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CENTER AND ITS SPECTRUM OF ALMOST ALL n-VERTEX GRAPHS OF GIVEN DIAMETER

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Abstract. We study *typical* (valid for almost all graphs of a class under consideration) properties of the center and its *spectrum* (the set of centers cardinalities) for n-vertex graphs of fixed diameter k. The spectrum of the center of all and almost all n-vertex connected graphs is found. The structure of the center of almost all n-vertex graphs of given diameter k is established. For $k = 1, 2$ any vertex is central, while for $k \geq 3$ we identified two types of central vertices, which are necessary and sufficient to obtain the centers of almost all such graphs; in addition, centers of constructed typical graphs are found explicitly.

It is proved that the center of almost all n-vertex graphs of diameter k has cardinality $n-2$ for $k = 3$, and for $k \geq 4$ the spectrum of the center is bounded by an interval of consecutive integers except no more than one value (two values) outside the interval for even diameter k (for odd diameter k) depending on k. For each center cardinality value outside this interval, we calculated an asymptotic fraction of the number of the graphs with such a center. The realizability of the found cardinalities spectrum as the spectrum of the center of typical n-vertex graphs of diameter k is established.

Keywords: graph, diameter, diametral vertices, radius, central vertices, center, spectrum of center, typical graphs, almost all graphs.

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Introduction

We study finite labeled ordinary graphs. For a connected graph G, the *distance* $\rho_G(u,v)$ between its vertices $u, v \in V(G)$ is defined as the length of the shortest path connecting these vertices. In this case, $e_G(v) = \max_{u\in V} \rho_G(v,u)$ is the *eccentricity* of the vertex v of the graph G, $d(G) = \max_{v\in V} e_G(v)$ is the *diameter* of the graph G, and $r(G) = \min_{v\in V} e_G(v)$ is the *radius* of the graph G. A vertex is called *central* if its eccentricity is equal to the radius of the graph. The *graph center* $\mathbb{C}(G)$ is the set of all central vertices of the graph G.

It is well known that for any graph H there is a connected graph G such that its subgraph induced by the center $\mathbb{C}(G)$ is isomorphic to H. This fact was established by G.N. Kopylov and E.A. Timofeev [13], its simple justification was also given by S.T. Hedetniemi (see [**?**]). And for any rational q, $0 < q \leq 1$, F. Buckley proved the existence of a graph G such that $|\mathbb{C}(G)| = q\,|V(G)|$ [2].

For an arbitrary class of connected graphs $\mathcal{K}$ through $\mathbb{S}p_c(\mathcal{K})$ we denote the *center spectrum* of graphs of this class, i.e. the set of cardinalities of graphs centers from the class $\mathcal{K}$. The class of all n-vertex connected graphs is naturally partitioned into subclasses of graphs determined by their diameter. Let $\mathcal{J}_{n,\,d=k}$, $\mathcal{J}_{n,\,d\geq k}$, $\mathcal{J}^*_{n,\,d\geq k}$ be the following classes of labeled n-vertex graphs: graphs of diameter k; connected graphs of diameter at least k and graphs (not necessarily connected) with a shortest path of length at least k, respectively. Then $\mathcal{J}_{n,\,d\geq 1}$ is the class of all n-vertex connected nontrivial graphs, and obviously, the following inclusions are fulfilled: $\mathcal{J}_{n,\,d=k} \subseteq \mathcal{J}_{n,\,d\geq k} \subseteq \mathcal{J}^*_{n,\,d\geq k}$.

In [13] all possible values of the parameters n, m and c are found for which there exists an n-vertex graph with m edges and c central vertices. The relations between these parameters are reduced to lower and upper bounds of the number of edges m in terms of the given parameters n and c (see Section 2, Theorem 2). Using this Theorem of G.N. Kopylov and E.A. Timofeev, one can find the center spectrum of all n-vertex connected graphs (see Section 2, Theorem 3):

$$\mathbb{S}p_c(\mathcal{J}_{n,\,d\geq 1}) = \big[\big[\,1,\,n\,\big]\big] \setminus \{n-1\},\ n \geq 2 \tag{1}$$

(here $\big[\big[\,x,\,y\,\big]\big]$ denotes an integer interval between two given numbers x and y, i.e. $\big[\big[\,x,\,y\,\big]\big] = [x,\,y] \cap \mathbb{Z}$). In addition, for almost all n-vertex connected graphs G, the following equality holds $|\mathbb{C}(G)| = n$. Obviously, $\mathbb{S}p_c(\mathcal{J}_{n,\,d=1}) = \{n\}$. Moreover, from the well-known result of J.W. Moon and L. Moser (almost all graphs have diameter 2 [14]) it is easy to obtain that almost all graphs from $\mathcal{J}_{n,\,d=2}$ have the radius equal to the diameter [8], and therefore, the cardinality of the center is also equal to n. Naturally the question arises as to the possible center spectrum of almost all n-vertex graphs of fixed diameter $k \geq 3$. The radius of almost all graphs of the class $\mathcal{J}_{n,\,d=k}$ is established by the author in [8]. For $k \geq 3$, almost all n-vertex graphs of diameter k have radius $\lceil \frac{k}{2} \rceil$.

Note that Yanan Hu and Xingzhi Zhan found the center spectrum of n-vertex graphs of given radius r [12]. As for the properties of the center spectrum of almost all n-vertex graphs of fixed diameter k (for large n), this result only implies inclusion $\mathbb{S}p_c(\mathcal{J}_{n,\,d=k}) \subseteq \big[\big[\,1,\,n\,\big]\big] \setminus \{n-1\}$ for all $n \geq (8k-2)/3$. This relation is a consequence of the equality (1), i.e. new restrictions for possible values of the center spectrum of the almost all graphs do not arise.

In [15] Dhruv Mubayi and Douglas B. West investigated the smallest $h_{n,k}(c)$ and the largest $f_{n,k}(c)$ number of vertices with eccentricity c in n-vertex graphs of diameter k. For individual cases that do not cover all possible relationships between parameters n, k and c, the values $f_{n,k}(c)$ and $h_{n,k}(c)$ are found. In particular, for $c = \lceil \frac{k}{2} \rceil$, which is the radius of almost all graphs from $\mathcal{J}_{n,\, d=k}$, $h_{n,k}(\lceil \frac{k}{2} \rceil) = 0$ and $f_{n,k}(\frac{k}{2}) = n-k$, $f_{n,k}(\frac{k+1}{2}) = n-k+1$. Such a lower bound of the center cardinality is reduced to trivial, and the upper bound, due to the definition, does not take into account possible jumps and gaps of center cardinality values in the interval $\big[\big[\, 1,\, f_{n,k}(\lceil \frac{k}{2} \rceil) \big]\big]$, defined by these estimates, and also turns out to be uninformative for the study of the distribution of the center cardinalities of the almost all graphs, when n tends to infinity.

In this paper, we investigate the center and its spectrum for almost all n-vertex graphs of fixed diameter k. Necessary preliminary information is contained in Section 1. There is also given a definition of the family of nested classes $\mathcal{F}_{n,k,p}$, $p \geq 1$ of n-vertex graphs of fixed diameter $k \geq 3$, possessing a number of metric properties and constructed by the author in [8]. It was previously established that $\mathcal{F}_{n,k,p}$ is a class of typical graphs for each of the classes $\mathcal{J}_{n,\, d=k}$, $\mathcal{J}_{n,\, d\geq k}$ and $\mathcal{J}^{*}_{n,\, d\geq k}$ [8] (Theorem 1 and its Corollaries). Hereinafter, we use this class of typical graphs.

In Section 2, we find the center spectrum of all and almost all n-vertex connected graphs (Theorem 3).

In Section 3, we establish the structure of the center of almost all graphs of a given diameter. For almost all graphs G of diameter $k = 1, 2$, every vertex is central, i.e. $\mathbb{C}(G) = V(G)$. For $k \geq 3$, we identified two types of central vertices, which are necessary and sufficient to obtain the centers of almost all n-vertex graphs of fixed diameter k. For odd k these are the central vertices of diametral paths of the graph and vertices equidistant at distance $\frac{k+1}{2}$ from their endpoints, while for even k these are only the central vertices of diametral paths (Theorem 4). Moreover, for typical graphs $G \in \mathcal{F}_{n,k,p}$ the center $\mathbb{C}(G)$ is explicitly distinguished (Lemma 5).

In Section 4, we asymptotically study the center spectrum of n-vertex graphs of a fixed diameter. It is proved that the center of almost all n-vertex graphs of diameter k has cardinality n for $k = 1, 2$, and $n - 2$ for $k = 3$, while for $k \geq 4$ the center spectrum is bounded by an interval of consecutive integers and additionally contains at most one value (two values) outside this interval for even diameter k (for odd diameter k) depending on the value k (Theorem 6). Note that the boundaries of the interval depend on predetermined arbitrary integer p and shrink when choosing a greater value p. For each value of the center cardinality outside this interval, the asymptotic fraction of the number of the graphs with such a center are calculated. Moreover, the graphs whose center cardinality belongs to the interval also have a nonzero asymptotic fraction (see Theorem 6 for more details). In Theorem 5 it is established realizability of the found cardinalities spectrum as the center spectrum $\mathbb{S}p_c(\mathcal{F}_{n,k,p})$ of typical n-vertex graphs of diameter k. Furthermore, typical graphs for graphs classes corresponding to the cardinality cases of the center in Theorem 6 are found in Corollaries 12, 13. Theorem 6 implies a number of properties of the centers of almost all graphs of fixed diameter k. For example, there are almost no graphs with a trivial center of diameter $k = 2, 4$ and odd diameter k, while for any even $k \geq 6$ this is not true. Similarly, there are almost no graphs with a 2-vertex center of diameter $k = 1, 3, 5$ and even diameter k, however, for every odd $k \geq 7$ this does not hold. Unexpected is the jump of the center cardinality outside the interval of the consecutive integer values both from above for odd diameter $k \geq 5$, and from below for even $k \geq 6$ and odd $k \geq 7$.

All obtained typical properties of the center and its spectrum for n-vertex graphs of fixed diameter $k \geq 2$ remain typical for connected graphs of diameter at least k, as well as for graphs (not necessarily connected) with a shortest path of length at least k. In particular, Corollary 6 is valid.

1. Preliminary information

The article uses the generally accepted concepts and notation of graph theory [4, 11], as well as the standard concepts of combinatorial analysis [10]. We consider only finite ordinary (i.e., without loops and multiple edges) graphs $G = (V, E)$ with set of vertices $V = \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$. As usual, denote by $G \setminus v$ the graph obtained as a result of *removing a vertex* v and all edges incident to it, $G \setminus V'$ is the graph obtained by removing all vertices from a subset $V' \subseteq V$, $G \setminus \{e_1, e_2, \ldots, e_k\}$ is the graph obtained as a result of *removing edges* $e_1, e_2, \ldots, e_k$ of the graph G, $G + H$ is the graph obtained by the *join operation* from graphs G and H, $deg_G\, v$ is the *degree of vertex* v of the graph G, $\delta(G)$ is the *minimum degree* of vertices of G, $B_i^G(v) = \{u \in V \mid \rho_G(v, u) \leq i\}$ is a *ball of radius* i *centered at a vertex* $v \in V$ in the metric space of the graph G with the metric ρ_G, $S_i^G(v) = \{u \in V \mid \rho_G(v, u) = i\}$ is a *sphere of radius* i *centered at a vertex* $v \in V$, K_n is a *complete* n-vertex graph, $K_{1,n-1}$ is an n-vertex *star*, C_n is an n-vertex *cycle*, P_n is an n-vertex *simple path*. For a shortest path P with endpoints v_0 and v_n, sequentially passing through vertices $v_0, v_1, \ldots, v_n$, we use the notation $P = (v_0, v_1, \ldots, v_n)$. A vertex of degree 1 is called *pendant*, a shortest path of length $d(G)$ is the *diametral path* of the graph G, and under by a *pair of diametral vertices* we mean an unordered sample of two vertices from the set V, the distance between which is equal to the diameter.

The graph $V_k(u, v)$ shown in Fig. 1a we call the *shuttlecock on the vertices* u, v [5]. A graph G (not necessarily connected) *has a shuttlecock*, if G has a subgraph $V_k(u, v)$ and $\deg_G u = \deg_G v = k+1$ (Fig. 1b). It is easy to see that a graph does not contain

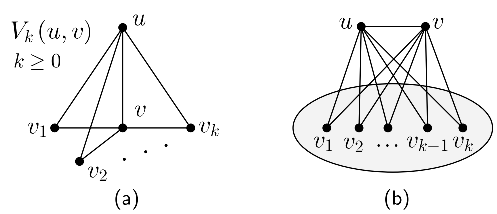

FIG. 1. The shuttlecock

shuttlecocks if and only if it does not contain coincident balls of radius 1 centered at different vertices [5].

By $(n)_k$ we denote the number of order placements from n elements by k, i.e. $(n)_k = n(n-1)\cdots(n-k+1)$, and wherein we define $(n)_k = 0$ for $n < k$ and $(n)_0 = (0)_0 = 1$. We will write $\lceil x \rceil$ ($\lfloor x \rfloor$) to denote the smallest (largest) integer greater (less) or equal to real nonnegative number x. To denote the *asymptotic equality* of functions $f(n)$ and $g(n)$ as $n \to \infty$, we use the notation $f(n) \sim g(n)$, which by definition means that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$ or, equivalently, $f(n) = g(n)(1 + r(n))$ for all large enough n, where $r(n) = o(1)$ is the *approximation error* of $g(n)$.

To estimate the measure of the number of graphs with a certain property, the concept of *almost all* is often used; in this approach, the studied property is considered for graphs with a large number of vertices. Let $\mathcal{J}_n$ be the class of labeled n-vertex graphs with the fixed set of vertices $V = \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$. Consider some property $\mathcal{P}$, by which each graph may or may not possess. Through $\mathcal{J}_n^{\mathcal{P}}$ denote the set of all graphs from $\mathcal{J}_n$ that possess the property $\mathcal{P}$. *Almost all graphs possess the property* $\mathcal{P}$ if $\lim_{n\to\infty} \frac{|\mathcal{J}_n^{\mathcal{P}}|}{|\mathcal{J}_n|} = 1$, i.e. $|\mathcal{J}_n^{\mathcal{P}}| \sim |\mathcal{J}_n|$, and *there are almost no graphs with the property* $\mathcal{P}$, if $\lim_{n\to\infty} \frac{|\mathcal{J}_n^{\mathcal{P}}|}{|\mathcal{J}_n|} = 0$.

In the study and selection of almost all graphs in the class of graphs under consideration it is often useful to define not characteristic properties themselves for the notion of almost all, but directly select a subclass of typical graphs itself (in [6,7] a more general concept of a class of typical combinatorial objects and an abstract typical combinatorial object for a given class of objects admitting the concept of dimension is formulated). Further we will also use this formal concept for graphs (when the dimension of a graph is understood as the number of its vertices). Let Ω be an arbitrary class of graphs such that $\Omega_n \neq \varnothing$ for all large enough n, where $\Omega_n = \Omega \cap \mathcal{J}_n$. A subclass $\Omega^* \subseteq \Omega$ is the *class of typical graphs of the class* Ω if

$$\lim_{n\to\infty} \frac{|\Omega_n^*|}{|\Omega_n|} = 1.$$

In [8] for every $k \geq 3$, a family of nested classes $\mathcal{F}_{n,k,p}$, $p \geq 1$ of n-vertex graphs of fixed diameter k is constructed. To define the class $\mathcal{F}_{n,k,p}$, first consider special graphs of diameter 3 and their properties. Let $x, y \in V$ and $\mathcal{F}_{n,3,p}(x, y)$ be the class of all graphs $F \in \mathcal{J}_n$ with the following properties:

a) the vertices x, y are not pendant in F;

b) $\rho_F(z, x) = \rho_F(z, y) = 2$ for some vertex $z \in V$ (a vertex with this property will be called the *pole of graph* F);

c) $d(F) = 3$, the graph F has the unique pair of diametral vertices x, y and does not contain shuttlecocks;

d) the following property of spheres holds:

$$|S_1^F(u) \cap S_1^F(v)| \geq p \ \ \forall u, v \in V \setminus \{x, y\} \text{ and } u \neq v,$$

$$|S_1^F(u) \cap S_1^F(v)| \geq p \ \ \forall v \in V \setminus \{x, y\} \ \ \forall u \in \{x, y\}.$$

Now, we define graphs of the class $\mathcal{F}_{n,k,p}$ as follows. Let $u = (u_0, u_1, \ldots, u_{k-2})$ be an arbitrary ordered sequence of different vertices from the set V. Fix an arbitrary pair of neighboring elements u_s and u_{s+1}, $0 \leq s \leq k-3$. On the set $V \setminus \{u_0, \ldots, u_{s-1}, u_{s+2}, \ldots, u_{k-2}\}$ of $n-k+3$ vertices, define an arbitrary graph F from the class $\mathcal{F}_{n-k+3,3,p}(u_s, u_{s+1})$. Finally, join by edges the vertices u_i, u_{i+1} for $i \neq s$ and $0 \leq i < k-2$. Denote the so-obtained graph by $G(u, s, F)$. Let $\mathcal{F}_{n,k,p}$ be the class of all constructed graphs $G(u, s, F)$ under condition $0 \leq s \leq \lfloor \frac{k-3}{2} \rfloor$, and let $\mathcal{F}_{n,k,p}^s$ denote the class of all graphs $G(u, s, F)$ for fixed $s \leq k-3$. In what follows, we will use the notation $G(u, s, F)$ for the graph constructed for given k, p, u, s and F, without detailing the properties $k \geq 3$, $p \geq 1$, $u = (u_0, u_1, \ldots, u_{k-2})$, $0 \leq s \leq \lfloor \frac{k-3}{2} \rfloor$ and $F \in \mathcal{F}_{n-k+3,3,p}(u_s, u_{s+1})$, unless otherwise specified.

In [8] for any fixed $k \geq 3$ and $p \geq 1$, it is proved that the class of graphs $\mathcal{F}_{n,k,p}$ is typical for each of the classes $\mathcal{J}_{n,\, d=k}$, $\mathcal{J}_{n,\, d\geq k}$ and $\mathcal{J}^*_{n,\, d\geq k}$, and also established asymptotically exact value $2^{\binom{n}{2}}\, \xi_{n,k}$ of the number of graphs in this class.

Theorem 1 [8]. *Let $k \geq 3$, $0 < \varepsilon < 1$ and $p \geq 1$ do not depend on n. Then there is a constant $c > 0$ independent of n and such that for every $n \in \mathbb{N}$ the following inequalities hold*

$$2^{\binom{n}{2}}\xi_{n,k}\Big(1 - c\Big(\tfrac{5+\varepsilon}{6}\Big)^{n-k+1}\Big) \leq |\mathcal{F}_{n,k,p}| \leq |\mathcal{J}_{n,\,d=k}|$$
$$\leq |\mathcal{J}_{n,\,d\geq k}| \leq |\mathcal{J}^*_{n,\,d\geq k}| \leq 2^{\binom{n}{2}}\xi_{n,k}\Big(1 + c\Big(\tfrac{5+\varepsilon}{6}\Big)^{n-k+1}\Big),$$

$$\textit{where } \xi_{n,k} = q_k\ (n)_{k-1}\left(\frac{3}{2^{k-1}}\right)^{n-k+1}, \quad q_k = \frac{1}{2}\,(k-2)\,2^{-\binom{k-1}{2}}.$$

Corollary 1 [8]. *Let $k \geq 3$ and $p \geq 1$ be independent of n. Then for $n \to \infty$*

$$|\mathcal{F}_{n,k,p}| \sim |\mathcal{J}_{n,\,d=k}| \sim |\mathcal{J}_{n,\,d\geq k}| \sim |\mathcal{J}^*_{n,\,d\geq k}| \sim 2^{\binom{n}{2}}\xi_{n,k}.$$

Corollary 2 [8]. *Let $k \geq 3$ and $p \geq 1$ be independent of n. Then $\mathcal{F}_{n,k,p}$ is the class of typical graphs of the class of n-vertex graphs of diameter k.*

Next, we need the following properties of the graphs $G(u, s, F)$.

Lemma 1 [8] (properties of $G(u, s, F)$). *Let $k \geq 3$, $p \geq 1$ and $G = G(u, s, F) \in \mathcal{F}_{n,k,p}$. Then the following properties hold:*

(i) $G \in \mathcal{J}_{n,d=k}$;

(ii) *the vertices u_s, u_{s+1} are not pendant in F;*

(iii) *u_0, u_{k-2} is the unique pair of diametral vertices of the graph G and u_0, $u_1, \ldots, u_{k-2} \in V(P)$ for every diametral path P.*

Corollary 3. *For a vertex v of graph $G(u, s, F) \in \mathcal{F}_{n,k,p}$, the following conditions are equivalent:*

(i) *v belongs to some diametral path;*

(ii) *$v \in S_1^F(u_s) \cup S_1^F(u_{s+1}) \cup \{u_0, u_1, \ldots, u_{k-2}\}$;*

(iii) *v is not a pole.*

Proof. In view of properties b), c) of the graph F, it is easy to understand the equivalence of statements (ii) and (iii). Moreover, property c) of the graph F implies that any vertex v from $S_1^F(u_s) \cup S_1^F(u_{s+1})$ belongs to some diametral path of the graph F with endpoints u_s, u_{s+1}. Extending it with two paths passing respectively through the vertices $u_0, \ldots, u_s$ and $u_{s+1}, \ldots, u_{k-2}$, we get the diametral path of the graph $G(u, s, F)$.

Further, note that if the vertex v is a pole, then $\rho_F(u_s, v) = \rho_F(v, u_{s+1}) = 2$. Therefore, v cannot belong to a diametral path with endpoints u_0, u_{k-2}, since this path contains u_s, u_{s+1} and $\rho_F(u_s, u_{s+1}) = 3$ by Lemma 1. □

Lemma 2. *Let $k \geq 3$, $p \geq 1$ and $0 \leq s \leq k-3$. Then the following properties hold:*

(i) $|\mathcal{F}_{n,k,p}| = \frac{1}{2}(k-2)(n)_{k-1}|\mathcal{F}_{n-k+3,3,p}(x, y)|$, *where $x \neq y$;*

(ii) $|\mathcal{F}^s_{n,k,p}| = \frac{\sigma(s)}{k-2}\,|\mathcal{F}_{n,k,p}|$, *where $\sigma(s) = 1$ for $s = \frac{k-3}{2}$ and $\sigma(s) = 2$ if $s \neq \frac{k-3}{2}$;*

(iii) $\mathcal{F}_{n,k,p} = \bigcup\limits_{s=0}^{\lfloor\frac{k-3}{2}\rfloor} \mathcal{F}^s_{n,k,p}$;

(iv) $\mathcal{F}^i_{n,k,p} \cap \mathcal{F}^j_{n,k,p} = \varnothing$ *if $i \neq j$ and $i, j \leq \lfloor\frac{k-3}{2}\rfloor$.*

Proof. Statements (i)-(ii) are proved in [8]. And statement (iii) follows from the definitions of the classes. Prove (iv). Let $G = G(u, s, F) \in \mathcal{F}_{n,k,p}$. By Lemma 1(iii), the graph G has the single pair of diametral vertices u_0, u_{k-2}. Moreover, a part of the diametral path of the graph G from a given diametral vertex to the first encountered vertex v from the graph F can be uniquely reconstructed knowing the edges of the graph G, because $\deg_F v \geq 2$ by Lemma 1(ii). Therefore, G has two such vertex-disjoint parts of its diametral path of length $s \leq \lfloor \frac{k-3}{2} \rfloor$ and $k-3-s \geq \lceil \frac{k-3}{2} \rceil$. Consequently, if $G(u, i, F) = G(u', j, F')$ and $i, j \leq \lfloor \frac{k-3}{2} \rfloor$, then $i = j$. □

Further, we use the following well-known fact.

Lemma 3 (see, for example, [9]). *The radius of a simple path of length k is equal to $\lceil \frac{k}{2} \rceil$, and the central vertices of the path are at distance $\lceil \frac{k}{2} \rceil$ and $\lfloor \frac{k}{2} \rfloor$ from its endpoints.*

2. Center spectrum of connected n-vertex graphs

Consider the class $\mathcal{J}_{n,\, d\geq 1}$ of all n-vertex connected graphs. Obviously, the diameter d of these graphs satisfies the inequalities $1 \leq d \leq n-1$. Now, we find center spectrum $\mathbb{S}p_c(\mathcal{J}_{n,\, d\geq 1})$ for $n = 2, 3, 4$.

Example 1: Let $n = 2$. In this case, there is only one connected graph K_2. Moreover, $r(K_2) = 1$ and $|\mathbb{C}(K_2)| = 2$. Thus, $\mathbb{S}p_c(\mathcal{J}_{2,\, d\geq 1}) = \{2\}$.

Example 2: Let $n = 3$. For $d = 1$ there is a unique graph K_3. Moreover, $r(K_3) = 1$ and $|\mathbb{C}(K_3)| = 3$. For $d = 2$ there is a single graph, a 3-vertex simple path P_3, for which $r(P_3) = 1$ and $|\mathbb{C}(P_3)| = 1$. Thus, $\mathbb{S}p_c(\mathcal{J}_{3,\, d\geq 1}) = \{1, 3\}$.

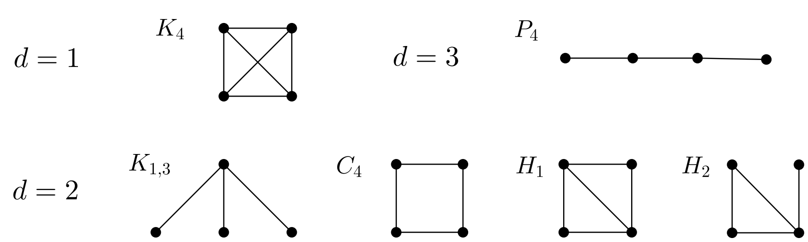

FIG. 2. 4-vertex connected graphs

Example 3: Let $n = 4$ (fig. 2). For $d = 1$ there is the only graph K_4 with $r(K_4) = 1$ and $|\mathbb{C}(K_4)| = 4$. For $d = 2$ there are four graphs: a star $K_{1,3}$, a cycle C_4, a square with diagonal H_1 and a graph-claw H_2, for which the following equalities hold:

$r(K_{1,3}) = 1$ and $|\mathbb{C}(K_{1,3})| = 1$;
$r(C_4) = 2$ and $|\mathbb{C}(C_4)| = 4$;
$r(H_1) = 1$ and $|\mathbb{C}(H_1)| = 2$;
$r(H_2) = 1$ and $|\mathbb{C}(H_2)| = 1$.

For $d = 3$ there is a single graph, a 4-vertex simple path P_4, herewith $r(P_4) = 2$ and $|\mathbb{C}(P_4)| = 2$. Thus, $\mathbb{S}p_c(\mathcal{J}_{4,\, d\geq 1}) = [[1, 4]] \setminus \{3\}$.

Now, find the center spectrum of all and almost all graphs of the class $\mathcal{J}_{n,\, d\geq 1}$. We need the following G.N. Kopylov and E.A. Timofeev's Theorem.

Theorem 2 [13]. *For $n \geq 2$, there exists a graph in the class of connected n-vertex graphs with m edges and c central vertices if and only if one of the following conditions holds:*

(i) $c \leq n,\ c \neq n-1$ *and* $\frac{c(c-1)}{2} + c(n-c) \leq m \leq \frac{n(n-2)+c}{2}$;

(ii) $c = n$ *and* $n \leq m \leq \frac{n(n-2)}{2}$;

(iii) $2 \leq c \leq n-2$ *and* $E(c,n) \leq m \leq \frac{(n-2)(n-3)}{2} + c$, *where*

$$E(c,n) = \begin{cases} n-1 & \text{if } c = 2, \\ n+1 & \text{if } c \text{ even and } n-3 \leq c \leq n-2, \\ n & \text{else.} \end{cases}$$

Theorem 3 (spectrum $\mathbb{S}p_c(\mathcal{J}_{n,\,d\geq 1})$). *The following properties are valid:*
(i) $\mathbb{S}p_c(\mathcal{J}_{n,\,d\geq 1}) = [[1,\, n]] \setminus \{n-1\}$ *for every* $n \geq 2$;
(ii) *almost all n-vertex connected graphs have a center of cardinality n.*

Proof. Prove statement (i). The case $n = 2$ is considered in Example 1. Therefore, we can assume that $n \geq 3$. Let $c \in [[1,\, n]] \setminus \{n-1\}$. Further, we find the value of the parameter m, satisfying one of conditions (i)-(iii) of Theorem 2, considering possible cases for the value c.

Let $c = 1$. Note that the inequality $n - 1 \leq \frac{n(n-2)+1}{2}$ holds for every $n \geq 3$, i.e. $m = n - 1$ satisfies condition (i) from Theorem 2.

Let $2 \leq c \leq n-4$. Then $n \geq 6$. Note that the inequality $n \leq \frac{(n-2)(n-3)}{2} + 2$ is valid for every $n \geq 6$, i.e. $m = n$ satisfies condition (iii) from Theorem 2 for every $n \geq 6$.

Let $c = n - i$, where $i = 0, 2, 3$. Then $n - i > 0$. Note that inequality

$$\frac{c(c-1)}{2} + c(n-c) \leq \frac{n(n-2)+c}{2}$$

is equivalent to inequality $(n-i)(n+i-2) \leq n(n-2)$. It is not hard to understand the validity of this inequality for $i = 0, 2, 3$, i.e. $m = (n-i)(n+i-1)/2$ satisfies condition (i) from Theorem 2.

Thus, due to Theorem 2, for all specified values of the parameter c, there exists a graph $G \in \mathcal{J}_{n,\,d\geq 1}$ such that $|\mathbb{C}(G)| = c$. To prove the converse, it suffices to note that if $G \in \mathcal{J}_{n,\,d\geq 1}$, then $1 \leq |\mathbb{C}(G)| \leq n$ and $n - 1 \notin \mathbb{S}p_c(\mathcal{J}_{n,\,d\geq 1})$ by Theorem 2.

Prove statement (ii). It is known that almost all n-vertex graphs have diameter and radius equal to 2 (see, for example, [4]). It remains to note that $\mathcal{J}_{n,\,d=2} \subseteq \mathcal{J}_{n,\,d\geq 1} \subseteq \mathcal{J}_n$ and $\{G \in \mathcal{J}_{n,\,d=2} |\, r(G) = d(G)\} \subseteq \{G \in \mathcal{J}_{n,\,d\geq 1} \mid |\mathbb{C}(G)| = n\}$. □

3. Center of almost all graphs from $\mathcal{J}_{n,\,d=k}$

Find out the structure of the center of almost all n-vertex graphs of a given diameter. A radius of almost all graphs of fixed diameter k was established in [8]. Almost all graphs of diameter $k = 1, 2$ have the radius equal to the diameter. Therefore, every vertex is central, i.e. $\mathbb{C}(G) = V(G)$ for almost all graphs G of diameter $k = 1, 2$. The radius of almost all graphs of fixed diameter $k \geq 3$ is equal to $\lceil \frac{k}{2} \rceil$. Investigate the center of such n-vertex graphs. For this, turn to the class of typical graphs $\mathcal{F}_{n,k,p}$, establish properties of their central vertices and find the center explicitly.

Lemma 4 [8]. *If $k \geq 3$, $p \geq 1$ and $G \in \mathcal{F}_{n,k,p}$, then $r(G) = \lceil \frac{k}{2} \rceil$.*

Corollary 4 [8]. *For each graph $G \in \mathcal{F}_{n,k,p}$, every central vertex of its arbitrary diametral path is the central vertex of the graph G.*

Note that the converse statement in Corollary 4, generally speaking, is not true; this will be shown below.

Corollary 5. *If P is an arbitrary diametral path of a graph $G \in \mathcal{F}_{n,k,p}$, then $\mathbb{C}(G) \cap V(P) = \mathbb{C}(P)$.*

Proof. From Corollary 4 we have $\mathbb{C}(P) \subseteq \mathbb{C}(G) \cap V(P)$. Moreover, by Lemmas 3 and 4, if $v \in P \setminus \mathbb{C}(P)$, then $e_G(v) \geq e_P(v) > r(P) = r(G)$ and, therefore, $v \notin \mathbb{C}(G)$. □

Lemma 5 (center $\mathbb{C}(G)$, $G \in \mathcal{F}_{n,k,p}$). *Let $G = G(u, s, F) \in \mathcal{F}_{n,k,p}$. Then*

(i) *if k is even, then*

$$\mathbb{C}(G) = \begin{cases} S_1^F(u_{s+1}) & \text{if } s = \frac{k}{2} - 2\,, \\ \{u_{\frac{k}{2}-2}\} & \text{else;} \end{cases}$$

(ii) *if k is odd, then*

$$\mathbb{C}(G) = \begin{cases} V \setminus \{u_0, \dots, u_{k-2}\} & \text{if } s = \frac{k-3}{2}\,, \\ B_1^F(u_{s+1}) & \text{if } s = \frac{k-5}{2}\,, \\ \{u_{\frac{k-5}{2}}, u_{\frac{k-3}{2}}\} & \text{else;} \end{cases}$$

(iii) *$\mathbb{C}(G)$ consists of all central vertices of diametral paths with endpoints u_0, u_{k-2} and all vertices, equidistant at distance $\frac{k+1}{2}$ from the vertices u_0, u_{k-2}.*

Proof. Prove statements (i) and (ii). In view of Lemma 1, a diametral path P of the graph G has the form $P = (u_0, \dots, u_s, u'_s, u'_{s+1}, u_{s+1}, \dots, u_{k-2})$. By Lemmas 3, 4 and Corollary 5, we obtain

$$r(G) = r(P) = \left\lceil \frac{k}{2} \right\rceil \text{ and } \mathbb{C}(G) \cap V(P) = \mathbb{C}(P). \tag{2}$$

Due to the properties of the graph F, the set $V(F) \setminus \{u_s, u_{s+1}\}$ consists of three types of vertices x, y, z such that

$$\begin{aligned} &\rho_G(x, u_s) = 1,\ \rho_G(x, u_{s+1}) = 2, \\ &\rho_G(y, u_{s+1}) = 1,\ \rho_G(y, u_s) = 2, \\ &\rho_G(z, u_s) = \rho_G(z, u_{s+1}) = 2. \end{aligned} \tag{3}$$

Further, by x, y, z we mean an arbitrary vertex from $F \setminus \{u_s, u_{s+1}\}$ with the above metric relations. From condition $s \leq \lfloor \frac{k-3}{2} \rfloor$ we have $s' = k-3-s \geq \lceil \frac{k-3}{2} \rceil$. Consider the possible cases.

Case 1. Let $s = s'$. Then k is odd and $s = \frac{k-3}{2}$. By (2), we have $\mathbb{C}(G) \cap V(P) = \{u'_s, u'_{s+1}\}$. Considering the form of the graph $G(u, s, F)$ and the relation (3), we obtain $e_G(x) = e_G(y) = e_G(z) = s + 2 = \lceil \frac{k}{2} \rceil$. Therefore, $x, y, z \in \mathbb{C}(G)$. Hence,

$$\mathbb{C}(G) = \{u'_s, u'_{s+1}\} \cup V(F) \setminus \{u_s, u_{s+1}\} = V \setminus \{u_0, \dots, u_{k-2}\}.$$

Case 2. Let k be even and $s < s'$. Then

$$s' \geq \frac{k}{2} - 1 > \frac{k}{2} - 2 \geq s. \tag{4}$$

It follows from (2) and (4) that

$$\mathbb{C}(G) \cap V(P) = \begin{cases} \{u'_{s+1}\} & \text{if } s' = \frac{k}{2} - 1\,, \\ \{u_{\frac{k}{2}-2}\} & \text{else.} \end{cases} \tag{5}$$

Reckoning the form of the graph $G(u,s,F)$, the relations (3) and (4), we obtain

$$\begin{aligned} e_G(x) &= \max\{s+1, s'+2\} = s'+2 \geq 0.5k+1, \\ e_G(y) &= \max\{s+2, s'+1\} = s'+1 \geq 0.5k, \\ e_G(z) &= \max\{s+2, s'+2\} = s'+2 \geq 0.5k+1. \end{aligned}$$

Therefore, $x, z \notin \mathbb{C}(G)$ due to (2) and the following equivalence holds

$$y \in \mathbb{C}(G) \Leftrightarrow s'+1 = \frac{k}{2} \Leftrightarrow s = \frac{k}{2} - 2.$$

Hence,

$$\mathbb{C}(G) \setminus V(P) = \begin{cases} S_1^F(u_{s+1}) & \text{if } s = \frac{k}{2} - 2\,, \\ \varnothing & \text{else.} \end{cases} \tag{6}$$

Thus, from (5), (6) we obtain the required form $\mathbb{C}(G)$ for even k.

Case 3. Let k be odd and $s < s'$. Then

$$s' > \frac{k-3}{2} > s. \tag{7}$$

It follows from (2) and (7) that

$$\mathbb{C}(G) \cap V(P) = \begin{cases} \{u'_{s+1}, u_{s+1}\} & \text{if } s' = \frac{k-1}{2}\,, \\ \{u_{\frac{k-5}{2}}, u_{\frac{k-3}{2}}\} & \text{else.} \end{cases} \tag{8}$$

Reckoning the form of the graph $G(u,s,F)$, the relations (3) and (7), we obtain

$$\begin{aligned} e_G(x) &= \max\{s+1, s'+2\} = s'+2 \geq 0.5(k+1)+1, \\ e_G(y) &= \max\{s+2, s'+1\} = s'+1 \geq 0.5(k+1), \\ e_G(z) &= \max\{s+2, s'+2\} = s'+2 \geq 0.5(k+1)+1. \end{aligned}$$

Therefore, $x, z \notin \mathbb{C}(G)$ by (2) and

$$y \in \mathbb{C}(G) \Leftrightarrow s'+1 = \frac{k+1}{2} \Leftrightarrow s = \frac{k-5}{2}.$$

Hence,

$$\mathbb{C}(G) \setminus V(P) = \begin{cases} S_1^F(u_{s+1}) & \text{if } s = \frac{k-5}{2}\,, \\ \varnothing & \text{else.} \end{cases} \tag{9}$$

It follows from (8) and (9) that

$$\mathbb{C}(G) = \begin{cases} B_1^F(u_{s+1}) & \text{if } s = \frac{k-5}{2}\,, \\ \{u_{\frac{k-5}{2}}, u_{\frac{k-3}{2}}\} & \text{else.} \end{cases}$$

Thus, the form $\mathbb{C}(G)$ required in statement (ii) for odd k is obtained in cases 1, 3.

Prove statement (iii). By Corollary 4, every central vertex of a diametral path of the graph G belongs to $\mathbb{C}(G)$. Now let v be equidistant at distance $\frac{k+1}{2}$ from the both vertices u_0, u_{k-2}. Then the vertex v does not belong to diametral paths of the graph G by Lemma 1(iii). By corollary 3, the vertex v is a pole of the graph F. Therefore, $s = \frac{k-3}{2}$. Hence, $v \in \mathbb{C}(G)$ by the obtained statement (ii).

Prove the converse statement. Let $v \in \mathbb{C}(G)$. Note that that if $s = \frac{k-3}{2}$ and the vertex v is a pole, then v is equidistant at distance $\frac{k+1}{2}$ from the diametral vertices u_0, u_{k-2}. Using statements (i) and (ii), it is easy to understand that in other cases the vertex v is at distance $\lceil \frac{k}{2} \rceil$ from one of the vertices u_0, u_{k-2} and belongs to

$V(P) \cup S_1^F(u_s) \cup S_1^F(u_{s+1})$. Hence, v is the central vertex of some diametral path of the graph G with endpoints u_0, u_{k-2} by Corollary 3 and Lemmas 1(iii), 3. □

Theorem 4 (Center of almost all graphs from $\mathcal{J}_{n,\,d=k}$). *Let $k \geq 3$ be a fixed integer. Then*

(i) *the center of almost all n-vertex graphs of even diameter k consists of all central vertices of diametral paths of the graph*;

(ii) *the center of almost all n-vertex graphs of odd diameter k consists of all central vertices of diametral paths of the graph and all vertices equidistant at distance $\frac{k+1}{2}$ from their endpoints. Furthermore, the proportion of such n-vertex graphs whose center consists only of central vertices of diametral paths of the graph is asymptotically equal to $\frac{k-3}{k-2}$.*

Proof. Directly from Corollary 1 and Lemmas 1(iii), 5(iii) we obtain the required properties of the central vertices of almost all n-vertex graphs of a given diameter.

By $\mathcal{K}_{n,k}$ we denote the class of all graphs from $\mathcal{J}_{n,\,d=k}$ whose center consist only of all central vertices of diametral paths of the graph. Find out a fraction of such graphs of odd diameter k. Let $s^* = \frac{k-3}{2}$. From Lemma 5 and the proof of its statement (iii) it follows that $\mathcal{F}^s_{n,k,p} \subseteq \mathcal{K}_{n,k}$ for every $s \neq s^*$ and $\mathcal{F}^{s^*}_{n,k,p} \cap \mathcal{K}_{n,k} = \varnothing$. Hence,

$$\bigcup_{0 \leq s < \frac{k-3}{2}} \mathcal{F}^s_{n,k,p} \subseteq \mathcal{K}_{n,k} \subseteq \mathcal{J}_{n,\,d=k} \setminus \mathcal{F}^{s^*}_{n,k,p}\,. \tag{10}$$

From Lemma 2 we obtain

$$\Big|\bigcup_{0 \leq s < \frac{k-3}{2}} \mathcal{F}^s_{n,k,p}\Big| = |\mathcal{F}_{n,k,p} \setminus \mathcal{F}^{s^*}_{n,k,p}| = |\mathcal{F}_{n,k,p}|\Big(1 - \frac{\sigma(s^*)}{k-2}\Big). \tag{11}$$

By Lemmas 1(i) and 2, we have

$$|\mathcal{J}_{n,\,d=k} \setminus \mathcal{F}^{s^*}_{n,k,p}| = |\mathcal{J}_{n,\,d=k}| - \frac{\sigma(s^*)}{k-2}\,|\mathcal{F}_{n,k,p}|. \tag{12}$$

Thus, from (10)-(12) and Corollary 1 as $n \to \infty$ we conclude

$$\frac{|\mathcal{K}_{n,k}|}{|\mathcal{J}_{n,\,d=k}|} \longrightarrow 1 - \frac{\sigma(s^*)}{k-2} = \frac{k-3}{k-2}.$$

□

Theorem 4 implies that to obtain almost all graphs G of odd diameter $k \geq 3$ in the center $\mathbb{C}(G)$ cannot do without vertices equidistant at distance $\frac{k+1}{2}$ from endpoints of its diametral paths (since the class $\mathcal{K}_{n,k}$ does not asymptotically cover the whole class $\mathcal{J}_{n,\,d=k}$), and for $k \geq 5$ it is also impossible to do without central vertices of diametral paths (because the class $\mathcal{K}_{n,k}$ has a nonzero asymptotic fraction). In addition, note that Fred Buckley and Martin Lewinter investigated the class of graphs (L'-graphs), centers of which do not contain vertices lying on diametral paths, and established characteristic limitations between the diameter and the radius for these graphs [**?**]. In particular, they showed that there are no such graphs of diameter 3. It means that for $k = 3$ in the center $\mathbb{C}(G)$ it is also impossible to do without central vertices of diametral paths.

4. Center spectrum of almost all graphs from $\mathcal{J}_{n,\,d=k}$

Let us investigate the center spectrum of almost all n-vertex graphs of a fixed diameter. For this, turn to the class of typical graphs $\mathcal{F}_{n,k,p}$.

Lemma 6. *Let $G = G(u, s, F) \in \mathcal{F}_{n,k,p}$. Then*
(i) *if k is even, then*

$$|\mathbb{C}(G)| = 1 \Leftrightarrow s \neq \frac{k}{2} - 2,$$

$$p + 1 \leq |\mathbb{C}(G)| \leq n - k - p - 1 \text{ for } s = \frac{k}{2} - 2;$$

(ii) *if k is odd, then*

$$|\mathbb{C}(G)| = 2 \Leftrightarrow s \neq \frac{k-3}{2} \text{ and } s \neq \frac{k-5}{2},$$

$$p + 2 \leq |\mathbb{C}(G)| \leq n - k - p \text{ for } s = \frac{k-5}{2},$$

$$|\mathbb{C}(G)| = n - k + 1 \Leftrightarrow s = \frac{k-3}{2}.$$

Proof. Let $P = (u_0, \ldots, u_s, u'_s, u'_{s+1}, u_{s+1}, \ldots, u_{k-2})$ be an arbitrary diametral path of the graph G. From the property of spheres of the graph F we have the inequalities $|S_1^F(u_s) \cap S_1^F(u'_s)| \geq p$ and $|S_1^F(u_{s+1}) \cap S_1^F(u'_{s+1})| \geq p$. Therefore,

$$|S_1^F(u_s)| \geq p + 1, \; |S_1^F(u_{s+1})| \geq p + 1. \tag{13}$$

From the property of a pole of the graph F we obtain

$$\exists z \in V(F) \setminus (B_1^F(u_s) \cup B_1^F(u_{s+1})).$$

It is also easy to see that the sets $\{u_0, \ldots, u_{k-2}\}$, $\{z\}$, $S_1^F(u_s)$, $S_1^F(u_{s+1})$ are pairwise disjoint. Hence,

$$|V| \geq |(S_1^F(u_s)| + |S_1^F(u_{s+1})| + k \geq 2p + k + 2. \tag{14}$$

Now, for even k, from Lemma 5 and the relations (13), (14) we obtain the required statement (i).

Let k be odd. Due to the inequality (14) and the condition $p \geq 1$, the sets $\{2\}$, $\{p+2, p+3, \ldots, n-k-p\}$, $\{n-k+1\}$, $\{n-k+1\}$ are pairwise disjoint. Now from Lemma 5 and the relations (13), (14) we obtain the statement (ii). □

Corollary 6. *If $G \in \mathcal{F}_{n,k,p}$, then $n \geq 2p + k + 2$.*

Let us show the realizability of all center cardinalities indicated in Lemma 6 in graphs of the class $\mathcal{F}_{n,k,p}$ for all large enough n. For this purpose, give a method of constructing the graphs $G(u, s, F)$ of the class $\mathcal{F}_{n,k,p}$ using two constructions that allow us to construct graphs $F \in \mathcal{F}_{n,3,p}(x, y)$.

First, for any $m \geq 1$ define m-vertex graph H_m in the following way. We put $H_1 = K_1$. For $m \geq 2$, fix pairwise non-adjacent edges $e_i = v_i v'_i$, $i = 1, 2, \ldots, \lfloor m/2 \rfloor$ in a complete graph $K_{2\lfloor m/2 \rfloor}$. If m even, put $H_m = K_{2\lfloor m/2 \rfloor} \setminus \{e_1, \ldots, e_{\lfloor m/2 \rfloor}\}$, else $H_m = K_1 + K_{2\lfloor m/2 \rfloor} \setminus \{e_1, \ldots, e_{\lfloor m/2 \rfloor}\}$.

Lemma 7 (properties H_m)**.** *The following properties of the graph H_m hold:*
(i) $\delta(H_m) \geq m - 2$;
(ii) *H_m does not contain shuttlecocks.*

Proof. Every vertex of the graph H_m is not joined by an edge with at most one vertex, therefore $\delta(H_m) \geq m-2$. Prove statement (ii). Let e be an edge of the graph H_m. Then $m \geq 3$. If the edge e is of the form $v_i v_j$ (the case $e = v_i' v_j'$ is similar), then e belongs to the shortest path (v_i, v_j, v_i') of length 2. In the case when e has the form $v_i v_j'$ (the case $e = v_i' v_j$ is similar), the edge e belongs to the shortest path (v_j', v_i, v_j) of length 2. There remains the case when m is odd and $e = v_0 v_i$ (similarly $e = v_0 v_i'$), where v_0 is a vertex adjacent to all other vertices. Then H_m contains the shortest path (v_i, v_0, v_i') of length 2. Hence, in all cases H_m does not contain a shuttlecock on the edge e. □

Let $P = (x, x', y', y)$ be a 4-vertex simple path with endpoints x, y, G_1 and G_2 are arbitrary graphs, and z is a new vertex, moreover, the sets $V(G_1)$, $V(G_2)$, $\{x, x', y', y\}$, $\{z\}$ are pairwise disjoint. Join each vertex of the graph G_1 by edges with vertices x, x', y', z and connect each vertex of the graph G_2 by edges with vertices x', y', y, z. Also, join each vertex of the graph G_1 by edges with all vertices of the graph G_2. The resulting graph denote by $F(G_1 + G_2)$ (see Fig. 3).

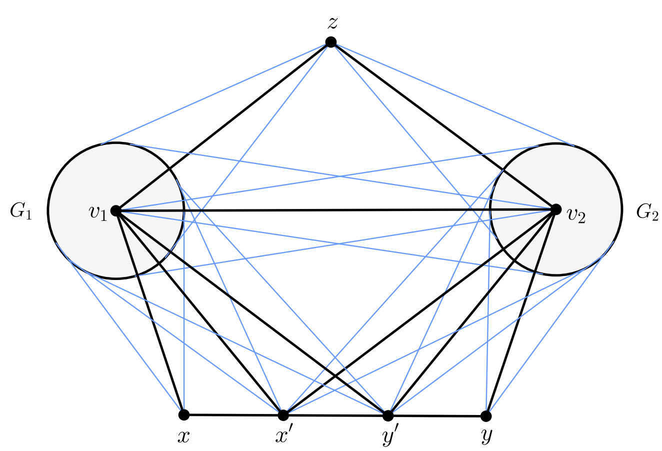

FIG. 3. Graph $F(G_1 + G_2)$

Lemma 8 (graph $F(G_1 + G_2)$). *Let graphs G_1, G_2 do not contain shuttlecocks and $\delta(G_1) \geq p-1$, $\delta(G_2) \geq p-1$, $p \geq 1$. Then $F(G_1 + G_2) \in \mathcal{F}_{n,3,p}(x, y)$, where $n = |V(G_1)| + |V(G_2)| + 5$.*

Proof. Denote the graph $F(G_1 + G_2)$ by F. It is directly verified that the graph F has diameter 3, z is its pole and x, y is the unique pair of diametral vertices.

Let us show that F does not contain shuttlecocks. Let $e \in E(F)$. If $e \in E(G_i)$, then the edge e in the graph G_i belongs to some shortest path of length 2, as far as G_i does not contain shuttlecocks. Since this shortest path is also the shortest in the graph F, the graph F does not contain a shuttlecock on the edge e. Now let $e \notin E(G_1) \cup E(G_2)$. Then in the graph F the edge e belongs to one of the following shortest paths of length 2: (z, v_1, u_1), (z, v_2, u_2), (x, x', y'), (x', y', y), (v_1, v_2, y), where $v_1 \in V(G_1)$, $v_2 \in V(G_2)$, $u_1 \in \{x, x', y'\}$ and $u_2 \in \{x', y', y\}$. Hence, F does not contain a shuttlecock on the edge e.

Check the property of spheres. Let $v_1 \in V(G_1)$ and $v_2 \in V(G_2) \cup \{x', y', z\}$. Then the following inequalities hold

$$|S_1^F(x) \cap S_1^F(v_2)| \geq |V(G_1)| \geq \delta(G_1) + 1 \geq p,$$

$$|S_1^F(x) \cap S_1^F(v_1)| \geq |S_1^{G_1}(v_1) \cup \{x'\}| \geq p.$$

The same property of spheres for the diametral vertex y is established similarly. Let $v_i, v_i' \in V(G_i) \cup \{x', y', z\}$, $w_i \in V(G_i)$ and $v_i \neq v_i'$, $i = 1, 2$. Then the following inequalities hold

$$|S_1^F(v_i) \cap S_1^F(v_i')| \geq |V(G_j)| \geq p, \text{ where } j \neq i,$$

$$|S_1^F(w_1) \cap S_1^F(w_2)| \geq |S_1^{G_1}(w_1) \cup \{z\}| \geq p.$$

□

Directly from Lemmas 7 and 8 we obtain the following consequences (here, for all specified values of the parameters, we define a graph $F \in \mathcal{F}_{n,3,p}(x, y)$ such that $|S_1^F(y)| = m + 1$).

Corollary 7 $(1 = p = m \leq n - 5 - p)$. *For all $n \geq 8$ the graph $F(H_{n-6} + K_1)$ belongs to the class $\mathcal{F}_{n,3,1}(x, y)$ and $|S_1^{F(H_{n-6}+K_1)}(y)| = 2$.*

Corollary 8 $(1 = p < m = n - 5 - p)$. *For all $n \geq 8$ the graph $F(H_{n-6} + K_1)$ belongs to the class $\mathcal{F}_{n,3,1}(y, x)$ and $|S_1^{F(H_{n-6}+K_1)}(x)| = n - 5$.*

Corollary 9 $(1 \leq p < m < n - 5 - p)$. *Let $p \geq 1$. Then for all $n \geq 2p + 7$ and any integer m such that $p < m < n - 5 - p$, the graph $F(H_{n-5-m} + H_m)$ belongs to the class $\mathcal{F}_{n,3,p}(x, y)$ and $|S_1^{F(H_{n-5-m}+H_m)}(y)| = m + 1$.*

For arbitrary graphs G_1 and G_2 such that $|V(G_1)| \geq |V(G_2)|$, define a graph $F(G_1 \circledast G_2)$ in the following way. Consider an arbitrary subset $W \subseteq V(G_1)$ of the cardinality $|V(G_2)|$ and bijection $\varphi : W \to V(G_2)$. Put $F(G_1 \circledast G_2) = F(G_1 + G_2) \setminus \{e \,|\, e$ is an edge with endpoints $w \in W$ and $\varphi(w) \in V(G_2)\}$ (see Fig. 4).

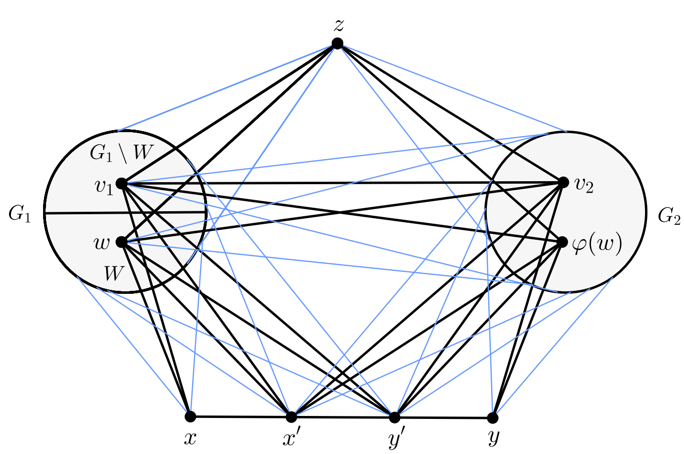

FIG. 4. Graph $F(G_1 \circledast G_2)$

Lemma 9 (graph $F(G_1 \circledast G_2)$)**.** *Let G_1 and G_2 be graphs such that $|V(G_1)| \geq |V(G_2)| \geq 2$, G_1 does not contain shuttlecocks and $\delta(G_1) \geq p-1$, $\delta(G_2) \geq p-1$, $p \geq 1$. Then $F(G_1 \circledast G_2) \in \mathcal{F}_{n,3,p}(x,y)$, where $n = |V(G_1)| + |V(G_2)| + 5$.*

Proof. Denote the graph $F(G_1 \circledast G_2)$ by F. Directly it is verified that the graph F has diameter 3, z is its pole, and x, y is the unique pair of diametral vertices.

Show that F does not contain shuttlecocks. Let $e \in E(F)$. The case $e \notin E(G_2)$ is considered in the same way as in Lemma 8. Now let e be an edge of the graph G_2 with endpoints $v_2, v_2' \in V(G_2)$. Then F contains the shortest path $(v_2, v_2', \varphi^{-1}(v_2))$ of length 2. Hence, F does not contain a shuttlecock on the edge e.

Check the property of spheres. Let $u \in \{x', y', z\}$ and $v_i \in V(G_i)$, $i = 1, 2$. Then the following inequalities hold

$$|S_1^F(x) \cap S_1^F(u)| \geq |V(G_1)| \geq p,$$
$$|S_1^F(x) \cap S_1^F(v_1)| \geq |S_1^{G_1}(v_1) \cup \{x'\}| \geq p,$$
$$|S_1^F(x) \cap S_1^F(v_2)| \geq |(V(G_1) \setminus \{\varphi^{-1}(v_2)\}) \cup \{x'\}| \geq p,$$
$$|S_1^F(y) \cap S_1^F(u)| \geq |V(G_2)| \geq p,$$
$$|S_1^F(y) \cap S_1^F(v_2)| \geq |S_1^{G_2}(v_2) \cup \{y'\}| \geq p,$$
$$|S_1^F(y) \cap S_1^F(v_1)| \geq |(V(G_2) \setminus V_2) \cup \{y'\}| \geq p, \text{ where}$$
$$V_2 = \begin{cases} \{\varphi(v_1)\} & \text{if } v_1 \in W, \\ \varnothing & \text{else.} \end{cases}$$

Let $u, u' \in \{x', y', z\}$, $u \neq u'$ and $v_i, v_i' \in V(G_i)$, $v_i \neq v_i'$, $i = 1, 2$. Reckoning the condition $|V(G_i)| \geq 2$, we obtain the following inequalities

$$|S_1^F(v_i) \cap S_1^F(v_i')| \geq |(V(G_j) \setminus V_j) \cup \{x', y'\}| \geq p,$$
$$|S_1^F(v_i) \cap S_1^F(u)| \geq |S_1^{G_i}(v_i) \cup (V(G_j) \setminus V_j')| \geq \delta(G_i) + 1 \geq p,$$
$$|S_1^F(v_1) \cap S_1^F(v_2)| \geq |S_1^{G_1}(v_1) \setminus \{\varphi^{-1}(v_2\} \cup \{x', y'\}| \geq \delta(G_1) + 1 \geq p,$$
$$|S_1^F(u) \cap S_1^F(u')| \geq |V(G_1)| \geq p, \text{ where } j \neq i \text{ and}$$

$$V_j = \begin{cases} \{\varphi^{-1}(v_i),\, \varphi^{-1}(v_i')\} & \text{if } i = 2, \\ \{\varphi(v_i),\, \varphi(v_i')\} & \text{if } i = 1 \text{ and } v_i, v_i' \in W, \\ \{\varphi(v_i)\} & \text{if } i = 1 \text{ and } v_i \in W,\ v_i' \notin W, \\ \{\varphi(v_i')\} & \text{if } i = 1 \text{ and } v_i' \in W,\ v_i \notin W, \\ \varnothing & \text{else;} \end{cases}$$

$$V_j' = \begin{cases} \{\varphi^{-1}(v_i)\} & \text{if } i = 2, \\ \{\varphi(v_i)\} & \text{if } i = 1 \text{ and } v_i \in W, \\ \varnothing & \text{else.} \end{cases}$$

□

Directly from Lemmas 7 and 9 we obtain the following consequences (here, for all specified values of the parameters, we define a graph $F \in \mathcal{F}_{n,3,p}(x,y)$ such that $|S_1^F(y)| = m + 1$).

Corollary 10 ($2 \leq p = m \leq n-5-p$)**.** *Let $p \geq 2$. Then for all $n \geq 2p+6$ the graph $F(H_{n-5-p} \circledast K_p)$ belongs to the class $\mathcal{F}_{n,3,p}(x,y)$ and $|S_1^{F(H_{n-5-p} \circledast K_p)}(y)| = p+1$.*

Corollary 11 ($2 \le p < m = n-5-p$)**.** *Let $p \ge 2$. Then for all $n \ge 2p+6$ the graph $F(H_{n-5-p} \circledast K_p)$ belongs to the class $\mathcal{F}_{n,3,p}(y,x)$ and $|S_1^{F(H_{n-5-p} \circledast K_p)}(x)| = n-4-p$.*

Theorem 5 (spectrum $\mathbb{S}p_c(\mathcal{F}_{n,k,p})$)**.** *Let $k \ge 3$ and $p \ge 1$. Then for every $n \ge 2p+k+4$ the following equalities hold*

(i) $\mathbb{S}p_c(\mathcal{F}_{n,k=3,p}) = \{n-2\}$;
(ii) $\mathbb{S}p_c(\mathcal{F}_{n,k=4,p}) = [[\,1+p,\, n-5-p\,]]$;
(iii) $\mathbb{S}p_c(\mathcal{F}_{n,k=5,p}) = [[\,2+p,\, n-5-p\,]] \cup \{n-4\}$;
(iv) $\mathbb{S}p_c(\mathcal{F}_{n,k,p}) = \{1\} \cup [[\,1+p,\, n-k-1-p\,]]$ *for every even* $k \ge 6$;
(v) $\mathbb{S}p_c(\mathcal{F}_{n,k,p}) = \{2\} \cup [[\,2+p,\, n-k-p\,]] \cup \{n-k+1\}$ *for every odd* $k \ge 7$.

Proof. From Lemma 6 we obtain the inclusion of the spectrum $\mathbb{S}p_c(\mathcal{F}_{n,k,p})$ into the set of values of the center cardinalities indicated in the statement of the theorem. Show that any such value is realized as the cardinality of the center of a suitable graph from the class $\mathcal{F}_{n,k,p}$ for all $n \ge 2p+k+4$.

Let $k \ge 4$ be even. By Lemmas 5 and 6, for every $k \ge 4$ and $n \ge 2p+k+4$, it is required to construct graphs $G(u,s,F)$ such that $s = \frac{k}{2}-2$, $F \in \mathcal{F}_{n-k+3,3,p}(u_s,u_{s+1})$, $|S_1^F(u_{s+1})| = m+1$ and m can take any value satisfying the inequalities $p \le m \le n-k-2-p$. Equivalently, for every $n \ge 2p+7$ and m, $p \le m \le n-5-p$, it is required to construct a graph $F \in \mathcal{F}_{n,3,p}(x,y)$ such that $|S_1^F(y)| = m+1$. For m the following cases are possible:

$1 = p = m \le n-5-p$ (Corollary 7),
$1 = p < m = n-5-p$ (Corollary 8),
$1 \le p < m < n-5-p$ (Corollary 9),
$2 \le p = m \le n-5-p$ (Corollary 10),
$2 \le p < m = n-5-p$ (Corollary 11).

For these cases, in the above corollaries it is shown the existence of the required graph F. Further, for every $k \ge 6$ and each graph $F \in \mathcal{F}_{n-k+3,3,p}(u_s,u_{s+1})$ (for example, for $F(H_{p+1}+H_{n-k-3-p})$ if $n \ge 2p+k+4$), we have $|\mathbb{C}(G)| = 1$ by Lemma 6, where $G = G(u,0,F) \in \mathcal{F}_{n,k,p}$. Thus, for any even diameter $k \ge 4$, each value of the center cardinality indicated in the statement of the theorem is realized in graphs of the class $\mathcal{F}_{n,k,p}$ for all $n \ge 2p+k+4$.

Now let $k \ge 3$ be odd. Note that for each graph $F \in \mathcal{F}_{n-k+3,3,p}(u_s,u_{s+1})$ (for example, for $F(H_{p+1}+H_{n-k-3-p})$ if $n \ge 2p+k+4$), we have $|\mathbb{C}(G)| = n-k+1$ by Lemma 6, where $G = G(u,\frac{k-3}{2},F) \in \mathcal{F}_{n,k,p}$.

By Lemmas 5 and 6, it is required for every $k \ge 5$ and $n \ge 2p+k+4$ to construct graphs $G(u,s,F)$ such that $s = \frac{k-5}{2}$, $F \in \mathcal{F}_{n-k+3,3,p}(u_s,u_{s+1})$, $|B_1^F(u_{s+1})| = m$ and m can take any value satisfying the inequalities $2+p \le m \le n-k-p$. Equivalently, for every $n \ge 2p+7$ and m, $p \le m \le n-5-p$, it is required to construct a graph $F \in \mathcal{F}_{n,3,p}(x,y)$ such that $|S_1^F(y)| = m+1$. The existence of such graph was proved above when considering the case of even k.

Further, for every $k \ge 7$, $n \ge 2p+k+4$ and each graph $F \in \mathcal{F}_{n-k+3,3,p}(u_s,u_{s+1})$, we have $|\mathbb{C}(G)| = 2$ by Lemma 6, where $G = G(u,\frac{k-7}{2},F) \in \mathcal{F}_{n,k,p}$. Thus, for any odd diameter $k \ge 3$ the spectrum $\mathbb{S}p_c(\mathcal{F}_{n,k,p})$ has the required form. □

Theorem 6 (spectrum of almost all graphs from $\mathcal{J}_{n,\,d=k}$)**.** *Let $k \ge 1$ and $p \ge 1$ be fixed integer constants. Then*

(i) $|\mathbb{C}(G)| = n$ *for almost all n-vertex graphs G of diameter $k = 1,2$;*

(ii) $|\mathbb{C}(G)| = n-2$ *for almost all n-vertex graphs G of diameter* 3;

(iii) $|\mathbb{C}(G)| \in \big[\big[1+p,\, n-5-p\big]\big]$ *for almost all n-vertex graphs G of diameter* 4;

(iv) $|\mathbb{C}(G)| \in \big[\big[2+p,\, n-5-p\big]\big] \cup \{n-4\}$ *for almost all n-vertex graphs G of diameter* 5; *moreover, the fraction of such graphs with an $(n-4)$-vertex center asymptotically equals* $\frac{1}{3}$;

(v) $|\mathbb{C}(G)| \in \{1\} \cup \big[\big[1+p,\, n-k-1-p\big]\big]$ *for almost all n-vertex graphs G of even fixed diameter $k \geq 6$; moreover, the fraction of such graphs with a trivial center asymptotically equals* $\frac{k-4}{k-2}$;

(vi) $|\mathbb{C}(G)| \in \{2\} \cup \big[\big[2+p,\, n-k-p\big]\big] \cup \{n-k+1\}$ *for almost all n-vertex graphs G of odd fixed diameter $k \geq 7$; moreover, the fraction of such graphs with a 2-vertex and an $(n-k+1)$-vertex center asymptotically equals* $\frac{k-5}{k-2}$ *and* $\frac{1}{k-2}$ *respectively.*

Proof. In Section 3, we noticed that $\mathbb{C}(G) = V(G)$ for almost all graphs G of diameter $k = 1, 2$. Further, directly from Corollary 1 and Theorem 5, we obtain the possible values of the center cardinality of almost all n-vertex graphs of fixed diameter $k \geq 3$ for the cases (ii)-(vi). Find the asymptotic fractions of the indicated classes of graphs. By $\mathbb{C}_{n,k,i}$ we denote the class of all graphs from $\mathcal{J}_{n,\,d=k}$ with an i-vertex center.

Let $k \geq 4$ be even and $s^* = \frac{k}{2} - 2$. Note that, by Lemmas 2 and 6, the following inclusions hold

$$\mathcal{F}_{n,k,p} \setminus \mathcal{F}^{s^*}_{n,k,p} \subseteq \mathbb{C}_{n,k,1} \subseteq \mathcal{J}_{n,\,d=k} \setminus \mathcal{F}^{s^*}_{n,k,p}. \tag{15}$$

Hence, by Lemmas 1(i) and 2, obtain

$$|\mathcal{F}_{n,k,p}|\Big(1 - \frac{\sigma(s^*)}{k-2}\Big) \leq |\mathbb{C}_{n,k,1}| \leq |\mathcal{J}_{n,\,d=k}| - \frac{\sigma(s^*)}{k-2}\, |\mathcal{F}_{n,k,p}|. \tag{16}$$

Thus, from (16), Lemma 2 and Corollary 1 as $n \to \infty$ we conclude

$$\frac{|\mathbb{C}_{n,k,1}|}{|\mathcal{J}_{n,\,d=k}|} \longrightarrow 1 - \frac{\sigma(s^*)}{k-2} = \frac{k-4}{k-2}.$$

Now, let $k \geq 3$ be odd and $s^* = \frac{k-3}{2}$. By Lemmas 2, 6 and Corollary 6, the following inclusions hold

$$\mathcal{F}^{s^*}_{n,k,p} \subseteq \mathbb{C}_{n,k,n-k+1} \subseteq \mathcal{J}_{n,\,d=k} \setminus (\mathcal{F}_{n,k,p} \setminus \mathcal{F}^{s^*}_{n,k,p}). \tag{17}$$

Hence, by Lemmas 1(i) and 2, obtain

$$\frac{\sigma(s^*)}{k-2}\, |\mathcal{F}_{n,k,p}| \leq |\mathbb{C}_{n,k,n-k+1}| \leq |\mathcal{J}_{n,\,d=k}| - |\mathcal{F}_{n,k,p}|\Big(1 - \frac{\sigma(s^*)}{k-2}\Big). \tag{18}$$

Thus, from (18), Lemma 2 and Corollary 1 as $n \to \infty$ we conclude

$$\frac{|\mathbb{C}_{n,k,n-k+1}|}{|\mathcal{J}_{n,\,d=k}|} \longrightarrow \frac{\sigma(s^*)}{k-2} = \frac{1}{k-2}.$$

Finally, let $k \geq 7$ be odd and $s^{**} = \frac{k-5}{2}$. By Lemmas 2 and 6, the following inclusions hold

$$\mathcal{F}_{n,k,p} \setminus (\mathcal{F}^{s^*}_{n,k,p} \cup \mathcal{F}^{s^{**}}_{n,k,p}) \subseteq \mathbb{C}_{n,k,2} \subseteq \mathcal{J}_{n,\,d=k} \setminus (\mathcal{F}^{s^*}_{n,k,p} \cup \mathcal{F}^{s^{**}}_{n,k,p}). \tag{19}$$

Hence, by Lemmas 1(i) and 2, obtain

$$|\mathcal{F}_{n,k,p}|\Big(1 - \frac{\sigma(s^*) + \sigma(s^{**})}{k-2}\Big) \leq |\mathbb{C}_{n,k,2}| \leq |\mathcal{J}_{n,\,d=k}| - \frac{\sigma(s^*) + \sigma(s^{**})}{k-2}\, |\mathcal{F}_{n,k,p}|. \tag{20}$$

Thus, from (20), Lemma 2 and Corollary 1 as $n \to \infty$ we conclude

$$\frac{|\mathbb{C}_{n,k,2}|}{|\mathcal{J}_{n,\,d=k}|} \longrightarrow 1 - \frac{\sigma(s^*) + \sigma(s^{**})}{k-2} = \frac{k-5}{k-2}.$$

□

Theorem 6 implies a number of properties of the centers of almost all graphs of fixed diameter k. For example, there are almost no graphs with a trivial center of diameter $k = 2, 4$ and odd diameter k, while for any even $k \geq 6$ this is not true. Similarly, there are almost no graphs with a 2-vertex center of diameter $k = 1, 3, 5$ and even diameter k, however, for every odd $k \geq 7$ this does not hold. Unexpected is the jump of the center cardinality outside the interval of consecutive integer values both from above from $n - k - p$ to $n - k + 1$ for odd $k \geq 5$, and from below from $1 + p$ to 1 for even $k \geq 6$ and from $2 + p$ to 2 for odd $k \geq 7$.

In the following corollaries we find typical graphs for graphs classes corresponding to the cardinality cases of the center in Theorem 6.

Corollary 12. *Let $k \geq 4$ be an even integer, $p \geq 1$ and $s^* = \frac{k}{2} - 2$. Then*

(i) *$\mathcal{F}^{s^*}_{n,k,p}$ is a class of typical graphs of the class of n-vertex graphs of fixed diameter $k \geq 4$ with a nontrivial center;*

(ii) *$\mathcal{F}_{n,k,p} \setminus \mathcal{F}^{s^*}_{n,k,p}$ is a class of typical graphs of the class of n-vertex graphs of fixed diameter $k \geq 6$ with a trivial center.*

Proof. Note that $\mathcal{F}_{n,k,p} \setminus \mathcal{F}^{s^*}_{n,k,p}$ is a subclass of $\mathbb{C}_{n,k,1}$ and $\mathcal{F}^{s^*}_{n,k,p}$ is a subclass of $\mathcal{J}_{n,\,d=k} \setminus \mathbb{C}_{n,k,1}$ by (15) and Lemma 1(i). Reckoning Lemma 2, Corollary 1 and Theorem 6, for $n \to \infty$ we obtain

$$\frac{|\mathcal{F}^{s^*}_{n,k,p}|}{|\mathcal{J}_{n,\,d=k} \setminus \mathbb{C}_{n,k,1}|} = \frac{2}{k-2}\,\frac{|\mathcal{F}_{n,k,p}|}{|\mathcal{J}_{n,\,d=k}|}\Big(1 - \frac{|\mathbb{C}_{n,k,1}|}{|\mathcal{J}_{n,\,d=k}|}\Big)^{-1} \longrightarrow 1,$$

$$\frac{|\mathcal{F}_{n,k,p} \setminus \mathcal{F}^{s^*}_{n,k,p}|}{|\mathbb{C}_{n,k,1}|} = \frac{k-4}{k-2}\,\frac{|\mathcal{F}_{n,k,p}|}{|\mathcal{J}_{n,\,d=k}|}\,\frac{|\mathcal{J}_{n,\,d=k}|}{|\mathbb{C}_{n,k,1}|} \longrightarrow 1.$$

□

Corollary 13. *Let $k \geq 3$ be an odd integer, $p \geq 1$ and $s^* = \frac{k-3}{2}$, $s^{**} = \frac{k-5}{2}$. Then*

(i) *$\mathcal{F}^{s^*}_{n,k,p}$ is a class of typical graphs of the class of n-vertex graphs of diameter k with an $(n - k + 1)$-vertex center;*

(ii) *$\mathcal{F}_{n,5,p} \setminus \mathcal{F}^{s^*}_{n,5,p}$ is a class of typical graphs of the class of n-vertex graphs of diameter 5, whose center cardinality is not equal to $n - 4$;*

(iii) *$\mathcal{F}_{n,k,p} \setminus (\mathcal{F}^{s^*}_{n,k,p} \cup \mathcal{F}^{s^{**}}_{n,k,p})$ is a class of typical graphs of the class of n-vertex graphs of fixed diameter $k \geq 7$ with a 2-vertex center;*

(iv) *$\mathcal{F}^{s^{**}}_{n,k,p}$ is a class of typical graphs of the class of n-vertex graphs of fixed diameter $k \geq 7$, whose the center cardinality is not equal to 2 and $n - k + 1$.*

Proof. By (17), (19) and Lemmas 1(i), 2, we have $\mathcal{F}^{s^*}_{n,k,p} \subseteq \mathbb{C}_{n,k,n-k+1}$, $\mathcal{F}_{n,5,p} \setminus \mathcal{F}^{s^*}_{n,5,p} \subseteq \mathcal{J}_{n,\,d=5} \setminus \mathbb{C}_{n,5,n-4}$ and $\mathcal{F}_{n,k,p} \setminus (\mathcal{F}^{s^*}_{n,k,p} \cup \mathcal{F}^{s^{**}}_{n,k,p}) \subseteq \mathbb{C}_{n,k,2}$, $\mathcal{F}^{s^{**}}_{n,k,p} \subseteq \mathcal{J}_{n,\,d=k} \setminus (\mathbb{C}_{n,k,2} \cup \mathbb{C}_{n,k,n-k+1})$ for $k \geq 7$. Reckoning Lemma 2, Corollary 1 and Theorem 6, we obtain

$$\frac{|\mathcal{F}^{s^*}_{n,k,p}|}{|\mathbb{C}_{n,k,n-k+1}|} = \frac{1}{k-2}\,\frac{|\mathcal{F}_{n,k,p}|}{|\mathcal{J}_{n,\,d=k}|}\,\frac{|\mathcal{J}_{n,\,d=k}|}{|\mathbb{C}_{n,k,n-k+1}|},$$

$$\frac{|\mathcal{F}_{n,5,p} \setminus \mathcal{F}^{s^*}_{n,5,p}|}{|\mathcal{J}_{n,\,d=5} \setminus \mathbb{C}_{n,5,n-4}|} = \frac{2}{3}\,\frac{|\mathcal{F}_{n,k,p}|}{|\mathcal{J}_{n,\,d=k}|}\Big(1 - \frac{|\mathbb{C}_{n,5,n-4}|}{|\mathcal{J}_{n,\,d=5}|}\Big)^{-1},$$

$$\frac{|\mathcal{F}_{n,k,p} \setminus (\mathcal{F}^{s^*}_{n,k,p} \cup \mathcal{F}^{s^{**}}_{n,k,p})|}{|\mathbb{C}_{n,k,2}|} = \frac{k-5}{k-2}\,\frac{|\mathcal{F}_{n,k,p}|}{|\mathcal{J}_{n,\,d=k}|}\,\frac{|\mathcal{J}_{n,\,d=k}|}{|\mathbb{C}_{n,k,2}|},$$

$$\frac{|\mathcal{F}^{s^{**}}_{n,k,p}|}{|\mathcal{J}_{n,\,d=k} \setminus (\mathbb{C}_{n,k,2} \cup \mathbb{C}_{n,k,n-k+1})|} = \frac{2}{k-2}\,\frac{|\mathcal{F}_{n,k,p}|}{|\mathcal{J}_{n,\,d=k}|}\Big(1 - \frac{|\mathbb{C}_{n,k,2}|}{|\mathcal{J}_{n,\,d=k}|} - \frac{|\mathbb{C}_{n,k,n-k+1}|}{|\mathcal{J}_{n,\,d=k}|}\Big)^{-1}.$$

□

In view of Corollary 1, the obtained properties of the centers are also valid for graphs of the classes $\mathcal{J}_{n,\,d\geq k}$ and $\mathcal{J}^*_{n,\,d\geq k}$.

Corollary 14. *For every fixed $k \geq 2$, almost all n-vertex graphs of each of the following classes $\mathcal{J}_{n,\,d\geq k}$, $\mathcal{J}^*_{n,\,d\geq k}$ are connected, have diameter k, and their center satisfies the properties stated in Theorem 6.*

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