S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 18, №1, стр. 474–478 (2021) УДК 515.1
DOI 10.33048/semi.2021.18.033 MSC 54B20, 54A25

LOCAL τ-DENSITY AND LOCAL WEAK τ-DENSITY IN TOPOLOGICAL SPACES

T.K. YULDASHEV, F.G. MUKHAMADIEV

Abstract. The article considers locally compact, metrizable, and linearly ordered spaces. In these spaces, the concepts of local τ-density and local weak τ-density are considered. It is shown that in locally compact, metrizable, and linearly ordered spaces the concepts of local τ-density and local weak τ-density coincide. It is proved that a locally weakly τ-dense subset of a compact set with the first axiom of countability is locally τ-dense.

Keywords: local τ-density, local weak τ-density, locally compact space, metrizable space, linearly ordered space.

1. Introduction

Set-theoretic topology, one of the prominent areas of research in general topology considers properties of topological spaces and their continious mappings, operations on said spaces and mappings as well as their classification. Neigbourhood, closure, compactness, density, separability, cardinal number, π-base of sets, sum, intersection and tikhonoff product can be considered, among many other, as key notions and concepts of this field. For a survey, covering the main steps of development of set-theoretic topology, see [1].

In this paper we consider an infinite cardinal number τ, local τ-density and local weak τ-density in various topological spaces, such as locally compact space, metrizable space and linearly ordered space.

A set $A \subset X$ is said to be dense (in X), if $[A] = X$. The density of the space X is defined as the smallest cardinal $|A|$, where A is a dense subset of X. This cardinal

Yuldashev, T.K., Mukhamadiev, F.G., Local τ-density and local weak τ-density in topological spaces.

Received June, 24, 2020, published April, 28, 2021.

is denoted by $d(X)$. If $d(X) = \tau,\ \tau \geq \aleph_0$, the space X is said to be τ-dense. If $d(X) \leq \aleph_0$, then X is said to be separable [2].

A topological space X is called locally τ-dense at the point $x \in X$, if τ is the smallest cardinal number, such that x has a neighbourhood of density τ in X [3]. Local density at x is denoted by $ld(x)$. Local density of the space X is defined as follows:

$$ld(X) = \sup\{ld(x) : x \in X\}.$$

It is clear that local density of a topological space cannot excede the density of said space, i.e. $ld(X) \leq d(X)$.

In the topological space X, consider the smallest cardinal number $\tau \geq \aleph_0$ such that X contains a π-base, which splits into τ centered families of open sets, i.e. there exists a π-base $B = \cup\{B_\alpha : \alpha \in A\}$, where B_α is a centered family of open sets for every $\alpha \in A$ и $|A| = \tau$. This cardinal number τ is said to be the weak density of X [4]. Weak density of a topological space X is denoted by $wd(X)$. If $wd(X) = \aleph_0$, then X is said to be weakly separable.

In [4], R. B. Beshimov shows that $wd(X) = d(X)$ for every a) locally compact space X; b) metric space X; c) topologically linearly ordered space X. It is also shown that if the relation $wd\ (X_s) \leq \tau$ holds for every $s \in S$ with $|S| \leq 2^\tau$, then

$$wd\left(\prod_{s\in S} X_s\right) \leq \tau.$$

A topological space X is said to be locally weakly τ-dense at the point $x \in X$ if τ is the smallest cardinal number such that x has a neighbourhood of weak density τ in X [3]. Local weak density at the point x is denoted by $lwd(x)$. The local weak density of the space X is a supremum of all cardinal numbers $lwd(x)$ for $x \in X$ and is denoted by $lwd(X) = \sup\{lwd(x) : x \in X\}$.

Consider a locally finite open cover ν, which is contained in some finite open cover of X by separable subsets. Define the equivalence relation E on ν. We say that relation UEU' takes place if there exists a sequence of elements $U_0, U_1, \ldots, U_k$ of ν, such that $U_0 = U$, $U_k = U'$ and $U_i \cap U_{i+1} \neq \emptyset$, $i = 1, 2, \ldots, k-1$. For each equivalence class $\nu_s \subset \nu$ of equivalence relation E we consider a union $X_s = \cup\nu_s$.

Now we take a look at a family of non-intersecting topological spaces, i.e.

$$\{X_\alpha : \alpha \in A\}$$

such that $X_\alpha \cap X_{\alpha'} = \emptyset$ for $\alpha \neq \alpha'$. Consider a set $X = \cup_\alpha \{X_\alpha : \alpha \in A\}$ and the family $\tilde{\mathrm{O}}$ of all sets $U \subset X$, such that $U \cap X_\alpha$ is open in X_α for every $\alpha \in A$. It is easy to see that the family $\tilde{\mathrm{O}}$ defines a certain topology on the set X. So X with that topology is called a sum of spaces $\{X_\alpha : \alpha \in A\}$ and is denoted by $\oplus\{X_\alpha : \alpha \in A\}$ [2].

We will now present a theorem, which was proved in [2] and [5].

Theorem 1. *For a locally separable metrizable space X the following relation holds: $X = \oplus\{X_\alpha : \alpha \in A\}$, where every X_α is a separable space.*

A topological space X is said to be locally compact, if for every point $x \in X$ there exists a neighbourhood U of this point, such that $\overline{U}$ is a compact subspace of X [2].

Let X be a set, and $<$ be some relation on X. We say that $<$ is a linear order on X if the relation $<$ satisfies the following properties: 1) If $x < y$ and $y < z$, then

$x < z$; 2) If $x < y$ then the relation $y < x$ does not hold; 3) If $x \neq y$ then either $x < y$ or $y < x$ holds. A set X together with some linear order defined on it is called a linearly ordered set.

In this paper we consider the questions of local τ-density and local weak τ-density in locally compact, metrizable and linearly ordered spaces. We prove that in these spaces these notions are equivalent. We will also show that a locally weakly τ-dense subset of a compact set with the first axiom of countability is locally τ-dense.

2. Main results

In the first part of our paper we presented some necessary notions and statements. In this section we prove the following theorems.

Theorem 2. *Suppose that a space X satisfies at least one of the following conditions:*

1) *X is locally compact,*
2) *X is metrizable,*
3) *X is a linearly ordered topological space with the interval topology.*

Then X is locally τ-dense if and only if it is locally weak τ-dense.

Proof. We will first prove the statement for the case when X is a locally compact space. Then the following statements are equivalent: 1) X is locally weakly τ-dense; 2) X is locally τ dense. Let $x \in X$; denote by Ox a weakly τ-dense neighbourhood of x. Every open subset of a locally compact space is locally compact, hence Ox is locally compact. Since weakly τ-dense locally compact space is τ-dense, we get that Ox is also τ-dense. Therefore, X is locally τ-dense. The converse follows from the fact that $wd\,(X) \leq d\,(X)$ [4].

We will now prove the theorem for the case of X being a metrizable space. Let X be a metrizable locally weak τ-dense space and consider $x \in X$. Then there exists a neighbourhood Ox of the point x, such that Ox is weakly τ-dense. Since metrizability of the space implies metrizability of every subset of that space we obtain that Ox is a metrizable weakly τ-dense subset. Then, since a weakly τ-dense metrizable subset is τ-dense, we get that Ox is τ-dense [4]. Hence, the set X is locally τ-dense. The converse follows from the fact that the inequality $wd\,(X) \leq d\,(X)$ holds in any metrizable space [4]. In the case of X being a linearly ordered topological space, the proof is similar to the proof of two previous cases. Thus Theorem 2 is proved. □

Corollary 1. *Let X be a compact Hausdorff space. Then the following conditions are equivalent:*

1) *X is locally weakly τ-dense,*
2) *X is locally τ-dense.*

By using Theorem 2 in conjuction with Theorem 1 we obtain the following result.

Corollary 2. *Let X be a locally weakly τ-dense metrizable space. Then*

$$X = \oplus \{X_\alpha : \alpha \in A\},$$

where every space X_α is τ-dense.

Corollary 2 can be proved in the similar way as Theorem 1 (see [2, page 435]), with additional use of Theorem 2.

Character of the point x in the topological space X is defined by the smallest cardinal number of type $|\beta(x)|$ and is denoted as $\chi(x, X)$ where $\beta(x)$ is a base of X at x. Character of a topological space X is defined as a supremum of all cardinal numbers $\chi(x, X)$ for $x \in X$ and is denoted by $\chi(X)$. If $\chi(X) \leq \aleph_0$, then it is said that X satisfies the first axiom of countability, i.e. there exists a countable base in every point $x \in X$ [2].

Theorem 3. *A locally weakly τ-dense subset of a compact set with character $\chi(X) \leq \tau$ is locally τ-dense.*

Proof. Let X be a compact set that satisfies the first axiom of countability. Also let $Y \subset X$ be a locally weakly τ-dense subset of X. Now consider $x \in Y$. There exists a neighbourhood $Ox \subset Y$ which is weakly τ-dense in Y. Let $B = \cup\{B_i : i \in K\}$ be a π-base in Ox, where $B_i = \left\{U_\alpha^i : \alpha \in A_i\right\}$ is a centered family of sets, which are open in the neighborhood Ox for every $i \in K$. Then, from [6, Theorem 1], it follows that $[Ox]_X$ is a τ-dense compact set that satisfies the first axiom of countability. Every centered family B_i contains the same contact point a_i in $[Ox]_X$ for every $i \in K$. We set $C = \{a_i : i \in K\}$. It is clear that the set C is dense in $[Ox]_X$ and the family $\mu = \{O_i a_n : i, n \in K\}$ is a base at the point a_n and $|\mu| \leq \tau$. We now choose a point $b_i^n \in O_i a_n \cap Ox$ for each pair $i, n \in K$. We will show that the set $F = \{b_i^n : i, n \in K, n \in K\}$ is dense in the neighbourhood Ox. Let G be a non-empty open set in Ox and consider $y \in G$. Then there exists a neighbourhood Oy, such that $Oy \subset G$. Since X is a regular space, there exists a neighbourhood $O_1 y$, such that $O_1 y \subset [O_1 y]_X \subset Oy$. From this we get that $O_1 y \subset [O_1 y]_X \subset Oy \subset G$. Since B is a π-base in the neighbourhood Ox, there exists $U_\alpha^n \in B_n \subset B$, such that $U_\alpha^n \subset O_1 y$. Then $[U_\alpha^n]_X \subset [O_1 y]_X$. We also have that $a_n \in [U_\alpha^n]_X$. Choose a point $b_i^n \in O_i a_n \bigcap U_\alpha^n$. From this, we get that $b_i^n \in U_\alpha^n \subset O_y \subset G$. Therefore, F is dense in Ox, i.e. a neighbourhood Ox is τ-dense. This implies that the set Y is locally τ-dense. Thus, the theorem is proved. □

Remark. The following example shows, that the requirement $\chi(X) \leq \tau$ for a compact set, stated in Theorem 3 cannot be dropped.

Example. There exists a locally weakly τ-dense subset Y of a compact set X, which is not locally τ-dense.

Indeed, let $\{X_\alpha : \alpha \in A\}$ be a family of topological spaces and consider a point $a = \{a_\alpha\}$ of the product $\prod\{X_\alpha : \alpha \in A\}$. By $\sum(a)$ denote a subspace of a product $\prod\{X_\alpha : \alpha \in A\}$, consisting of all points $\{x_\alpha\}$, such that the relation $x_\alpha \neq a_\alpha$ is true only for a countable set of indices $\alpha \in A$. A subspace $\sum(a)$ of the space $\prod\{X_\alpha : \alpha \in A\}$ is called a $\sum$-product of spaces $\{X_\alpha : \alpha \in A\}$. Consider $\tau = \aleph_0$, i.e. assume that X satisfies the first axiom of countability. We show that, in that case, there exists a locally weakly separable subset of X (denoted by Y), which is not locally separable. Let $I_\alpha = [0, 1]$ and $X = I^c = \prod\{I_\alpha : \alpha \in A\}$, where A is a set of indices of cardinality c. By the Hew-Marchevsky-Pondicerri theorem, I^c is a separable compact set (see [2, Theorem 2.3.15]). Choose an arbitrary point $a = \{a_\alpha\} \in I^c$. Now consider a $\sum$-product $Y = \sum(a)$ at the point a. Then $\sum(a)$ is dense in I^c. Since $\sum(a)$ is a weakly separable subset of a compact set I^c, we get that $d(\sum(a)) = c$ (see [4, Example 2]). We show that $\sum(a)$ is not locally separable. Consider an arbitraty neighbourhood $O(a) \subset \sum(a)$ of the point a. Then $\chi(O(a)) = c$. On the other hand $\chi(O(a)) \leq d(O(a))$. From this we obtain that and arbitrary neighbourhood $O(a)$ of the point a is not separable. Hence $\sum(a)$ is not locally separable at the point a.

References

[1] P.S. Aleksandrov, V.V. Fedorchuk, V.I. Zajtsev, *The main aspects in the development of set-theoretical topology*, Russ. Math. Surv., **33**:3 (1978), 1–53. Zbl 0409.54001

[2] R. Engelking, *General Topology*, Heldermann Verlag, Berlin, 1989. Zbl 0684.54001

[3] R.B. Beshimov, F.G. Mukhamadiev, N.K. Mamadaliev, *The local density and the local weak density of hyperspaces*, Int. J. Geom., **4**:1 (2015), 42–49. Zbl 1350.54008

[4] R.B. Beshimov, *On the weak density of topological spaces*, Dokl. AN Uzb., **2000**:11 (2000), 10–13.

[5] P.S. Aleksandrov, *Über die Metrization der im kleinen kompakten topologischen Räume*, Math. Ann., **92** (1924), 294–301. JFM 50.0128.04

[6] R.B. Beshimov, *Weakly separable spaces and their separability compactifications*, Dokl. AN Uzb., **1997**:1 (1997), 15–18. Zbl 0952.54015

Tursun Kamaldinovich Yuldashev
National University of Uzbekistan,
4, Universitetskaya str.,
Tashkent, 100174, Uzbekistan
Email address: tursun.k.yuldashev@gmail.com

Farkhod Gafurjanovich Mukhamadiev
National University of Uzbekistan,
4, Universitetskaya str.,
100174, Tashkent, Uzbekistan
Email address: farhod8717@mail.ru