2$ for P may be written as

$$\sum_{v\in V}(d(v)-6)=-12. \tag{1}$$

Every vertex v contributes the *charge* $\mu(v)=d(v)-6$ to (1), so only the charges of 5^--vertices are negative. Using the properties of M as a counterexample, we define a local redistribution of μ's, preserving their sum, such that the *new charge* $\mu'(v)$ is non-negative for all $v\in V$. This will contradict the fact that the sum of the new charges is, by (1), equals to -12.

Throughout the paper, we denote the vertices adjacent to a vertex v in a cyclic order by $v_1,\dots,v_{d(v)}$.

Our rules of discharging are as follows:

R1. *Every 6-vertex gives $\frac{1}{2}$ to an adjacent 5^--vertex.*

R2. *Every 7^+-vertex v gives $\frac{2}{3}$ to an adjacent 4-vertex.*

R3. *Every 7^+-vertex gives $\frac{1}{4}$ to an adjacent 5-vertex.*

R4. *Every 7^+-vertex v gives $\frac{1}{6}$ to a 6-vertex v_2 if $d(v_1)\geq 7$ and $d(v_3)\geq 7$.*

2.1.3. *Checking $\mu'(v)\geq 0$ whenever $v\in V$.*

CASE 1. $d(v)=4$. If v has a 5^--neighbor, then it has three 7^+-neighbors by non-$(5,4,6)$!, so we have $\mu'(v)=4-6+3\times\frac{2}{3}=0$ by R2.

Further suppose v is completely surrounded by 6^+-vertices. Note that if v has a 6-neighbor v_1, then $d(v_2)\geq 7$ and $d(v_4)\geq 7$ by non-$(4,6,6)$!, so v_1 gives $\frac{1}{2}$ to v by R1. The same is true for v_3 if $d(v_3)=6$. In view of R2, we have $\mu'(v)\geq -2+2\times\frac{1}{2}+2\times\frac{2}{3}>0$.

CASE 2. $d(v)=5$. If v has a 5^--neighbor, then it has four 7^+-neighbors by non-$(5,5,6)$!, so $\mu'(v)=5-6+4\times\frac{1}{2}=0$ by R3.

Next suppose v has no 5^--neighbors. If v has a 6-neighbor v_1, then $d(v_2)\geq 7$ and $d(v_5)\geq 7$ by non-$(5,6,6)$!, which means that v receives $\frac{1}{2}+2\times\frac{1}{4}$ from v_1,v_2,v_5 by R1 and R3, and hence $\mu'(v)\geq 0$.

If v has no 6^--neighbors at all, then $\mu'(v)=-1+5\times\frac{1}{4}>0$ by R3.

CASE 3. $d(v)=6$. If v has no 5^--neighbors, then it does not give charge away by R1, which means that $\mu'(v)=\mu(v)=0$.

Suppose $d(v_1)\leq 5$. Now v has five 7^+-neighbors by non-$(5,6,6)$!. Note that each of v_3,v_4,v_5 gives $\frac{1}{6}$ to v by R4, while v gives $\frac{1}{2}$ to $v_1$by R1, which results in $\mu'(v)=0+3\times\frac{1}{6}-\frac{1}{2}=0$.

CASE 4. $d(v)=7$. If v has a 4-neighbor, v_1, then v has no other 5^--neighbors due to non-$(4,7,5)$!; so v gives $\frac{2}{3}$ to v_1 by R2. Furthermore, v can only make at most two donations of $\frac{1}{6}$ to $v_3,\dots,v_6$ by R4.

Indeed, each such a donation to v_k with $3\leq k\leq 6$ forbids a donation to v_{k-1} and v_{k+1}, as said in R4. As a result, $\mu'(v)=7-6-\frac{2}{3}-2\times\frac{1}{6}=0$, as desired.

Now suppose v has no 4-neighbor. Now at most four donations of $\frac{1}{4}$ are possible to 5-neighbors by R3 due to non-$(5,5,5)$!. Also, donations of $\frac{1}{6}$ by R4 are possible.

We note that donations of $\frac{1}{4}$ and $\frac{1}{6}$ can occur along at most two consecutive edges at v, which means that the total number of donations is at most $\lfloor\frac{2\times 7}{2+1}\rfloor = 4$. This yields $\mu'(v) \geq 1 - 4 \times \frac{1}{4} = 0$, and we are done.

CASE 5. $8 \leq d(v) \leq 9$. By the same reasons as in Case 4, our v can make at most $\lfloor\frac{2\times 9}{3}\rfloor = 6$ donations at all, including at most one donation of $\frac{2}{3}$ to a 4-vertex by R2. Therefore, $\mu'(v) \geq d(v) - 6 - \frac{2}{3} - (6-1) \times \frac{1}{4} > d(v) - 8 \geq 0$.

CASE 6. $d(v) \geq 10$. Still, v can make at most $\lfloor\frac{2\times d(v)}{3}\rfloor$ donations by R2–R4 , and each of them is of at most $\frac{2}{3}$. For $d(v) = 10$ we have $\mu'(v) \geq 10 - 6 - \lfloor\frac{20}{3}\rfloor \times \frac{2}{3} = 4 - 6 \times \frac{2}{3} = 0$. For $d(v) \geq 11$, it holds $\mu'(v) \geq d(v) - 6 - \lfloor\frac{2\times d(v)}{3}\rfloor \times \frac{2}{3} \geq d(v) - 6 - \frac{2d(v)}{3} \times \frac{2}{3} = \frac{5d(v)-54}{9} > \frac{5(d(v)-11)}{9} \geq 0$.

Thus we have proved $\mu'(v) \geq 0$ for every $v \in V$, which contradicts (1):

$$0 \leq \sum_{v\in V} \mu'(v) = \sum_{v\in V} \mu(v) = -12.$$

■

2.2. **Proving that each of** $(td2), \ldots, (td4)$ **is a description of** 3**-paths in** $\mathbf{P_4}$**.** Note that, by definition, each (i,j,k)-path is also an (i',j',k')-path if $i' \geq i$, $j' \geq j$, and $k' \geq k$.

Therefore, for each of the sets $(td2), \ldots, (td4)$ it suffices to check that all triplets in each of them together cover all triplets in $(td1)$.

For example, the only triplet $(5,9,6)$ in $(td4)$ covers each of the triplets in $(td1)$. ■

2.3. **Proving that each of descriptions** $(td1), \ldots, (td4)$ **is tight.** This is based on the properties of H_1–H_3. Namely, each of $(td1), \ldots, (td4)$ must contain triplets $(5^+, 6^+, 6^+)$, $(4^+, 7^+, 5^+)$, and $(4^+, 9^+, 4^+)$, due to H_1, H_2, and H_3, respectively.

For example, an attempt to decrease 9 in any of $(td1), \ldots, (td4)$ is prevented from by H_3, in which every non-minor 3-path is centered at a 9^+-vertex.

Also, we cannot replace the central 7 in $(td1)$ or $(td2)$ by 6, since otherwise the thus reduced set of triplets fails to cover H_2, in which each 7-vertex has just one 4-neighbor. By the same reason, we cannot replace 5 in $(4,7,5)$ by 4.

Finally, we cannot replace the non-central 6 in any of $(td1), \ldots, (td4)$ due to H_1, which has no two 5^--vertices with a common neighbor. ■

2.4. **Proving that there are no tight major descriptions of 3-paths in** $\mathbf{P_4}$ **other than** $(td1), \ldots, (td4)$**.** Suppose $D = \{(x_1,y_1,z_1), \ldots (x_k,y_k,z_k)\}$ is a a major tight description of 3-paths in $\mathbf{P_5}$. This means that

(1) every $P_4 \in \mathbf{P_4}$ has a (x_i,y_i,z_i)-path for at least one i with $1 \leq i \leq k$, and

(2) if we delete any term (x_i,y_i,z_i) from D or decrease any parameter in D by one (but not decrease any y_i below 6) without changing the other $3k-1$ parameters, then the new description is not satisfied by at least one $P_4 \in \mathbf{P_4}$.

Note that, due to its tightness, the description D cannot have triplets (X,Y,Z) and (X',Y',Z') such that $X \leq X'$, $Y \leq Y'$, and $Z \leq Z'$, because $D' = D \setminus \{(X,Y,Z)\}$ is equivalent to D but shorter.

Also, all parameters in D should be at least 4 since we deal with $\mathbf{P_4}$ and, moreover, $y_i \geq 6$ whenever $1 \leq i \leq k$. By symmetry, we can assume that $x_i \leq z_i$ whenever $1 \leq i \leq k$.

Note that D must contain a term $(4^+, 9^+, 4^+)$ to be able to describe H_3. Therefore, our case analysis splits into Cases 1–3.

Case 1. D has a term $(x_1, y_1, z_1) = (5^+, 9^+, 6^+)$. Due to Subsection 2.3, D must coincide with the tight description $(td4)$, so $D = \{(5, 9, 6)\}$.

Case 2. D has a term $(x_1, y_1, z_1) = (4, 9^+, 5^+)$. Due to H_1, which has no 4-vertices, there should be a term $(x_2, y_2, z_2) = (5^+, 6^+, 6^+)$ in D, and hence D coincides with the tight description $\{(4, 9, 5), (5, 6, 6)\}$, which is $(td3)$.

CASE 3. D has a term $(x_1, y_1, z_1) = (4, 9^+, 4)$. Due to H_1 again, there should be a term $(x_2, y_2, z_2) = (5^+, 6^+, 6^+)$.

SUBCASE 3.1. $(x_2, y_2, z_2) = (5^+, 7^+, 6^+)$. Here, $D = \{(4, 9, 4), (5, 7, 6)\}$, which is a tight description $(td2)$.

SUBCASE 3.2. $(x_2, y_2, z_2) = (5^+, 6, 6^+)$. Now the first two terms of D do not cover H_2, which has 7-vertices, but none of them has two 4-neighbors. This mean that there is a term $(x_3, y_3, z_3) = (4^+, 7^+, 5^+)$, and so D coincides with the tight description $(td1)$: $\{(4, 9, 4), (4, 7, 5), (5, 6, 6)\}$. ■

This completes the proof of Theorem 1.

REFERENCES

[1] V.A. Aksenov, O.V. Borodin, A.O. Ivanova, *Weight of 3-paths in sparse plane graphs*, Electron. J. Comb., **22**:3 (2015), Paper #P3.28. Zbl 1323.05058

[2] K. Ando, S. Iwasaki, A. Kaneko, *Every 3-connected planar graph has a connected subgraph with small degree sum* (Japanese), Annual Meeting of Mathematical Society of Japan (1993).

[3] O.V. Borodin, *Solution of Kotzig-Grünbaum problems on separation of a cycle in planar graphs*, Mat. Zametki, **46**:5 (1989), 9–12. Zbl 0694.05027

[4] O.V. Borodin, *Structural properties of planar maps with the minimal degree 5*, Math. Nachr., **158** (1992), 109–117. Zbl 0776.05035

[5] O.V. Borodin, *Minimal vertex degree sum of a 3-path in plane maps*, Discuss. Math., Graph Theory, **17**:2 (1997), 279–284. Zbl 0906.05017

[6] O.V. Borodin, A.O. Ivanova, *Describing tight descriptions of 3-paths in triangle-free normal plane maps*, Discrete Math., **338**:11 (2015), 1947–1952. Zbl 1314.05042

[7] O.V. Borodin, A.O. Ivanova, *An analogue of Franklin's theorem*, Discrete Math., **339**:10 (2016), 2553–2556. Zbl 1339.05067

[8] O.V. Borodin, A.O. Ivanova, *New results about the structure of plane graphs: a survey*, AIP Conference Proceedings, **1907** (2017), 030051.

[9] O.V. Borodin, A.O. Ivanova, *All tight descriptions of 3-paths centered at 2-vertices in plane graphs with girth at least 6*, Sib. Électron. Mat. Izv, **16** (2019), 1334–1344. Zbl 1429.05046

[10] O.V. Borodin, A.O. Ivanova, *An extension of Franklin's theorem*, Sib. Électron. Math. Izv., **17** (2020), 1516–1521. Zbl 1443.05160

[11] O.V. Borodin, A.O. Ivanova, *A tight description of 3-polytopes by major 3-paths*, Siberian Math. J. (to appear).

[12] O.V. Borodin, A.O. Ivanova, T.R. Jensen, A.V. Kostochka, M.P. Yancey, *Describing 3-paths in normal plane maps*, Discrete Math., **313**:23 (2013), 2702–2711. Zbl 1280.05026

[13] O.V. Borodin, A.O. Ivanova, A.V. Kostochka, *Tight descriptions of 3-paths in normal plane maps*, J. Graph Theory, **85**:1 (2017), 115–132. Zbl 1365.05150

[14] P. Franklin, *The four color problem*, Amer. J. Math., **44** (1922), 225–236. JFM 48.0664.02

[15] S. Jendrol', *Paths with restricted degrees of their vertices in planar graphs*, Czech. Math. J., **49**:3 (1999), 481–490. Zbl 1003.05055

[16] S. Jendrol', T. Madaras, *On light subgraphs in plane graphs with minimum degree five*, Discuss. Math., Graph Theory, **16**:2 (1996), 207–217. Zbl 0877.05050

[17] S. Jendrol', H.-J. Voss, *Light subgraphs of graphs embedded in the plane. A survey*, Discrete Math., **313**:4 (2013) 406–421. Zbl 1259.05045

[18] S. Jendrol', M. Maceková, *Describing short paths in plane graphs of girth at least* 5, Discrete Math., **338**:2 (2015) 149–158. Zbl 1302.05040

[19] T. Madaras, *Note on the weight of paths in plane triangulations of minimum degree* 4 *and* 5, Discuss. Math., Graph Theory, **20**:2 (2000), 173–180. Zbl 0982.05043

[20] T. Madaras, *Two variations of Franklin's theorem*, Tatra Mt. Math. Publ., **36** (2007), 61–70.

[21] B. Mohar, R. Škrekovski, H.-J. Voss, *Light subgraphs in planar graphs of minimum degree* 4 *and edge-degree* 9, J. Graph Theory, **44**:4 (2003), 261–295. Zbl 1031.05075

[22] P. Wernicke, *Über den kartographischen Vierfarbensatz*, Math. Ann., **58** (1904), 413–426. JFM 35.0511.01

Tsyndyma Chemit-Dorzhievna Batueva
Sobolev Institute of Mathematics,
4, Koptyuga ave.,
Novosibirsk, 630090, Russia
Email address: tsyn.batueva@gmail.com

Oleg Veniaminovich Borodin
Sobolev Institute of Mathematics,
4, Koptyuga ave.,
Novosibirsk, 630090, Russia
Email address: brdnoleg@math.nsc.ru

Anna Olegovna Ivanova
Sobolev Institute of Mathematics,
4, Koptyuga ave.,
Novosibirsk, 630090, Russia
Email address: shmgnanna@mail.ru

Dmitrii Vladislavovich Nikiforov
Sobolev Institute of Mathematics,
4, Koptyuga ave.,
Novosibirsk, 630090, Russia
Email address: zerorebelion@mail.ru