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ON THE TRANSCENDENTAL SOLUTIONS OF FERMAT TYPE DELAY-DIFFERENTIAL AND c-SHIFT EQUATIONS WITH SOME ANALOGOUS RESULTS

A. BANERJEE, T. BISWAS

Abstract. In this paper, we mainly investigate on the finite order transcendental entire solutions of two Fermat types delay-differential and one Fermat type c-shift equations, as these types were not considered earlier. Our results improve those of [13] in some sense. In addition, we also extend some recent results obtained in [18]. A handful number of examples have been provided by us to justify our certain assertion as and when required.

Keywords: Fermat type equation, delay-differential equations, shift equation, entire and meromorphic solutions, finite order, Nevanlinna theory.

1. Introduction and some basic definitions

At the outset, we assume that the readers are familiar with the basic terms and notations of Nevanlinna's value distribution theory of meromorphic functions in the complex plane $\mathbb{C}$. So for such a meromorphic function f, terms like $T(r,f)$, $N(r,f)$, $m(r,f)$ etc., we refer to [6, 8]. The notation $S(r,f)$, is defined to be any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$, possibly outside a set E of r of finite logarithmic measure. The order of f is defined by

$$\rho(f) = \limsup_{r\longrightarrow\infty} \frac{\log T(r,f)}{\log r}.$$

Moreover, the shift and difference operator of a function are represented by $f(z+c)$ and $\Delta(f) = f(z+c) - f(z),\ c \in \mathbb{C}\backslash\{0\}$, respectively.

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In this paper, by linear c-shift operator in $f(z)$, we mean

$$L_c(z,f)=\sum_{j=0}^{\tau}a_jf(z+jc), \tag{1.1}$$

where $\tau\geq 1$, $a_\tau\neq 0$, a_j's are any constants. On the other hand, for delay-differential operator in f, we mean, a finite sum of products of f, shifts of f, derivatives of f and derivative of their shifts $f(z+jc),(c\in\mathbb{C})$, with constants coefficients.

We organize our paper as follows: In Section 2, we investigate on the entire solutions of Fermat type delay-differential and c-shift equations and extend some previous results. The non-existence conditions of meromorphic solutions of certain non-linear c-shift equations will be considered in Section 3, which extend [18].

2. Fermat type Delay-Differential and c-Shift Equations

Initially, Fermat type equations were investigated by Gross [3, 4], Montel [16]. Yang [21] investigated the Fermat type equation and obtained the following result:

Theorem A. [21] *Let m, n be positive integers satisfying $\frac{1}{m}+\frac{1}{n}<1$. Then there are no non-constant entire solutions $f(z)$ and $g(z)$ that satisfy*

$$a(z)f^n(z)+b(z)g^m(z)=1, \tag{2.1}$$

where $a(z)$, $b(z)$ are small functions of $f(z)$.

From *Theorem A* it is clear that either $m\geq 2$, $n>2$ or $m>2$, $n\geq 2$. So, it is natural that the case $m=n=2$ can be treated when $f(z)$ and $g(z)$ have some special relationship in (2.1). This was the starting point of a new era about the solution of Fermat type equations. As a result, successively several papers were published (see [1, 9, 10, 11, 12, 15, 19, 20, 22, 24]).

In 2007, Tang-Liao [19] investigated on the transcendental meromorphic solutions of the following non-linear differential equations

$$f(z)^2+P(z)^2(f^{(k)}(z))^2=Q(z), \tag{2.2}$$

where $P(z)$, $Q(z)$ are non-zero polynomials.

In 2013, Liu-Yang [14] considered the existence of solutions of the analogous difference equations of (2.2) namely

$$f(z)^2+P(z)^2(f(z+c)-f(z))^2=Q(z), \tag{2.3}$$

In this paper, we wish to investigate on the existence of solutions of certain Fermat type delay-differential equation as follows:

$$f^2(z)+R^2(z)(f^{(k)}(z+c)-f^{(k)}(z))^2=Q(z), \tag{2.4}$$

where $R(z)$, $Q(z)$ are and non-zero polynomials.

Liu-Yang [14] proved that (2.3) has no finite order transcendental entire solution, that's why in (2.4), it will be natural to investigate the case for $k\geq 1$.

Theorem 2.1. *If the non-linear delay-differential equation (2.4) has a transcendental entire solution of finite order, then $f(z)$ takes the form*

$$f(z)=\frac{Q_1(z)e^{az+b}+Q_2(z)e^{-(az+b)}}{2},$$

$a,b\in\mathbb{C}$ such that $Q_1(z)Q_2(z)=Q(z)$. Moreover, one of the following conclusions hold:

(i) *If* $e^{ac} \neq 1$; *then* k *must be odd,* $Q(z)$ *and* $R(z)$ *reduce to constants satisfying the equation* $2iRa^k + 1 = 0$.

(ii) *If* $e^{ac} = 1$; *then* $\deg R(z) = 1$, *none of* $Q_1(z)$, $Q_2(z)$ *be constants. Also,*

$$R(z) = \frac{Q_1(z)}{P(Q_1)} = \frac{Q_2(z)}{(-1)^{k-l+1}P(Q_2)},$$

such that $P(x) = i\sum_{l=0}^{k} \binom{k}{l} a^{k-l}[x^{(l)}(z+c) - x^{(l)}(z)]$.

The following examples clarify that both the cases of *Theorem 2.1* actually hold.

Example 2.1. *The function* $f(z) = \dfrac{2e^{3z+2} + 3e^{-(3z+2)}}{2}$ *satisfies the Fermat equation* $f^2(z) - \frac{1}{36}(f'(z+c) - f'(z))^2 = 6$, *where* $c = \pi i$, $R = -\frac{1}{6i}$.

Example 2.2. *The function* $f(z) = \dfrac{\alpha z e^{az+b} + \beta z e^{-(az+b)}}{2}$ *satisfies the Fermat equation* $f^2(z) - \frac{z^2}{a^2c^2}(f'(z+c) - f'(z))^2 = \alpha\beta z^2$ *such that* $e^{ac} = 1$, $Q_1(z) = \alpha z$, $Q_2(z) = \beta z$, *where* $\alpha, \beta \in \mathbb{C}\backslash\{0\}$. *Here,* $R = \dfrac{\alpha z}{ia(\alpha.(z+c) - \alpha.z)} = \dfrac{\beta z}{ia(\beta.(z+c) - \beta.z)}$, *i.e.,* $R = \dfrac{z}{iac}$.

In 2015, Liu-Dong [13] investigated on

$$D^2 f^2(z) + (Af(z+c) + Bf(z))^2 = 1 \tag{2.5}$$

and proved that if there exist finite order transcendental entire solutions of (2.5), then $A^2 = B^2 + D^2$. In that paper, they also discussed about

$$f^2(z) + (Af^{(m)}(z) + Bf^{(n)}(z))^2 = 1 \tag{2.6}$$

that (2.6) admits transcendental entire solution when $m + n$ is an even and m, n are odds.

In this paper, we wish to investigate on the following Fermat type equation as this type of equation was not dealt earlier:

$$f^2(z) + R^2(z)(Af^{(m)}(z+c) + Bf^{(n)}(z))^2 = 1, \tag{2.7}$$

where $m, n \in \mathbb{N}$, $R(z)$ be a non-zero polynomial, A and B are non-zero constants and prove the following theorem.

Theorem 2.2. *If the non-linear delay-differential equation (2.7) has a transcendental entire solution of finite order, then* $R(z)$ *reduces to constant, namely,* R *and* $f(z)$ *takes the form*

$$f(z) = \frac{e^{az+b} + e^{-(az+b)}}{2}$$

such that when

(I) m, n *are even, then* $a^{m-n} \neq \pm\frac{B}{A}$, $R^2 = \dfrac{1}{a^{2m}A^2 - a^{2n}B^2}$,

$$e^{ac} = \frac{-a^n B \pm \sqrt{(a^n B)^2 - (a^m A)^2}}{a^m A}.\ \textit{Also, } e^{ac} \notin \left\{\pm 1, -\left(\frac{a^m A}{a^n B}\right)^{\pm 1}\right\};$$

(II) m, n *are odd, then* $e^{ac} = \pm 1$, $a^{m-n} \neq \mp\frac{B}{A}$ *and* $R = \dfrac{-i}{a^n B \pm a^m A}$;

(III) *m is even, n is odd, then* $e^{ac} = \pm i$, $a^{m-n} \neq \pm i\frac{B}{A}$ *and* $R = \dfrac{-i}{a^nB \pm ia^mA}$;

(IV) *n is even, m is odd, then* $a^{m-n} \neq \pm i\frac{B}{A}$, $R^2 = -\dfrac{1}{a^{2n}B^2 + a^{2m}A^2}$,

$$e^{ac} = \frac{-a^nB \pm \sqrt{(a^nB)^2 + (a^mA)^2}}{a^mA}. \text{ Also, } e^{ac} \notin \left\{\pm 1, \frac{a^mA}{a^nB}, -\frac{a^nB}{a^mA}\right\}.$$

Remark 2.1. *Adopting the same procedure of* Case (I) *in Theorem 2.2, for* $m = n = 0$, *if we choose* $f(z) = Dg(z)$ *and* $R = 1/D$ *in (2.7), then we have* $R^2 = \dfrac{1}{A^2 - B^2} \implies A^2 = B^2 + D^2$, *i.e., we get* [13, Theorem 1.13]. *So our theorem is a significant extension of the same.*

Following example shows that in *Theorem 2.1*, each of the cases (I)-(IV) actually occurs.

Example 2.3. **(I)** *Let* m, n *both be even. Consider the function* $f(z) = \frac{e^{2z+3}+e^{-(2z+3)}}{2}$. *It is easy to see that* f *satisfies the Fermat equation* $f^2(z) - \frac{1}{2^7}(f''(z+c) + 3f''(z))^2 = 1$, *such that* $e^{2c} = -3 + 2\sqrt{2}$.
(II) *Let* m, n *be both odd. Choose* $f(z) = \frac{e^{3z+4}+e^{-(3z+4)}}{2}$, *which satisfies the Fermat equation* $f^2(z) - \frac{1}{12^2}(5f'(z+c) + f'''(z))^2 = 1$, *such that* $e^{3c} = -1$.
(III) *Let* m *be even and* n *be odd. The function* $f(z) = \frac{e^{z+2}+e^{-(z+2)}}{2}$ *satisfies the Fermat equation* $f^2(z) - \frac{1}{8+6i}(f''(z+c) + 3f'''(z))^2 = 1$, *such that* $e^c = i$.
(IV) *Let* m *be odd,* n *be even. Consider the function* $f(z) = \frac{e^{2z+1}+e^{-(2z+1)}}{2}$. *The function* f *satisfies the Fermat equation* $f^2(z) - \frac{1}{100}(3f'(z+c) + 2f''(z))^2 = 1$, *such that* $e^{2c} = \frac{1}{3}$.

In [14], Liu-Yang obtained the following result:

Theorem B. *There is no finite order transcendental entire solution of (2.3), where P(z), Q(z) are non-zero polynomials.*

In this paper, we partially extend the above result in the following manner.

Theorem 2.3. *The non-linear c-shift equation*

$$f^2(z) + L_c^2(z, f) = 1 \tag{2.8}$$

has finite order transcendental entire solution of the form

$$f(z) = \frac{e^{az+b} + e^{-(az+b)}}{2},$$

satisfying following two equations:

$$\begin{cases} a_0 + a_1e^{ac} + a_2e^{2ac} + \ldots + a_\tau e^{\tau ac} &= -i, \\ a_0 + a_1e^{-ac} + a_2e^{-2ac} + \ldots + a_\tau e^{-\tau ac} &= i. \end{cases} \tag{2.9}$$

Here $e^{ac} \neq \pm 1$. *Also if* $\tau = 1$, $a_0 \neq \pm a_1$ *is required.*

Remark 2.2. *In* Theorem 2.3, *if we take* $\tau = 1$ *and* $a_1 = -a_0 = 1$, *from (2.9), it is clear that when* $L_c(z, f) \equiv \Delta_c f(z)$, *there exists no finite order transcendental entire solution, which includes special case of* Theorem B.

Example 2.4. *Consider the function* $f(z) = \sin(\frac{\pi z}{2c})$. *Here* $a = \frac{i\pi}{2c}$, $e^b = \frac{1}{i}$. *Then* $f(z)$ *be a solution of the equation* $f^2(z) + (L_c(z, f))^2 = 1$, *provided that* τ *is an odd* *integer say $\tau = 2m+1,\ m \geq 1$ and the coefficients of $L_c(z,f)$ satisfy the following simultaneous equations:*

$$\begin{cases} a_0 - a_2 + a_4 - a_6 + \ldots + (-1)^m a_{2m} & = & 0, \\ a_1 - a_3 + a_5 - a_7 + \ldots + (-1)^m a_{2m+1} & = & -1; \end{cases}$$

and when τ is an even integer say $\tau = 2m,\ m \geq 1$, the coefficients of $L_c(z,f)$ satisfy the following simultaneous equations:

$$\begin{cases} a_0 - a_2 + a_4 - a_6 + \ldots + (-1)^m a_{2m} & = & 0, \\ a_1 - a_3 + a_5 - a_7 + \ldots + (-1)^{m-1} a_{2m-1} & = & -1. \end{cases}$$

The following lemma plays an important part for the proof in this section:

Lemma 2.1. [23] *Suppose $f_j(z)$ $(j = 1,2,...,n+1)$ and $g_k(z)$ $(k = 1,2,...,n)$ $(n \geq 1)$ are entire functions satisfying the following conditions:*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv f_{n+1}(z)$,

(ii) *The order of $f_j(z)$ is less than the order of $e^{g_k(z)}$ for $1 \leq j \leq n+1$, $1 \leq k \leq n$ and furthermore, the order of $f_j(z)$ is less than the order of $e^{g_h(z)-g_k(z)}$ for $n \geq 2$ and $1 \leq j \leq n+1$, $1 \leq h < k \leq n$.*

Then $f_j(z) \equiv 0$, $(j = 1,2,...,n+1)$.

<u>Proof of Theorem 2.1</u>. Assume that $f(z)$ is a finite order transcendental entire solution of (2.4), then

$$\begin{aligned} &[f(z) + iR(z)(f^{(k)}(z+c) - f^{(k)}(z))] \\ &\times[f(z) - iR(z)(f^{(k)}(z+c) - f^{(k)}(z))] = Q(z). \end{aligned} \tag{2.10}$$

Thus both of $f(z)+iR(z)(f^{(k)}(z+c)-f^{(k)}(z))$ and $f(z)-iR(z)(f^{(k)}(z+c)-f^{(k)}(z))$ have finitely many zeros. Combining (2.10), with Hadamard factorization theorem, we assume that

$$f(z) + iR(z)(f^{(k)}(z+c) - f^{(k)}(z)) = Q_1(z)e^{P(z)}$$

and

$$f(z) - iR(z)(f^{(k)}(z+c) - f^{(k)}(z)) = Q_2(z)e^{-P(z)},$$

where $P(z)$ is a non-constant polynomial, otherwise $f(z)$ will be a polynomial and $Q_1(z)Q_2(z) = Q(z)$, where $Q_1(z)$, $Q_2(z)$ are non-zero polynomials. We denote the degrees of $Q_1(z)$, $Q_2(z)$ and $P(z)$ by k_1, k_2 and k_3, respectively. Thus, we have,

$$f(z) = \frac{Q_1(z)e^{P(z)} + Q_2(z)e^{-P(z)}}{2} \tag{2.11}$$

and

$$f^{(k)}(z+c) - f^{(k)}(z) = \frac{Q_1(z)e^{P(z)} - Q_2(z)e^{-P(z)}}{2iR(z)}. \tag{2.12}$$

From (2.11), we have

$$f^{(k)}(z) = \frac{p_1(z)e^{P(z)} + p_2(z)e^{-P(z)}}{2}, \tag{2.13}$$

where,

$$\begin{aligned} p_1(z) &= Q_1(z)\left[P'(z)^k + M_{1,k}\left(P', P'', \ldots, P^{(k)}\right)\right] \\ &\quad + Q_1'(z) M_{2,k-1}\left(P', P'', \ldots, P^{(k-1)}\right) \\ &\quad + \ldots + Q_1^{(k-1)}(z) M_{k,1}(P') + Q_1^{(k)}(z), \end{aligned} \tag{2.14}$$

$$\begin{aligned} p_2(z) &= Q_2(z)\left[(-1)^k P'(z)^k + N_{1,k}\left(P', P'', \ldots, P^{(k)}\right)\right] \\ &\quad + (-1)^{k-1} Q_2'(z) N_{2,k-1}\left(P', P'', \ldots, P^{(k-1)}\right) \\ &\quad + \ldots + Q_2^{(k-1)}(z) N_{k,1}(P') + Q_2^{(k)}(z), \end{aligned} \tag{2.15}$$

where $M_{j,k-j+1}(N_{j,k-j+1})$ $(j = 1, 2)$ are differential polynomials of P' with degree $k-1$. $M_{j,k-j+1}(N_{j,k-j+1})$ are differential polynomial of P' with degree $k-j+1$ $(j = 3, 4, \ldots, k)$. It follows that $p_1(z)$, $p_2(z)$ are polynomials with degree $k_1 + k(k_3 - 1) \geq k_1$ and $k_2 + k(k_3 - 1) \geq k_2$, respectively. Using (2.12) and (2.13) we have

$$\begin{aligned} &\frac{Q_1(z)e^{P(z)} - Q_2(z)e^{-P(z)}}{2iR(z)} \\ &= \frac{p_1(z+c)e^{P(z+c)} + p_2(z+c)e^{-P(z+c)}}{2} - \frac{p_1(z)e^{P(z)} + p_2(z)e^{-P(z)}}{2} \\ &= \frac{(p_1(z+c)e^{\Delta_c P(z)} - p_1(z))e^{P(z)} + (p_2(z+c)e^{-\Delta_c P(z)} - p_2(z))e^{-P(z)}}{2}. \end{aligned} \tag{2.16}$$

Then (2.16) can be written as

$$\begin{aligned} &\left[p_1(z+c)e^{\Delta_c P(z)} - p_1(z) - \frac{Q_1(z)}{iR(z)}\right] e^{P(z)} \\ &+ \left[p_2(z+c)e^{-\Delta_c P(z)} - p_2(z) + \frac{Q_2(z)}{iR(z)}\right] e^{-P(z)} = 0. \end{aligned} \tag{2.17}$$

Applying *Lemma 2.1* on (2.17), we have

$$p_1(z+c)e^{\Delta_c P(z)} - p_1(z) - \frac{Q_1(z)}{iR(z)} = 0 \tag{2.18}$$

and

$$p_2(z+c)e^{-\Delta_c P(z)} - p_2(z) + \frac{Q_2(z)}{iR(z)} = 0. \tag{2.19}$$

Now we show that $P(z)$ is a one-degree polynomial. If not, suppose that $\deg(P(z)) \geq 2$. Then applying *Lemma 2.1* on (2.18) and (2.19), we have $p_1(z+c) = 0$ and $p_2(z+c) = 0$, a contradiction, which implies that $P(z)$ is a one-degree polynomial, say,

$$P(z) = az + b, \; a, b \in \mathbb{C}. \tag{2.20}$$

In view of (2.20), (2.17) yields

$$\begin{aligned} &\left[p_1(z+c)e^{ac} - p_1(z) - \frac{Q_1(z)}{iR(z)}\right] e^{P(z)} \\ &+ \left[p_2(z+c)e^{-ac} - p_2(z) + \frac{Q_2(z)}{iR(z)}\right] e^{-P(z)} = 0. \end{aligned} \tag{2.21}$$

Using (2.20), the expressions of $p_1(z)$ and $p_2(z)$ given by (2.14) and (2.15) reduec to

$$p_1(z)=\sum_{l=0}^{k}\binom{k}{l}a^{k-l}Q_1^{(l)}(z)\ \textit{and}\ p_2(z)=\sum_{l=0}^{k}\binom{k}{l}(-a)^{k-l}Q_2^{(l)}(z), \tag{2.22}$$

respectively. Now we have to consider the following two cases:

Case 1: If $e^{ac}\neq 1$. Then considering the degrees of p_1, Q_1 and p_2, Q_2 of the equation (2.21), one can conclude that $R(z)$ is constant, say, R. Thus, using (2.20) and (2.22), (2.18) and (2.19) become

$$iR\sum_{l=0}^{k}\binom{k}{l}a^{k-l}\left[e^{ac}Q_1^{(l)}(z+c)-Q_1^{(l)}(z)\right]=Q_1(z) \tag{2.23}$$

and

$$iR\sum_{l=0}^{k}\binom{k}{l}(-a)^{k-l}\left[Q_2^{(l)}(z)-e^{-ac}Q_2^{(l)}(z+c)\right]=Q_2(z). \tag{2.24}$$

Considering the highest degree on both sides of (2.23) and (2.24), we have

$$iRa^k(e^{ac}-1)=1\ \textit{and}\ iR(-a)^k(1-e^{-ac})=1,$$

which implies $e^{ac}=(-1)^k$. Since $e^{ac}\neq 1$, k must be odd and

$$a=\frac{(2m+1)\pi i}{c}, \tag{2.25}$$

m is any integer. So,

$$2iRa^k+1=0. \tag{2.26}$$

Eliminating $e^{\Delta_c P(z)}$ from (2.18) and (2.19), we have

$$\begin{aligned}&\frac{iRp_1(z)+Q_1(z)}{iRp_1(z+c)}=\frac{iRp_2(z+c)}{iRp_2(z)-Q_2(z)} \\ \Longrightarrow\quad &(p_1(z)p_2(z)-p_1(z+c)p_2(z+c))R^2+(p_1(z)Q_2(z)-p_2(z)Q_1(z))iR \\ &+Q_1(z)Q_2(z)=0.\end{aligned} \tag{2.27}$$

First suppose $Q(z)$ be constant, then $Q_1(z)$ and $Q_2(z)$ are constants, then from (2.22) and (2.27) we have, $(2iRa^k+1)Q_1Q_2=0$, which is possible from (2.26).

Suppose that $Q(z)$ be one-degree polynomial and let $Q_1(z)=\alpha_1 z+\alpha_0$, $\alpha_1\neq 0$ and $Q_2(z)$ be constant. Using $e^{ac}=-1$, (2.25), (2.26), we have from (2.23) that

$$\begin{aligned}&iR[a^k\{-(\alpha_1(z+c)+\alpha_0)-(\alpha_1 z+\alpha_0)\}-2ka^{k-1}\alpha_1]=\alpha_1 z+\alpha_0\\ \Longrightarrow\quad &iR[a^k\{2(\alpha_1 z+\alpha_0)+\alpha_1 c\}+2ka^{k-1}\alpha_1]+(\alpha_1 z+\alpha_0)=0\\ \Longrightarrow\quad &(2iRa^k+1)(\alpha_1 z+\alpha_0)+iRa^{k-1}\alpha_1(ac+2k)=0\\ \Longrightarrow\quad &2k=-(2m+1)\pi i,\end{aligned}$$

a contradiction. Similarly, considering $Q_1(z)$ and $Q_2(z)$ respectively as constant and one-degree polynomial, we can get a contradiction in a similar way. So, $Q(z)$ cannot be one-degree polynomial.

Next suppose that $Q(z)$ is a polynomial of degree 2. If $Q_1(z)$ and $Q_2(z)$ both are one-degree polynomials, then from previous argument, we get a contradiction.

Now let $Q_1(z)=\beta_2z^2+\beta_1z+\beta_0$, $\beta_2\neq 0$ and $Q_2(z)$ be a constant. Using $e^{ac}=-1$, (2.25), (2.26), we have from (2.23) that

$$\begin{aligned}
& iR[a^k\{-(\beta_2(z+c)^2+\beta_1(z+c)+\beta_0)-(\beta_2z^2+\beta_1z+\beta_0)\}\\
& +ka^{k-1}\{-(2\beta_2(z+c)+\beta_1)-(2\beta_2z+\beta_1)\}-2k(k-1)a^{k-2}\beta_2]\\
& =\beta_2z^2+\beta_1z+\beta_0\\
\Longrightarrow\quad & (2iRa^k+1)(\beta_2z^2+\beta_1z+\beta_0)+2iRa^{k-1}\beta_2(ac+2k)z+P_0(z)=0,
\end{aligned}$$

where $P_0(z)$ is a polynomial of degree 0. Now, comparing the coefficient of z from both side of the above equation, again we have $2k=-(2m+1)\pi i$, a contradiction. Similarly, considering $Q_1(z)$ and $Q_2(z)$ respectively as constant and two-degree polynomial, again we get a contradiction. Thus $Q(z)$ cannot be a second degree polynomial.

Now suppose that $Q(z)$ is a polynomial of degree $n\geq 3$. Let $Q_1(z)=\gamma_jz^j+\gamma_{j-1}z^{j-1}+\ldots+\gamma_0$, $\gamma_j\neq 0$ and $Q_2(z)$ be of degree $n-j$, $3\leq j\leq n$. Again using $e^{ac}=-1$, (2.25), (2.26), we have from (2.23) that

$$(2iRa^k+1)Q_1(z)+niRa^{k-1}\gamma_n(ac+2k)z^{j-1}+P_{j-2}(z)=0,$$

$P_{j-2}(z)$ is a polynomial of degree $j-2$. Here, comparing the coefficient of z^{j-1} from both sides of the above equation, again we have $2k=-(2m+1)\pi i$, a contradiction. Similarly, $Q_2(z)$ cannot be a polynomial of degree $n-j$, as in that case also we get a contradiction. So, $Q(z)$ cannot be a polynomial of degree $n\geq 3$.

Thus $Q(z)$ must be constant, say Q. Therefore, we must have the form of the solution is

$$f(z)=\frac{Q_1e^{az+b}+Q_2e^{-az-b}}{2},$$

such that $Q_1Q_2=Q$.

Case 2: If $e^{ac}=1$, then (2.21) becomes

$$\begin{aligned}
(2.28)\qquad & \left[p_1(z+c)-p_1(z)-\frac{Q_1(z)}{iR(z)}\right]e^{P(z)}\\
& +\left[p_2(z+c)-p_2(z)+\frac{Q_2(z)}{iR(z)}\right]e^{-P(z)}=0.
\end{aligned}$$

Applying *Lemma 2.1* and using (2.22) on (2.28), we have

$$(2.29)\qquad iR(z)\sum_{l=0}^{k}\binom{k}{l}a^{k-l}\left[Q_1^{(l)}(z+c)-Q_1^{(l)}(z)\right]=Q_1(z)$$

and

$$(2.30)\qquad iR(z)\sum_{l=0}^{k}\binom{k}{l}(-a)^{k-l}\left[Q_2^{(l)}(z)-Q_2^{(l)}(z+c)\right]=Q_2(z).$$

Note that the highest degree of $\sum\limits_{l=0}^{k}\binom{k}{l}a^{k-l}[Q_1^{(l)}(z+c)-Q_1^{(l)}(z)]$ and $\sum\limits_{l=0}^{k}\binom{k}{l}(-a)^{k-l}[Q_2^{(l)}(z)-Q_2^{(l)}(z+c)]$ are k_1-1 and k_2-1 respectively. Also we see that none of $Q_1(z)$ and $Q_2(z)$ be constants, i.e., $\deg Q(z)\geq 2$. Comparing the total degree of the equations (2.29) and (2.30), we can deduce that $\deg R(z) = 1$. Moreover,

$$R(z) = \frac{Q_1(z)}{P(Q_1)} = \frac{Q_2(z)}{(-1)^{k-l+1}P(Q_2)}$$

such that

$$P(x) = i\sum_{l=0}^{k}\binom{k}{l}a^{k-l}\left[x^{(l)}(z+c) - x^{(l)}(z)\right].$$

$\square$

Proof of Theorem 2.2. Assume that $f(z)$ is a finite order transcendental entire solution of (2.7), then

$$\begin{aligned}[f(z) + iR(z)(Af^{(m)}(z+c) + Bf^{(n)}(z))]\\ \times[f(z) - iR(z)(Af^{(m)}(z+c) + Bf^{(n)}(z))] = 1.\end{aligned} \tag{2.31}$$

Thus both of $f(z)+iR(z)(Af^{(m)}(z+c)+Bf^{(n)}(z))$ and $f(z)-iR(z)(Af^{(m)}(z+c)+Bf^{(n)}(z))$ have no zeros. Combining (2.31), with Hadamard factorization theorem, we assume that

$$f(z) + iR(z)(Af^{(m)}(z+c) + Bf^{(n)}(z)) = e^{P(z)}$$

and

$$f(z) - iR(z)(Af^{(m)}(z+c) + Bf^{(n)}(z)) = e^{-P(z)},$$

where $P(z)$ is a non-constant polynomial, otherwise $f(z)$ will be constant. Thus we have,

$$f(z) = \frac{e^{P(z)} + e^{-P(z)}}{2} \tag{2.32}$$

and

$$Af^{(m)}(z+c) + Bf^{(n)}(z) = \frac{e^{P(z)} - e^{-P(z)}}{2iR(z)}. \tag{2.33}$$

Using (2.32) in (2.33), we have

$$\begin{aligned}\frac{e^{P(z)} - e^{-P(z)}}{2iR(z)} &= \frac{A}{2}\left[p_1(z+c)e^{P(z+c)} + p_2(z+c)e^{-P(z+c)}\right]\\ &\quad+\frac{B}{2}\left[q_1(z)e^{P(z)} + q_2(z)e^{-P(z)}\right]\end{aligned} \tag{2.34}$$

such that

$$p_1(z+c) = P'(z+c)^m + M_{1,m-1}\left(P'(z+c), P''(z+c), \ldots, P^{(m)}(z+c)\right),$$

$$p_2(z+c) = (-1)^m P'(z+c)^m + M_{2,m-1}\left(P'(z+c), P''(z+c), \ldots, P^{(m)}(z+c)\right),$$

$$q_1(z) = P'(z)^n + N_{1,n-1}\left(P', P'', \ldots, P^{(n)}\right),$$

$$q_2(z) = (-1)^n P'(z)^n + N_{2,n-1}\left(P', P'', \ldots, P^{(n)}\right),$$

where $M_{i,m-1}$, $N_{i,n-1}$ $(i=1,2)$ are differential polynomials of P' with degree $m-1$, $n-1$. Then (2.34) can be written as

$$\Big[iAR(z)p_1(z+c)e^{\Delta_c P(z)} + iBR(z)q_1(z) - 1\Big]\, e^{P(z)} + \Big[iAR(z)p_2(z+c)e^{-\Delta_c P(z)} + iBR(z)q_2(z) + 1\Big]\, e^{-P(z)} = 0. \tag{2.35}$$

Applying *Lemma 2.1* on (2.35), we have

$$iAR(z)p_1(z+c)e^{\Delta_c P(z)} + iBR(z)q_1(z) - 1 = 0 \tag{2.36}$$

and

$$iAR(z)p_2(z+c)e^{-\Delta_c P(z)} + iBR(z)q_2(z) + 1 = 0. \tag{2.37}$$

Now we show that $P(z)$ is a one-degree polynomial. If not, suppose that $\deg(P(z)) \geq 2$. Applying *Lemma 2.1* on (2.36) and (2.37), we have $p_1(z+c) = 0$ and $p_2(z+c) = 0$, a contradiction, which concludes that $P(z)$ is a one-degree polynomial, say,

$$P(z) = az + b. \tag{2.38}$$

Using (2.38), we have

$$p_1(z) = a^m,\ p_2(z) = (-a)^m,\ \ q_1(z) = a^n,\ \ q_2(z) = (-a)^n. \tag{2.39}$$

Using (2.38) and (2.39) in (2.35), we have

$$\begin{aligned}&\left[iAa^m R(z)e^{ac} + iBa^n R(z) - 1\right] e^{az+b}\\ &+ \left[iA(-a)^m R(z)e^{-ac} + iB(-a)^n R(z) + 1\right] e^{-az-b} = 0.\end{aligned} \tag{2.40}$$

Again applying *Lemma 2.1* on (2.40), we have

$$\begin{cases} iAa^m R(z)e^{ac} + iBa^n R(z) - 1 = 0, \\ iA(-a)^m R(z)e^{-ac} + iB(-a)^n R(z) + 1 = 0. \end{cases} \tag{2.41}$$

From here, we can conclude that $R(z)$ is constant, say, R. So, (2.41) becomes

$$iR(a^m e^{ac} A + a^n B) = 1 \tag{2.42}$$

and

$$iR((-a)^m e^{-ac} A + (-a)^n B) = -1. \tag{2.43}$$

Case I: Let m, n be even. Then (2.43) becomes

$$iR(a^m e^{-ac} A + a^n B) = -1. \tag{2.44}$$

Now eliminating e^{ac} from (2.42) and (2.44) we get, $R^2(a^{2m}A^2 - a^{2n}B^2) = 1$, which implies, $\dfrac{a^m A}{a^n B} \neq \pm 1$, i.e., $a^{m-n} \neq \pm\frac{B}{A}$.

Again from (2.42) and (2.44) we get, $e^{ac} = \dfrac{-a^n B \pm \sqrt{(a^n B)^2 - (a^m A)^2}}{a^m A}$.

Also, $e^{ac} \notin \left\{\pm 1, \left(\dfrac{a^m A}{a^n B}\right)^{\pm 1}\right\}$.

Case II: Let m, n be odd. Then (2.43) becomes

$$iR(a^m e^{-ac} A + a^n B) = 1. \tag{2.45}$$

So, in this case, from (2.42) and (2.45) we get, $e^{ac} = \pm 1$. Also, $a^{m-n} \neq \mp\frac{B}{A}$.
Then $R = \dfrac{-i}{a^n B \pm a^m A}$.

Case III: Let m be even and n be odd. Then (2.43) becomes

$$iR(a^m e^{-ac}A - a^n B) = -1. \tag{2.46}$$

So, in this case, from (2.42) and (2.46) we get, $e^{ac} = \pm i$ and $a^{m-n} \neq \pm i\frac{B}{A}$.

Then $R = \dfrac{-i}{a^n B \pm i a^m A}$.

Case IV: Let m be odd and n be even. Then (2.43) becomes

$$iR(a^m e^{-ac}A - a^n B) = 1. \tag{2.47}$$

Now eliminating e^{ac} from (2.42) and (2.47) we get, $R^2(a^{2n}B^2 + a^{2m}A^2) = -1$, which implies, $\dfrac{a^m A}{a^n B} \neq \pm i$.

Again from (2.42) and (2.47) we get, $e^{ac} = \dfrac{-a^n B \pm \sqrt{(a^n B)^2 + (a^m A)^2}}{a^m A}$.

Also, $e^{ac} \notin \left\{\pm 1, \dfrac{a^m A}{a^n B}, -\dfrac{a^n B}{a^m A}\right\}$.

$\square$

Proof of Theorem 2.3. Assume that $f(z)$ is a finite order transcendental entire solution of (2.8), then

$$[f(z) + iL_c(z,f)][f(z) - iL_c(z,f)] = 1. \tag{2.48}$$

Proceeding in the same way as done in the previous theorem, from (2.48), we have,

$$f(z) = \frac{e^{P(z)} + e^{-P(z)}}{2} \tag{2.49}$$

and

$$L_c(z,f) = \frac{e^{P(z)} - e^{-P(z)}}{2i}. \tag{2.50}$$

From (1.1), (2.49) and (2.50), we have

$$\frac{e^{P(z)} - e^{-P(z)}}{2i} = \sum_{j=0}^{\tau} a_j \frac{e^{P(z+jc)} + e^{-P(z+jc)}}{2}$$

$$\implies \sum_{j=1}^{\tau} a_j \left(e^{P(z+jc)} + e^{-P(z+jc)}\right) = -(a_0 + i)e^{P(z)} - (a_0 - i)e^{-P(z)}. \tag{2.51}$$

Then (2.51) can be written as

$$\left(a_0 + i + \sum_{j=1}^{\tau} a_j e^{\Delta_{jc} P(z)}\right) e^{P(z)} + \left(a_0 - i + \sum_{j=1}^{\tau} a_j e^{-\Delta_{jc} P(z)}\right) e^{-P(z)} = 0. \tag{2.52}$$

Applying *Lemma 2.1* on (2.52), we have

$$a_0 + i + \sum_{j=1}^{\tau} a_j e^{\Delta_{jc} P(z)} = 0 \tag{2.53}$$

and

$$a_0 - i + \sum_{j=1}^{\tau} a_j e^{-\Delta_{jc} P(z)} = 0. \tag{2.54}$$

Now we show that $P(z)$ is a one-degree polynomial. On the contrary, suppose that $\deg(P(z)) \geq 2$. Applying *Lemma 2.1* on (2.53) and (2.54), we have $a_j = 0$ for all $1 \leq j \leq \tau$, which in view of (2.8) implies that $f(z)$ is constant, a contradiction. So, $P(z)$ is a one-degree polynomial, say, $P(z) = az + b$. Then using (2.53) and (2.54), the relation between a, c and a_j, $0 \leq j \leq \tau$, can be determined by (2.9). Also from (2.9), it is clear that $e^{ac} \neq \pm 1$. If $\tau = 1$, $a_0 \neq \pm a_1$ is required. □

3. Non-linear c-Shift Equations

For the existence of solutions of non-linear c-shift equation, in 2011, Qi [17] obtained the following theorems:

Theorem C. [17] *Let $q(z)$, $p(z)$ be polynomials and let n, m be distinct positive integers. Then the equation*

$$f^m(z) + q(z)f(z+c)^n = p(z) \tag{3.1}$$

has no transcendental entire solutions of finite order.

In 2015, Qi-Liu-Yang [18] obtained the meromorphic variant of *Theorem C* and improved this as follows:

Theorem D. [18] *Let $f(z)$ be a transcendental meromorphic function with finite order, m and n be two positive integers such that $m \geq n+4$, $p(z)$ be a meromorphic function satisfying $\overline{N}\left(r, \frac{1}{p(z)}\right) = S(r, f)$ and $q(z)$ be a non-zero meromorphic function satisfying that $T(r, q(z)) = S(r, f)$. Then, $f(z)$ is not a solution of equation*

$$f^m(z) + q(z)f(z+c)^n = p(z). \tag{3.2}$$

Theorem E. [18] *Let $f(z)$ be a transcendental entire function with finite order, m and n be two positive integers such that $m \geq n+2$, $p(z)$ be a meromorphic function satisfying $\overline{N}\left(r, \frac{1}{p(z)}\right) = S(r, f)$ and $q(z)$ be a non-zero meromorphic function satisfying that $T(r, q(z)) = S(r, f)$. Then $f(z)$ is not a solution of equation (3.2).*

In this paper we extend *Theorems D-E* at the expense of replacing $f(z+c)$ by $L_c(z, f)$.

Theorem 3.1. *Let $f(z)$ be a transcendental meromorphic function with finite order, m and n be two positive integers such that $m \geq (\tau+1)(n+2)+2$, $p(z)$ be a meromorphic function satisfying $\overline{N}\left(r, \frac{1}{p(z)}\right) = S(r, f)$ and $q(z)$ be a non-zero meromorphic function satisfying that $T(r, q(z)) = S(r, f)$. Then, $f(z)$ is not a solution of the non-linear c-shift equation*

$$f^m(z) + q(z)(L_c(z, f))^n = p(z). \tag{3.3}$$

Corollary 3.1. *Let $f(z)$ be a transcendental entire function with finite order, m and n be two positive integers such that $m \geq n+2$, $p(z)$ be a meromorphic function satisfying $\overline{N}\left(r, \frac{1}{p(z)}\right) = S(r, f)$ and $q(z)$ be a non-zero meromorphic function satisfying that $T(r, q(z)) = S(r, f)$. Then, $f(z)$ is not a solution of the non-linear c-shift equation (3.3).*

The next examples show that if the condition $m \geq n+2$ is omitted then the equation (3.3) can admit a transcendental entire solution.

First considering $n = 1$ and $m = 2$ we have the following examples.

Example 3.1. *For an odd integer s, the function $f(z) = e^{\frac{s\pi iz}{c}} + z$ is a solution of the equation $f^2(z) - zL_c(z, f) = e^{\frac{2s\pi iz}{c}}$, for $k \geq 2$, provided that the coefficients of $L_c(z, f)$ satisfy the following simultaneous equations:*

$$\begin{cases} a_0 - a_1 + a_2 - a_3 + \ldots + (-1)^k a_k &= 2, \\ a_0 + a_1 + a_2 + a_3 + a_4 + \ldots + a_k &= 1, \\ a_1 + 2a_2 + 3a_3 + 4a_4 + \ldots + ka_k &= 0. \end{cases}$$

Next considering $m = n = 1$ we have the following example.

Example 3.2. *The function $f(z) = ze^{\frac{\pi iz}{c}}$ satisfies the equation $f(z)+\frac{1}{z+1}L_c(z, f) = \frac{z(z+2)}{z+1}e^{\frac{\pi iz}{c}}$, where the coefficients of $L_c(z, f)$ is chosen such that they satisfy simultaneously the equations*

$$\begin{cases} a_0 - a_1 + a_2 - \ldots + (-1)^k a_k &= 1, \\ -a_1 + 2a_2 - 3a_3 + \ldots + k(-1)^k a_k &= 0. \end{cases}$$

To proceed further we require the following lemmas:

Lemma 3.1. [2, Lemma 5.1] *Let $f(z)$ be a finite order meromorphic function and $\varepsilon > 0$, then $T(r, f(z + c)) = T(r, f(z)) + o(r^{\sigma-1+\varepsilon}) + O(\log r)$ and $\sigma(f(z + c)) = \sigma(f(z))$. Thus, if $f(z)$ is a transcendental meromorphic function with finite order, then we know $T(r, f(z + c)) = T(r, f) + S(r, f)$.*

Lemma 3.2. [5, Theorem 2.1] *Let $f(z)$ be a meromorphic function with finite order, and let $c \in \mathbb{C}$ and $\delta \in (0, 1)$. Then $m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = o\left(\frac{T(r,f)}{r^\delta}\right) = S(r, f)$.*

Lemma 3.3. [7] *Let f be a non-constant meromorphic function of finite order and $c \in \mathbb{C}$. Then*

$$N(r, \infty; f(z + c)) \leq N(r, \infty; f(z)) + S(r, f),$$

$$\overline{N}(r, \infty; f(z + c)) \leq \overline{N}(r, \infty; f) + S(r, f).$$

Proof of Theorem 3.1. Suppose by contradiction that $f(z)$ is a transcendental meromorphic function with finite order satisfying equation (3.3).

If $T(r, p(z)) = S(r, f)$, then applying *Lemma 3.1* to equation (3.3), we have

$$\begin{aligned} m.T(r, f) &= T(r, f^m) \\ &= T(r, p(z) - q(z)(L_c(z, f))^n) \\ &= T(r, L_c(z, f)^n) + S(r, f) \\ &\leq (\tau + 1)n.T(r, f) + S(r, f), \end{aligned}$$

which contradicts the assumption that $m \geq (\tau + 1)(n + 2) + 2$.

If $T(r, p(z)) \neq S(r, f)$, differentiating equation (3.3), we get

$$(f^m(z))' + (q(z)(L_c(z, f))^n)' = p'(z). \tag{3.4}$$

Next dividing (3.4) by (3.3) we have

$$p'(z)[f^m(z) + q(z)(L_c(z, f))^n] = p(z)[(f^m(z))' + (q(z)(L_c(z, f))^n)']$$

$$\Longrightarrow \quad f^m(z) = \frac{\frac{p'(z)}{p(z)}q(z)(L_c(z, f))^n - (q(z)(L_c(z, f))^n)'}{\frac{(f^m(z))'}{f^m(z)} - \frac{p'(z)}{p(z)}}. \tag{3.5}$$

First observe that $\frac{(f^m(z))'}{f^m(z)} - \frac{p'(z)}{p(z)}$ cannot vanish identically. Indeed, if $\frac{(f^m(z))'}{f^m(z)} - \frac{p'(z)}{p(z)} \equiv 0$, then we get $p(z) = \alpha f^m(z)$, where α is a non-zero constant. Substituting the above equality to equation (3.3), we have $q(z)(L_c(z,f))^n = (\alpha-1)f^m(z)$. From *Lemma 3.1* and the above equation, we immediately see as above that $mT(r,f) \leq (\tau+1)nT(r,f)+S(r,f)$, which is a contradiction to $m \geq (\tau+1)(n+2)+2$. From equation (3.5), we know

$$\begin{aligned}(3.6)\qquad & mT(r,f) = T(r,f^m)\\ \leq\ & m\left(r, q(z)(L_c(z,f))^n\right) + m\left(r, \frac{p'(z)}{p(z)} - \frac{(q(z)(L_c(z,f))^n)'}{q(z)(L_c(z,f))^n}\right)\\ +\ & N\left(r, \frac{p'(z)}{p(z)}\, q(z)(L_c(z,f))^n - (q(z)(L_c(z,f))^n)'\right)\\ +\ & m\left(r, \frac{(f^m(z))'}{f^m(z)} - \frac{p'(z)}{p(z)}\right) + N\left(r, \frac{(f^m(z))'}{f^m(z)} - \frac{p'(z)}{p(z)}\right) + S(r,f).\end{aligned}$$

As *Lemma 3.1* together with equation (3.3) implies that

$$(m-(\tau+1)n)T(r,f)+S(r,f) \leq T(r,p(z)) \leq (m+(\tau+1)n)T(r,f)+S(r,f),$$

we conclude that

$$(3.7)\qquad S(r,p(z)) = S(r,f).$$

Applying *Lemmas 3.1*, *3.2* and (3.7) to equation (3.6), we obtain that

$$\begin{aligned}(3.8)\ mT(r,f) \quad \leq\ & nm(r,f) + N\left(r, \frac{p'(z)}{p(z)} q(z)(L_c(z,f))^n - (q(z)(L_c(z,f))^n)'\right)\\ & + N\left(r, \frac{(f^m(z))'}{f^m(z)} - \frac{p'(z)}{p(z)}\right) + S(r,f).\end{aligned}$$

Let

$$(3.9)\qquad H(z) = \frac{p'(z)}{p(z)} q(z)(L_c(z,f))^n - (q(z)(L_c(z,f))^n)'$$

and

$$(3.10)\qquad G(z) = \frac{(f^m(z))'}{f^m(z)} - \frac{p'(z)}{p(z)}.$$

First of all, we deal with $N(r,H(z))$. From (3.3) and (3.9), we know the poles of $H(z)$ are at the zeros of $p(z)$ and at the poles of $f(z)$, $f(z+jc)$, $(j=1,2,\ldots,\tau)$ and $q(z)$. Poles of $p(z)$ will not contribute towards the poles of $H(z)$ as from the equation (3.3) we know that the poles of $p(z)$ should be at the poles of $f(z)$, $f(z+jc)$, $(j=1,2,\ldots,\tau)$ and $q(z)$. We note that $T(r,q(z)) = S(r,f)$.

If z_0 is a zero of $p(z)$ then by (3.9), z_0 is at most a simple pole of $H(z)$. If z_0 is a pole of $f(z)$ of multiplicity t but not a pole of $f(z+jc)$, $j=1,\ldots,\tau$, then z_0 will be a pole of $H(z)$ of multiplicity at most $tn+1$. Next suppose z_1 be any pole of $f(z)$ of multiplicity t_0 and a pole of at least one $f(z+jc)$, $j=1,2,\ldots,\tau$, of multiplicity $t_j \geq 0$. Then z_1 may or may not be a pole of $L_c(z,f)$. From the above arguments and our assumption, we conclude that

$$\begin{aligned}(3.11)\ N(r,H) \quad \leq\ & \overline{N}\left(r, \frac{1}{p(z)}\right) + N(r,(L_c(z,f))^n) + \overline{N}(r,L_c(z,f)) + S(r,f)\\ \leq\ & nN(r,L_c(z,f)) + (\tau+1)\overline{N}(r,f) + S(r,f).\end{aligned}$$

Next, we turn our attention towards the poles of $G(z)$. We know from (3.3) and (3.10) that the poles of $G(z)$ are at the zeros of $p(z)$ and $f(z)$ and at the poles of $f(z)$, $f(z+jc)$, $j=1,2,\ldots,\tau$. If z_0 is a zero of $p(z)$, zero of $f(z)$, or pole of $f(z+jc)$, $j=1,2,\ldots,\tau$, then by (3.10) we know z_0 will be at most a simple pole of $G(z)$. If z_0 is a pole of $f(z)$ but not a pole of $f(z+jc)$, $j=1,2,\ldots,\tau$, then by the Laurent expansion of $G(z)$ at z_0, we obtain that $G(z)$ is analytic at z_0. Therefore, from our assumption and the discussions above, we know

$$
\begin{aligned}
(3.12)\qquad N(r,G) &\leq \overline{N}\left(r,\frac{1}{p(z)}\right)+\overline{N}(r,L_c(z,f))+\overline{N}\left(r,\frac{1}{f}\right)+S(r,f)\\
&\leq \overline{N}(r,L_c(z,f))+\overline{N}\left(r,\frac{1}{f}\right)+S(r,f).
\end{aligned}
$$

Using *Lemma 3.3*, from equations (3.8), (3.11) and (3.12) we have

$$
\begin{aligned}
mT(r,f) &\leq nm(r,f)+nN(r,L_c(z,f))+(\tau+1)\overline{N}(r,f)+\overline{N}(r,L_c(z,f))\\
&\quad +\overline{N}\left(r,\frac{1}{f}\right)+S(r,f)\\
&\leq nm(r,f)+n(\tau+1)N(r,f)+(\tau+1)\overline{N}(r,f)+(\tau+1)\overline{N}(r,f)\\
&\quad +\overline{N}\left(r,\frac{1}{f}\right)+S(r,f)\\
&\leq \{(\tau+1)(n+2)+1\}T(r,f)+S(r,f),
\end{aligned}
$$

which contradicts the assumption that $m \geq (\tau+1)(n+2)+2$. This completes the proof of the theorem. □

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Abhijit Banerjee
Department of Mathematics,
University of Kalyani,
Nadia, West Bengal, 741235, India
Email address: abanerjee_kal@yahoo.co.in

Tania Biswas
Department of Mathematics,
University of Kalyani,
Nadia, West Bengal, 741235, India
Email address: taniabiswas2394@gmail.com