CENTER AND ITS SPECTRUM OF ALMOST ALL
n-VERTEX GRAPHS OF GIVEN DIAMETER

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Abstract. We study typical (valid for almost all graphs of a class under consideration) properties of the center and its spectrum (the set of centers cardinalities) for n-vertex graphs of fixed diameter k. The spectrum of the center of all and almost all n-vertex connected graphs is found. The structure of the center of almost all n-vertex graphs of given diameter k is established. For k = 1, 2 any vertex is central, while for k ≥ 3 we identified two types of central vertices, which are necessary and sufficient to obtain the centers of almost all such graphs; in addition, centers of constructed typical graphs are found explicitly.

It is proved that the center of almost all n-vertex graphs of diameter k has cardinality n − 2 for k = 3, and for k ≥ 4 the spectrum of the center is bounded by an interval of consecutive integers except no more than one value (two values) outside the interval for even diameter k (for odd diameter k) depending on k. For each center cardinality value outside this interval, we calculated an asymptotic fraction of the number of the graphs with such a center. The realizability of the found cardinalities spectrum as the spectrum of the center of typical n-vertex graphs of diameter k is established.

Keywords: graph, diameter, diametral vertices, radius, central vertices, center, spectrum of center, typical graphs, almost all graphs.
Introduction

We study finite labeled ordinary graphs. For a connected graph $G$, the distance $p_G(u,v)$ between its vertices $u,v \in V(G)$ is defined as the length of the shortest path connecting these vertices. In this case, $e_G(v) = \max_{u \in V} p_G(v,u)$ is the eccentricity of the vertex $v$ of the graph $G$, $d(G) = \max_{v \in V} e_G(v)$ is the diameter of the graph $G$, and $r(G) = \min_{v \in V} e_G(v)$ is the radius of the graph $G$. A vertex is called central if its eccentricity is equal to the radius of the graph. The graph center $C(G)$ is the set of all central vertices of the graph $G$.

It is well known that for any graph $H$ there is a connected graph $G$ such that its subgraph induced by the center $C(G)$ is isomorphic to $H$. This fact was established by G.N. Kopystov and E.A. Timofeev [13], its simple justification was also given by S.T. Hedetniemi (see [7]). And for any rational $q$, $0 < q \leq 1$, F. Buckley proved the existence of a graph $G$ such that $|C(G)| = q|V(G)|$ [2].

For an arbitrary class of connected graphs $\mathcal{K}$ through $\mathcal{SP}_c(\mathcal{K})$ we denote the center spectrum of graphs of this class, i.e. the set of cardinalities of graphs centers from the class $\mathcal{K}$. The class of all $n$-vertex connected graphs is naturally partitioned into subclasses of graphs determined by their diameter. Let $J_{n,d=k}$, $J_{n,d=k}^*$, $J_{n,d \geq k}$ be the following classes of labeled $n$-vertex graphs: graphs of diameter $k$; connected graphs of diameter at least $k$ and graphs (not necessarily connected) with a shortest path of length at least $k$, respectively. Then $J_{n,d=1}$ is the class of all $n$-vertex connected nontrivial graphs, and obviously, the following inclusions are fulfilled: $J_{n,d=k} \subseteq J_{n,d=k}^* \subseteq J_{n,d \geq k}$.

In [13] all possible values of the parameters $n$, $m$ and $c$ are found for which there exists an $n$-vertex graph with $m$ edges and $c$ central vertices. The relations between these parameters are reduced to lower and upper bounds of the number of edges $m$ in terms of the given parameters $n$ and $c$ (see Section 2, Theorem 2). Using this Theorem of G.N. Kopystov and E.A. Timofeev, one can find the center spectrum of all $n$-vertex connected graphs (see Section 2, Theorem 3):

\[
\mathcal{SP}_c(J_{n,d \geq 1}) = \left[\left[1, n\right]\right] \setminus \{n-1\}, \quad n \geq 2
\]  

(1)

(here $\left[x, y\right]$ denotes an integer interval between two given numbers $x$ and $y$, i.e. $\left[x, y\right] = [x, y] \cap \mathbb{Z}$). In addition, for almost all $n$-vertex connected graphs $G$, the following equality holds $|C(G)| = n$. Obviously, $\mathcal{SP}_c(J_{n,d=1}) = \{n\}$. Moreover, from the well-known result of J.W. Moon and L. Moser (almost all graphs have diameter 2 [14]) it is easy to obtain that almost all graphs from $J_{n,d=2}$ have the radius equal to the diameter [8], and therefore, the cardinality of the center is also equal to $n$. Naturally the question arises as to the possible center spectrum of almost all $n$-vertex graphs of fixed diameter $k \geq 3$. The radius of almost all graphs of the class $J_{n,d=k}$ is established by the author in [8]. For $k \geq 3$, almost all $n$-vertex graphs of diameter $k$ have radius $\left\lfloor \frac{k}{2} \right\rfloor$.

Note that Yanan Hu and Xingzhi Zhan found the center spectrum of $n$-vertex graphs of given radius $r$ [12]. As for the properties of the center spectrum of almost all $n$-vertex graphs of fixed diameter $k$ (for large $n$), this result only implies inclusion $\mathcal{SP}_c(J_{n,d=k}) \subseteq \left[\left[1, n\right]\right] \setminus \{n-1\}$ for all $n \geq (8k-2)/3$. This relation is a consequence of the equality (1), i.e. new restrictions for possible values of the center spectrum of the almost all graphs do not arise.

In [15] Dhruv Mubayi and Douglas B. West investigated the smallest $h_{n,k}(c)$ and the largest $f_{n,k}(c)$ number of vertices with eccentricity $c$ in $n$-vertex graphs of...
diameter \( k \). For individual cases that do not cover all possible relationships between parameters \( n, k \) and \( c \), the values \( f_{n,k}(c) \) and \( h_{n,k}(c) \) are found. In particular, for \( c = \left\lceil \frac{k}{2} \right\rceil \), which is the radius of almost all graphs from \( J_{n,d=k} \), \( h_{n,k}(\left\lceil \frac{k}{2} \right\rceil) = 0 \) and \( f_{n,k}(\frac{k}{2}) = n-k, f_{n,k}(\frac{k+1}{2}) = n-k+1 \). Such a lower bound of the center cardinality is reduced to trivial, and the upper bound, due to the definition, does not take into account possible jumps and gaps of center cardinality values in the interval \([1, f_{n,k}(\frac{k}{2})]\), defined by these estimates, and also turns out to be uninformative for the study of the distribution of the center cardinalities of the almost all graphs, when \( n \) tends to infinity.

In this paper, we investigate the center and its spectrum for almost all \( n \)-vertex graphs of fixed diameter \( k \). Necessary preliminary information is contained in Section 1. There is also given a definition of the family of nested classes \( F_{n,k,p} \), \( p \geq 1 \) of \( n \)-vertex graphs of fixed diameter \( k \geq 3 \), possessing a number of metric properties and constructed by the author in [8]. It was previously established that \( F_{n,k,p} \) is a class of typical graphs for each of the classes \( J_{n,d=k}, J_{n,d \geq k} \) and \( J_{n,d \geq k}^\ast \) (Theorem 1 and its Corollaries). Hereinafter, we use this class of typical graphs.

In Section 2, we find the center spectrum of all and almost all \( n \)-vertex connected graphs (Theorem 3).

In Section 3, we establish the structure of the center of almost all graphs of a given diameter. For almost all graphs \( G \) of diameter \( k = 1, 2 \), every vertex is central, i.e. \( C(G) = V(G) \). For \( k \geq 3 \), we identified two types of central vertices, which are necessary and sufficient to obtain the centers of almost all \( n \)-vertex graphs of fixed diameter \( k \). For odd \( k \) these are the central vertices of diametral paths of the graph and vertices equidistant at distance \( \frac{k+1}{2} \) from their endpoints, while for even \( k \) these are only the central vertices of diametral paths (Theorem 4). Moreover, for typical graphs \( G \in F_{n,k,p} \) the center \( C(G) \) is explicitly distinguished (Lemma 5).

In Section 4, we asymptotically study the center spectrum of \( n \)-vertex graphs of a fixed diameter. It is proved that the center of almost all \( n \)-vertex graphs of diameter \( k \) has cardinality \( n \) for \( k = 1, 2 \), and \( n - 2 \) for \( k = 3 \), while for \( k \geq 4 \) the center spectrum is bounded by an interval of consecutive integers and additionally contains at most one value (two values) outside this interval for even diameter \( k \) (for odd diameter \( k \)) depending on the value \( k \) (Theorem 6). Note that the boundaries of the interval depend on predetermined arbitrary integer \( p \) and shrink when choosing a greater value \( p \). For each value of the center cardinality outside this interval, the asymptotic fraction of the number of the graphs with such a center are calculated. Moreover, the graphs whose center cardinality belongs to the interval also have a nonzero asymptotic fraction (see Theorem 6 for more details). In Theorem 5 it is established realizability of the found cardinalities spectrum as the center spectrum \( S_{p,c}(F_{n,k,p}) \) of typical \( n \)-vertex graphs of diameter \( k \). Furthermore, typical graphs for graphs classes corresponding to the cardinality cases of the center in Theorem 6 are found in Corollaries 12, 13. Theorem 6 implies a number of properties of the centers of almost all graphs of fixed diameter \( k \). For example, there are almost no graphs with a trivial center of diameter \( k = 2, 4 \) and odd diameter \( k \), while for any even \( k \geq 6 \) this is not true. Similarly, there are almost no graphs with a 2-vertex center of diameter \( k = 1, 3, 5 \) and even diameter \( k \), however, for every odd \( k \geq 7 \) this does not hold. Unexpected is the jump of the center cardinality outside the interval of the consecutive integer values both from above for odd diameter \( k \geq 5 \), and from below for even \( k \geq 6 \) and odd \( k \geq 7 \).
All obtained typical properties of the center and its spectrum for $n$-vertex graphs of fixed diameter $k \geq 2$ remain typical for connected graphs of diameter at least $k$, as well as for graphs (not necessarily connected) with a shortest path of length at least $k$. In particular, Corollary 6 is valid.

1. Preliminary Information

The article uses the generally accepted concepts and notation of graph theory [4, 11], as well as the standard concepts of combinatorial analysis [10]. We consider only finite ordinary (i.e., without loops and multiple edges) graphs $G = (V, E)$ with set of vertices $V = \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$. As usual, denote by $G \setminus v$ the graph obtained as a result of removing a vertex $v$ and all edges incident to it, $G \setminus V'$ is the graph obtained by removing all vertices from a subset $V' \subseteq V$, $G \setminus \{e_1, e_2, \ldots, e_k\}$ is the graph obtained as a result of removing edges $e_1, e_2, \ldots, e_k$ of the graph $G$, $G + H$ is the graph obtained by the join operation from graphs $G$ and $H$, $\deg_G v$ is the degree of vertex $v$ of the graph $G$, $\delta(G)$ is the minimum degree of vertices of $G$, $B^G_i(v) = \{u \in V \setminus \rho_G(v, u) \leq i\}$ is a ball of radius $i$ centered at a vertex $v \in V$ in the metric space of the graph $G$ with the metric $\rho_G$, $S^G_i(v) = \{u \in V \setminus \rho_G(v, u) = i\}$ is a sphere of radius $i$ centered at a vertex $v \in V$, $K_n$ is a complete $n$-vertex graph, $K_{1,n-1}$ is an $n$-vertex star, $C_n$ is an $n$-vertex cycle, $P_n$ is an $n$-vertex simple path. For a shortest path $P$ with endpoints $v_0$ and $v_n$, sequentially passing through vertices $v_0, v_1, \ldots, v_n$, we use the notation $P = (v_0, v_1, \ldots, v_n)$. A vertex of degree 1 is called pendant, a shortest path of length $d(G)$ is the diametral path of the graph $G$, and under by a pair of diametral vertices we mean an unordered sample of two vertices from the set $V$, the distance between which is equal to the diameter.

The graph $V_k(u, v)$ shown in Fig. 1a we call the shuttlecock on the vertices $u, v$ [5]. A graph $G$ (not necessarily connected) has a shuttlecock, if $G$ has a subgraph $V_k(u, v)$ and $\deg_G u = \deg_G v = k + 1$ (Fig. 1b). It is easy to see that a graph does not contain shuttlecocks if and only if it does not contain coincident balls of radius 1 centered at different vertices [5].

By $(n)_k$ we denote the number of order placements from $n$ elements by $k$, i.e. $(n)_k = n(n - 1) \cdots (n - k + 1)$, and wherein we define $(n)_k = 0$ for $n < k$ and $(n)_0 = 1$. We will write $[x]$ (\lfloor x \rfloor) to denote the smallest (largest) integer greater (less) or equal to real nonnegative number $x$. To denote the asymptotic equality of functions $f(n)$ and $g(n)$ as $n \to \infty$, we use the notation $f(n) \sim g(n)$, which by definition means that $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1$ or, equivalently, $f(n) = g(n)(1 + o(n))$ for all large enough $n$, where $o(n) = o(1)$ is the approximation error of $g(n)$.

To estimate the measure of the number of graphs with a certain property, the concept of almost all is often used; in this approach, the studied property
is considered for graphs with a large number of vertices. Let \( \mathcal{J}_n \) be the class of labeled \( n \)-vertex graphs with the fixed set of vertices \( V = \{1, 2, \ldots, n\} \), \( n \in \mathbb{N} \). Consider some property \( \mathcal{P} \) by which each graph may or may not possess. Through \( \mathcal{J}_n^\mathcal{P} \) denote the set of all graphs from \( \mathcal{J}_n \) that possess the property \( \mathcal{P} \). Almost all graphs possess the property \( \mathcal{P} \) if \( \lim_{n \to \infty} \frac{|\mathcal{J}_n^\mathcal{P}|}{|\mathcal{J}_n|} = 1 \), i.e. \( |\mathcal{J}_n^\mathcal{P}| \sim |\mathcal{J}_n| \), and there are almost no graphs with the property \( \mathcal{P} \), if \( \lim_{n \to \infty} \frac{|\mathcal{J}_n^\mathcal{P}|}{|\mathcal{J}_n|} = 0 \).

In the study and selection of almost all graphs in the class of graphs under consideration it is often useful to define not characteristic properties themselves for the notion of almost all, but directly select a subclass of typical graphs itself (in \([6,7]\) a more general concept of a class of typical combinatorial objects and an abstract typical combinatorial object for a given class of objects admitting the concept of dimension is formulated). Further we will also use this formal concept for graphs (when the dimension of a graph is understood as the number of its vertices). Let \( \Omega \) be an arbitrary class of graphs such that \( \Omega_n \neq \emptyset \) for all large enough \( n \), where \( \Omega_n = \Omega \cap \mathcal{J}_n \). A subclass \( \Omega^* \subseteq \Omega \) is the class of typical graphs of the class \( \Omega \) if
\[
\lim_{n \to \infty} \frac{|\Omega^*_n|}{|\Omega_n|} = 1.
\]

In \([8]\) for every \( k \geq 3 \), a family of nested classes \( \mathcal{F}_{n,k,p}, p \geq 1 \) of \( n \)-vertex graphs of fixed diameter \( k \) is constructed. To define the class \( \mathcal{F}_{n,k,p} \), first consider special graphs of diameter 3 and their properties. Let \( x, y \in V \) and \( \mathcal{F}_{n,3,p}(x, y) \) be the class of all graphs \( F \in \mathcal{J}_n \) with the following properties:
   
   a) the vertices \( x, y \) are not pendant in \( F \);
   
   b) \( \rho_F(x, y) = 2 \) for some vertex \( z \in V \) (a vertex with this property will be called the pole of graph \( F \));
   
   c) \( d(F) = 3 \), the graph \( F \) has the unique pair of diametral vertices \( x, y \) and does not contain shuttlecocks;
   
   d) the following property of spheres holds:
   
   \[
   |S^F_1(u) \cap S^F_1(v)| \geq p \quad \forall u, v \in V \setminus \{x, y\} \text{ and } u \neq v, 
   \]
   
   \[
   |S^F_i(u) \cap S^F_i(v)| \geq p \quad \forall v \in V \setminus \{x, y\} \forall u \in \{x, y\}. 
   \]

Now, we define graphs of the class \( \mathcal{F}_{n,k,p} \) as follows. Let \( u = (u_0, u_1, \ldots, u_{k-2}) \) be an arbitrary ordered sequence of different vertices from the set \( V \). Fix an arbitrary pair of neighboring elements \( u_s \) and \( u_{s+1} \), \( 0 \leq s \leq k - 3 \). On the set \( V \setminus \{u_0, \ldots, u_{s-1}, u_{s+2}, \ldots, u_{k-2}\} \) of \( n - k + 3 \) vertices, define an arbitrary graph \( F \) from the class \( \mathcal{F}_{n-k+3,3,p}(u_s, u_{s+1}) \). Finally, join by edges the vertices \( u_i, u_{i+1} \) for \( i \neq s \) and \( 0 \leq i < k - 2 \). Denote the so-obtained graph by \( G(u, s, F) \). Let \( \mathcal{F}_{n,k,p} \) be the class of all constructed graphs \( G(u, s, F) \) under condition \( 0 \leq s \leq \lfloor \frac{k-3}{2} \rfloor \), and let \( \mathcal{F}_{n,k,p}^s \) denote the class of all graphs \( G(u, s, F) \) for fixed \( s \leq k - 3 \). In what follows, we will use the notation \( G(u, s, F) \) for the graph constructed for given \( k, p, u, s \) and \( F \), without detailing the properties \( k \geq 3, p \geq 1, u = (u_0, u_1, \ldots, u_{k-2}), 0 \leq s \leq \lfloor \frac{k-3}{2} \rfloor \) and \( F \in \mathcal{F}_{n-k+3,3,p}(u_s, u_{s+1}) \), unless otherwise specified.

In \([8]\) for any fixed \( k \geq 3 \) and \( p \geq 1 \), it is proved that the class of graphs \( \mathcal{F}_{n,k,p} \) is typical for each of the classes \( \mathcal{J}_n, d=k \), \( \mathcal{J}_n, d \geq k \) and \( \mathcal{J}_n^*, d \geq k \), and also established asymptotically exact value \( 2(\xi) \xi_{n,k} \) of the number of graphs in this class.
Theorem 1 [8]. Let \( k \geq 3, 0 < \varepsilon < 1 \) and \( p \geq 1 \) do not depend on \( n \). Then there is a constant \( c > 0 \) independent of \( n \) and such that for every \( n \in \mathbb{N} \) the following inequalities hold

\[
2^{\frac{3}{2}}\xi_{n,k}\left(1 - c\left(\frac{5+\varepsilon}{6}\right)^{n-k+1}\right) \leq |\mathcal{F}_{n,k,p}| \leq |\mathcal{J}_{n,d=k}|
\]

\[
\leq |\mathcal{J}_{n,d \geq k}| \leq |\mathcal{J}_{n,d \geq k}^*| \leq 2^{\frac{3}{2}}\xi_{n,k}\left(1 + c\left(\frac{5+\varepsilon}{6}\right)^{n-k+1}\right),
\]

where \( \xi_{n,k} = q_k \left(n\right)^{-k-1} \left(\frac{3}{2k-1}\right)^{n-k+1} \), \( q_k = \frac{1}{2} \left(k - 2\right)2^{-(k-1)} \).

Corollary 1 [8]. Let \( k \geq 3 \) and \( p \geq 1 \) be independent of \( n \). Then for \( n \to \infty \)

\[
|\mathcal{F}_{n,k,p}| \sim |\mathcal{J}_{n,d=k}| \sim |\mathcal{J}_{n,d \geq k}| \sim |\mathcal{J}_{n,d \geq k}^*| \sim 2^{\frac{3}{2}}\xi_{n,k}.\]

Corollary 2 [8]. Let \( k \geq 3 \) and \( p \geq 1 \) be independent of \( n \). Then \( \mathcal{F}_{n,k,p} \) is the class of typical graphs of the class of \( n \)-vertex graphs of diameter \( k \).

Next, we need the following properties of the graphs \( G(u, s, F) \).

Lemma 1 [8] (properties of \( G(u, s, F) \)). Let \( k \geq 3, p \geq 1 \) and \( G = G(u, s, F) \in \mathcal{F}_{n,k,p} \). Then the following properties hold:

(i) \( G \in \mathcal{J}_{n,d=k} \);

(ii) the vertices \( u_s, u_{s+1} \) are not pendant in \( F \);

(iii) \( u_0, u_{k-2} \) is the unique pair of diametral vertices of the graph \( G \) and \( u_0, u_1, \ldots, u_{k-2} \in V(P) \) for every diametral path \( P \).

Corollary 3. For a vertex \( v \) of graph \( G(u, s, F) \in \mathcal{F}_{n,k,p} \), the following conditions are equivalent:

(i) \( v \) belongs to some diametral path;

(ii) \( v \in S_1^F(u_s) \cup S_1^F(u_{s+1}) \cup \{u_0, u_1, \ldots, u_{k-2}\} \);

(iii) \( v \) is not a pole.

Proof. In view of properties b), c) of the graph \( F \), it is easy to understand the equivalence of statements (ii) and (iii). Moreover, property c) of the graph \( F \) implies that any vertex \( v \) from \( S_1^F(u_s) \cup S_1^F(u_{s+1}) \) belongs to some diametral path of the graph \( F \) with endpoints \( u_h, u_{s+1} \). Extending it with two paths passing respectively through the vertices \( u_0, \ldots, u_s \) and \( u_{s+1}, \ldots, u_{k-2} \), we get the diametral path of the graph \( G(u, s, F) \).

Further, note that if the vertex \( v \) is a pole, then \( \rho_F(u_s, v) = \rho_F(v, u_{s+1}) = 2 \).

Therefore, \( v \) cannot belong to a diametral path with endpoints \( u_0, u_{k-2} \), since this path contains \( u_s, u_{s+1} \) and \( \rho_F(u_s, u_{s+1}) = 3 \) by Lemma 1.

□

Lemma 2. Let \( k \geq 3, p \geq 1 \) and \( 0 \leq s \leq k - 3 \). Then the following properties hold:

(i) \( |\mathcal{F}_{n,k,p}^s| = \frac{1}{2}(k-2)(n)_{k-1}|\mathcal{F}_{n-k+3,3,p}(x, y)| \), where \( x \neq y \);

(ii) \( |\mathcal{F}_{n,k,p}^s| = \sigma(s)_{k-2} |\mathcal{F}_{n,k,p}| \), where \( \sigma(s) = 1 \) for \( s = \frac{k-3}{2} \) and \( \sigma(s) = 2 \) if \( s \neq \frac{k-3}{2} \);

(iii) \( \mathcal{F}_{n,k,p} = \bigcup_{s=0}^{k-3} \mathcal{F}_{n,k,p}^s \);

(iv) \( \mathcal{F}_{n,k,p}^i \cap \mathcal{F}_{n,k,p}^j = \emptyset \) if \( i \neq j \) and \( i, j \leq \lfloor \frac{k-3}{2} \rfloor \).
Proof. Statements (i)-(ii) are proved in [8]. And statement (iii) follows from the definitions of the classes. Prove (iv). Let $G = G(u, s, F) \in \mathcal{F}_{n,k,p}$. By Lemma 1(iii), the graph $G$ has the single pair of diametral vertices $u_0, u_{n-2}$. Moreover, a part of the diametral path of the graph $G$ from a given diametral vertex to the first encountered vertex $v$ from the graph $F$ can be uniquely reconstructed knowing the edges of the graph $G$, because $\deg_F v \geq 2$ by Lemma 1(ii). Therefore, $G$ has two such vertex-disjoint parts of its diametral path of length $s \leq \left\lfloor \frac{k-3}{2} \right\rfloor$ and $k-3-s \geq \left\lceil \frac{k-5}{2} \right\rceil$. Consequently, if $G(u, i, F) = G(u', j, F')$ and $i, j \leq \left\lfloor \frac{k-3}{2} \right\rfloor$, then $i = j$.

Further, we use the following well-known fact.

Lemma 3 (see, for example, [9]). The radius of a simple path of length $k$ is equal to $\left\lfloor \frac{k}{2} \right\rfloor$, and the central vertices of the path are at distance $\left\lfloor \frac{k}{2} \right\rfloor$ and $\left\lceil \frac{k}{2} \right\rceil$ from its endpoints.

2. Center spectrum of connected $n$-vertex graphs

Consider the class $\mathcal{J}_{n,d \geq 1}$ of all $n$-vertex connected graphs. Obviously, the diameter $d$ of these graphs satisfies the inequalities $1 \leq d \leq n-1$. Now, we find center spectrum $\mathcal{S}_c(\mathcal{J}_{n,d \geq 1})$ for $n = 2, 3, 4$.

Example 1: Let $n = 2$. In this case, there is only one connected graph $K_2$. Moreover, $r(K_2) = 1$ and $|C(K_2)| = 2$. Thus, $\mathcal{S}_c(\mathcal{J}_{2,d \geq 1}) = \{2\}$.

Example 2: Let $n = 3$. For $d = 1$ there is a unique graph $K_3$. Moreover, $r(K_3) = 1$ and $|C(K_3)| = 3$. For $d = 2$ there is a single graph, a 3-vertex simple path $P_3$, for which $r(P_3) = 1$ and $|C(P_3)| = 1$. Thus, $\mathcal{S}_c(\mathcal{J}_{3,d \geq 1}) = \{1, 3\}$.

![Fig. 2. 4-vertex connected graphs](image)

Example 3: Let $n = 4$ (fig. 2). For $d = 1$ there is the only graph $K_4$ with $r(K_4) = 1$ and $|C(K_4)| = 4$. For $d = 2$ there are four graphs: a star $K_{1,3}$, a cycle $C_4$, a square with diagonal $H_1$ and a graph-claw $H_2$, for which the following equalities hold:

- $r(K_{1,3}) = 1$ and $|C(K_{1,3})| = 1$;
- $r(C_4) = 2$ and $|C(C_4)| = 4$;
- $r(H_1) = 1$ and $|C(H_1)| = 2$;
- $r(H_2) = 1$ and $|C(H_2)| = 1$.

For $d = 3$ there is a single graph, a 4-vertex simple path $P_4$, for which $r(P_4) = 2$ and $|C(P_4)| = 2$. Thus, $\mathcal{S}_c(\mathcal{J}_{4,d \geq 1}) = \{1, 4\} \setminus \{3\}$.

Now, find the center spectrum of all and almost all graphs of the class $\mathcal{J}_{n,d \geq 1}$. We need the following G.N. Kopylov and E.A. Timofeev’s Theorem.
Theorem 2 [13]. For \( n \geq 2 \), there exists a graph in the class of connected \( n \)-vertex graphs with \( m \) edges and \( c \) central vertices if and only if one of the following conditions holds:

(i) \( c \leq n, c \neq n - 1 \) and \( \frac{c(c-1)}{2} + c(n-c) \leq m \leq \frac{n(n-2)+c}{2} \);

(ii) \( c = n \) and \( n \leq m \leq \frac{n(n-2)}{2} \);

(iii) \( 2 \leq c \leq n-2 \) and \( E(c,n) \leq m \leq \frac{(n-2)(n-3)}{2} + c \), where

\[
E(c,n) = \begin{cases} 
  n-1 & \text{if } c = 2, \\
  n+1 & \text{if } c \text{ even and } n-3 \leq c \leq n-2, \\
  n & \text{else}.
\end{cases}
\]

Theorem 3 (spectrum \( Sp_c(J_{n,d \geq 1}) \)). The following properties are valid:

(i) \( Sp_c(J_{n,d \geq 1}) = \{ 1, n \} \setminus \{ n-1 \} \) for every \( n \geq 2 \);

(ii) almost all \( n \)-vertex connected graphs have a center of cardinality \( n \).

Proof. Prove statement (i). The case \( n = 2 \) is considered in Example 1. Therefore, we can assume that \( n \geq 3 \). Let \( c \in \{ 1, n \} \setminus \{ n-1 \} \). Further, we find the value of the parameter \( m \), satisfying one of conditions (i)-(iii) of Theorem 2, considering possible cases for the value \( c \).

Let \( c = 1 \). Note that the inequality \( n-1 \leq \frac{n(n-2)+1}{2} \) holds for every \( n \geq 3 \), i.e. \( m = n-1 \) satisfies condition (i) from Theorem 2.

Let \( 2 \leq c \leq n-4 \). Then \( n \geq 6 \). Note that the inequality \( n \leq \frac{(n-2)(n-3)}{2} + 2 \) is valid for every \( n \geq 6 \), i.e. \( m = n \) satisfies condition (iii) from Theorem 2 for every \( n \geq 6 \).

Let \( c = n-i \), where \( i = 0, 2, 3 \). Then \( n-i > 0 \). Note that inequality

\[
\frac{c(c-1)}{2} + c(n-c) \leq \frac{n(n-2)+c}{2}
\]

is equivalent to inequality \( (n-i)(n+i-2) \leq n(n-2) \). It is not hard to understand the validity of this inequality for \( i = 0, 2, 3 \), i.e. \( m = (n-i)(n+i-1)/2 \) satisfies condition (i) from Theorem 2.

Thus, due to Theorem 2, for all specified values of the parameter \( c \), there exists a graph \( G \in J_{n,d \geq 1} \) such that \( |C(G)| = c \). To prove the converse, it suffices to note that if \( G \in J_{n,d \geq 1} \) then \( 1 \leq |C(G)| \leq n \) and \( n-1 \notin Sp_c(J_{n,d \geq 1}) \) by Theorem 2.

Prove statement (ii). It is known that almost all \( n \)-vertex graphs have diameter \( d \) equal to 2 (see, for example, [4]). It remains to note that \( J_{n,d=2} \subseteq J_{n,d \geq 1} \subseteq J_n \) and \( \{ G \in J_{n,d \geq 2} \mid r(G) = d(G) \} \subseteq \{ G \in J_n \mid |C(G)| = n \} \).

3. CENTER OF ALMOST ALL GRAPHS FROM \( J_{n,d=k} \)

Find out the structure of the center of almost all \( n \)-vertex graphs of a given diameter. A radius of almost all graphs of fixed diameter \( k \) was established in [8]. Almost all graphs of diameter \( k = 1, 2 \) have the radius equal to the diameter. Therefore, every vertex is central, i.e. \( C(G) = V(G) \) for almost all graphs \( G \) of diameter \( k = 1, 2 \). The radius of almost all graphs of fixed diameter \( k \geq 3 \) is equal to \( \left\lceil \frac{k}{2} \right\rceil \). Investigate the center of such \( n \)-vertex graphs. For this, turn to the class of typical graphs \( F_{n,k,p} \), establish properties of their central vertices and find the center explicitly.

Lemma 4 [8]. If \( k \geq 3 \), \( p \geq 1 \) and \( G \in F_{n,k,p} \), then \( r(G) = \left\lceil \frac{k}{2} \right\rceil \).
Corollary 4 [8]. For each graph $G \in \mathcal{F}_{n,k,p}$, every central vertex of its arbitrary diametral path is the central vertex of the graph $G$.

Note that the converse statement in Corollary 4, generally speaking, is not true; this will be shown below.

Corollary 5. If $P$ is an arbitrary diametral path of a graph $G \in \mathcal{F}_{n,k,p}$, then $C(G) \cap V(P) = C(P)$.

Proof. From Corollary 4 we have $C(P) \subseteq C(G) \cap V(P)$. Moreover, by Lemmas 3 and 4, if $v \in P \setminus C(P)$, then $e_G(v) \geq e_P(v) > r(P) = r(G)$ and, therefore, $v \notin C(G)$. \qed

Lemma 5 (center $C(G)$, $G \in \mathcal{F}_{n,k,p}$). Let $G = G(u,s,F) \in \mathcal{F}_{n,k,p}$. Then

(i) if $k$ is even, then
$$C(G) = \begin{cases} S_1^F(u_{s+1}) & \text{if } s = \frac{k}{2} - 2, \\ \{u_{\frac{k}{2}}\} & \text{else}; \end{cases}$$

(ii) if $k$ is odd, then
$$C(G) = \begin{cases} V \setminus \{u_0, \ldots, u_{k-2}\} & \text{if } s = \frac{k-3}{2}, \\ B_{s+1}^F(u_{s+1}) & \text{if } s = \frac{k-5}{2}, \\ \{u_{\frac{k}{2} - s}, u_{s+2}\} & \text{else}; \end{cases}$$

(iii) $C(G)$ consists of all central vertices of diametral paths with endpoints $u_0, u_{k-2}$ and all vertices, equidistant at distance $\frac{k+1}{2}$ from the vertices $u_0, u_{k-2}$.

Proof. Prove statements (i) and (ii). In view of Lemma 1, a diametral path $P$ of the graph $G$ has the form $P = (u_0, \ldots, u_s, u_s', u_{s+1}', u_{s+2}, \ldots, u_{k-2})$. By Lemmas 3, 4 and Corollary 5, we obtain
$$r(G) = r(P) = \left\lceil \frac{k}{2} \right\rceil \quad \text{and } C(G) \cap V(P) = C(P). \quad (2)$$

Due to the properties of the graph $F$, the set $V(F) \setminus \{u_s, u_{s+1}\}$ consists of three types of vertices $x, y, z$ such that
$$\rho_G(x, u_s) = 1, \quad \rho_G(x, u_{s+1}) = 2,$$
$$\rho_G(y, u_{s+1}) = 1, \quad \rho_G(y, u_s) = 2,$$
$$\rho_G(z, u_s) = \rho_G(z, u_{s+1}) = 2. \quad (3)$$

Further, by $x, y, z$ we mean an arbitrary vertex from $F \setminus \{u_s, u_{s+1}\}$ with the above metric relations. From condition $s \leq \left\lfloor \frac{k-3}{2} \right\rfloor$ we have $s' = k-3-s \geq \left\lfloor \frac{k-3}{2} \right\rfloor$. Consider the possible cases.

Case 1. Let $s = s'$. Then $k$ is odd and $s = \frac{k-3}{2}$. By (2), we have $C(G) \cap V(P) = \{u_s', u_{s+1}'\}$. Considering the form of the graph $G(u,s,F)$ and the relation (3), we obtain $e_G(x) = e_G(y) = e_G(z) = s + 2 = \left\lceil \frac{k}{2} \right\rceil$. Therefore, $x, y, z \in C(G)$. Hence,
$$C(G) = \{u_s', u_{s+1}'\} \cup V(F) \setminus \{u_s, u_{s+1}\} = V \setminus \{u_0, \ldots, u_{k-2}\}.$$

Case 2. Let $k$ be even and $s < s'$. Then
$$s' \geq \frac{k}{2} - 1 > \frac{k}{2} - 2 \geq s. \quad (4)$$

It follows from (2) and (4) that
$$C(G) \cap V(P) = \begin{cases} \{u_{s+1}'\} & \text{if } s' = \frac{k}{2} - 1, \\ \{u_{\frac{k}{2} - 2}\} & \text{else}. \end{cases} \quad (5)$$
Reckoning the form of the graph $G(u, s, F)$, the relations (3) and (4), we obtain
\[ e_G(x) = \max\{s + 1, s' + 2\} = s' + 2 \geq 0.5k + 1, \]
\[ e_G(y) = \max\{s + 2, s' + 1\} = s' + 1 \geq 0.5k, \]
\[ e_G(z) = \max\{s + 2, s' + 2\} = s' + 2 \geq 0.5k + 1. \]

Therefore, $x, z \notin \mathbb{C}(G)$ due to (2) and the following equivalence holds
\[ y \in \mathbb{C}(G) \iff s' + 1 = \frac{k}{2} \iff s = \frac{k}{2} - 2. \]

Hence,
\[ \mathbb{C}(G) \setminus V(P) = \begin{cases} S^F_1(u_{s+1}) & \text{if } s = \frac{k-1}{2}, \\ \emptyset & \text{else.} \end{cases} \tag{6} \]

Thus, from (5), (6) we obtain the required form $\mathbb{C}(G)$ for even $k$.

Case 3. Let $k$ be odd and $s < s'$. Then
\[ s' > \frac{k-3}{2} > s. \tag{7} \]

It follows from (2) and (7) that
\[ \mathbb{C}(G) \cap V(P) = \begin{cases} \{u_{s+1}, u_{s+1}\} & \text{if } s' = \frac{k-1}{2}, \\ \{u_{k-5}, u_{k-3}\} & \text{else.} \end{cases} \tag{8} \]

Reckoning the form of the graph $G(u, s, F)$, the relations (3) and (7), we obtain
\[ e_G(x) = \max\{s + 1, s' + 2\} = s' + 2 \geq 0.5(k + 1) + 1, \]
\[ e_G(y) = \max\{s + 2, s' + 1\} = s' + 1 \geq 0.5(k + 1), \]
\[ e_G(z) = \max\{s + 2, s' + 2\} = s' + 2 \geq 0.5(k + 1) + 1. \]

Therefore, $x, z \notin \mathbb{C}(G)$ by (2) and
\[ y \in \mathbb{C}(G) \iff s' + 1 = \frac{k + 1}{2} \iff s = \frac{k - 5}{2}. \]

Hence,
\[ \mathbb{C}(G) \setminus V(P) = \begin{cases} S^F_1(u_{s+1}) & \text{if } s = \frac{k-5}{2}, \\ \emptyset & \text{else.} \end{cases} \tag{9} \]

It follows from (8) and (9) that
\[ \mathbb{C}(G) = \begin{cases} B^F_1(u_{s+1}) & \text{if } s = \frac{k-5}{2}, \\ \{u_{k-5}, u_{k-3}\} & \text{else.} \end{cases} \]

Thus, the form $\mathbb{C}(G)$ required in statement (ii) for odd $k$ is obtained in cases 1, 3.

Prove statement (iii). By Corollary 4, every central vertex of a diametral path of the graph $G$ belongs to $\mathbb{C}(G)$. Now let $v$ be equidistant at distance $\frac{k+1}{2}$ from the both vertices $u_0$, $u_{k-2}$. Then the vertex $v$ does not belong to diametral paths of the graph $G$ by Lemma 1(iii). By corollary 3, the vertex $v$ is a pole of the graph $F$. Therefore, $s = \frac{k-3}{2}$. Hence, $v \in \mathbb{C}(G)$ by the obtained statement (ii).

Prove the converse statement. Let $v \in \mathbb{C}(G)$. Note that if $s = \frac{k-3}{2}$ and the vertex $v$ is a pole, then $v$ is equidistant at distance $\frac{k+1}{2}$ from the diametral vertices $u_0, u_{k-2}$. Using statements (i) and (ii), it is easy to understand that in other cases the vertex $v$ is at distance $\left\lfloor \frac{k}{2} \right\rfloor$ from one of the vertices $u_0, u_{k-2}$ and belongs to
Theorem 4 (Center of almost all graphs from $J_{n,d}=k$). Let $k \geq 3$ be a fixed integer. Then

(i) the center of almost all $n$-vertex graphs of even diameter $k$ consists of all central vertices of diametral paths of the graph;

(ii) the center of almost all $n$-vertex graphs of odd diameter $k$ consists of all central vertices of diametral paths of the graph and all vertices equidistant at distance $\frac{k+1}{2}$ from their endpoints. Furthermore, the proportion of such $n$-vertex graphs whose center consists only of central vertices of diametral paths of the graph is asymptotically equal to $\frac{k-3}{k-2}$.

Proof. Directly from Corollary 1 and Lemmas 1(iii), 5(iii) we obtain the required properties of the central vertices of almost all $n$-vertex graphs of a given diameter.

By $K_{n,k}$ we denote the class of all graphs from $J_{n,d}=k$ whose center consist only of all central vertices of diametral paths of the graph. Find out a fraction of such graphs of odd diameter $k$. Let $s^* = \frac{k-3}{2}$. From Lemma 5 and the proof of its statement (iii) it follows that $F_{n,k,p}^{s^*} \subseteq K_{n,k}$ for every $s \neq s^*$ and $F_{n,k,p}^{s^*} \cap K_{n,k} = \emptyset$. Hence,

$$\bigcup_{0 \leq s < \frac{k-3}{2}} F_{n,k,p}^{s} \subseteq K_{n,k} \subseteq J_{n,d=k} \setminus F_{n,k,p}^{s^*}.$$  \hspace{1cm} (10)

From Lemma 2 we obtain

$$|\bigcup_{0 \leq s < \frac{k-3}{2}} F_{n,k,p}^{s}| = |F_{n,k,p} \setminus F_{n,k,p}^{s^*}| = |F_{n,k,p}| \left(1 - \frac{\sigma(s^*)}{k-2}\right).$$  \hspace{1cm} (11)

By Lemmas 1(i) and 2, we have

$$|J_{n,d=k} \setminus F_{n,k,p}^{s^*}| = |J_{n,d=k}| - \frac{\sigma(s^*)}{k-2} |F_{n,k,p}|.$$  \hspace{1cm} (12)

Thus, from (10)-(12) and Corollary 1 as $n \to \infty$ we conclude

$$\frac{|K_{n,k}|}{|J_{n,d=k}|} \to 1 - \frac{\sigma(s^*)}{k-2} = \frac{k-3}{k-2}.$$  \hspace{1cm} \hfill \Box

Theorem 4 implies that to obtain almost all graphs $G$ of odd diameter $k \geq 3$ in the center $C(G)$ cannot do without vertices equidistant at distance $\frac{k+1}{2}$ from endpoints of its diametral paths (since the class $K_{n,k}$ does not asymptotically cover the whole class $J_{n,d=k}$), and for $k \geq 5$ it is also impossible to do without central vertices of diametral paths (because the class $K_{n,k}$ has a nonzero asymptotic fraction). In addition, note that Fred Buckley and Martin Lewinter investigated the class of graphs ($L'$-graphs), centers of which do not contain vertices lying on diametral paths, and established characteristic limitations between the diameter and the radius for these graphs [?]. In particular, they showed that there are no such graphs of diameter 3. It means that for $k = 3$ in the center $C(G)$ it is also impossible to do without central vertices of diametral paths.
4. Center spectrum of almost all graphs from $\mathcal{J}_{n,d}$

Let us investigate the center spectrum of almost all $n$-vertex graphs of a fixed diameter. For this, turn to the class of typical graphs $\mathcal{F}_{n,k,p}$.

**Lemma 6.** Let $G = G(u, s, F) \in \mathcal{F}_{n,k,p}$. Then

(i) if $k$ is even, then

$$|C(G)| = 1 \Leftrightarrow s \neq \frac{k}{2} - 2,$$

$$p + 1 \leq |C(G)| \leq n - k - p - 1 \text{ for } s = \frac{k}{2} - 2;$$

(ii) if $k$ is odd, then

$$|C(G)| = 2 \Leftrightarrow s \neq \frac{k - 3}{2} \text{ and } s \neq \frac{k - 5}{2},$$

$$p + 2 \leq |C(G)| \leq n - k - p \text{ for } s = \frac{k - 5}{2},$$

$$|C(G)| = n - k + 1 \Leftrightarrow s = \frac{k - 3}{2}.$$

**Proof.** Let $P = (u_0, \ldots, u_s, u'_s, u'_{s+1}, u_{s+1}, \ldots, u_{k-2})$ be an arbitrary diametral path of the graph $G$. From the property of spheres of the graph $F$ we have the inequalities $|S^F(u_s) \cap S^F(u'_s)| \geq p$ and $|S^F(u_{s+1}) \cap S^F(u'_{s+1})| \geq p$. Therefore,

$$|S^F(u_s)| \geq p + 1, \ |S^F(u_{s+1})| \geq p + 1. \quad (13)$$

From the property of a pole of the graph $F$ we obtain

$$\exists z \in V(F) \setminus (B^F_1(u_s) \cup B^F_1(u_{s+1})).$$

It is also easy to see that the sets $\{u_0, \ldots, u_{k-2}\}, \{z\}, S^F(u_s), S^F(u_{s+1})$ are pairwise disjoint. Hence,

$$|V| \geq |S^F(u_s)| + |S^F(u_{s+1})| + k \geq 2p + k + 2. \quad (14)$$

Now, for even $k$, from Lemma 5 and the relations (13), (14) we obtain the required statement (i).

Let $k$ be odd. Due to the inequality (14) and the condition $p \geq 1$, the sets $\{2\}$, $\{p + 2, p + 3, \ldots, n - k - p\}$, $\{n - k + 1\}$, $\{n - k + 1\}$ are pairwise disjoint. Now from Lemma 5 and the relations (13), (14) we obtain the statement (ii). \qed

**Corollary 6.** If $G \in \mathcal{F}_{n,k,p}$, then $n \geq 2p + k + 2$.

Let us show the realizability of all center cardinalities indicated in Lemma 6 in graphs of the class $\mathcal{F}_{n,k,p}$ for all large enough $n$. For this purpose, give a method of constructing the graphs $G(u, s, F)$ of the class $\mathcal{F}_{n,k,p}$ using two constructions that allow us to construct graphs $F \in \mathcal{F}_{n,3}(x, y)$.

First, for any $m \geq 1$ define $m$-vertex graph $H_m$ in the following way. We put $H_1 = K_1$. For $m \geq 2$, fix pairwise non-adjacent edges $e_i = v_i, v'_i$, $i = 1, 2, \ldots, |m|/2$ in a complete graph $K_{|m|/2}$. If $m$ even, put $H_m = K_{|m|/2} \setminus \{e_1, \ldots, e_{|m|/2}\}$, else $H_m = K_1 + K_{|m|/2} \setminus \{e_1, \ldots, e_{|m|/2}\}$.

**Lemma 7** (properties $H_m$). The following properties of the graph $H_m$ hold:

(i) $\delta(H_m) \geq m - 2$;

(ii) $H_m$ does not contain shuttlecocks.
**Proof.** Every vertex of the graph $H_m$ is not joined by an edge with at most one vertex, therefore $\delta(H_m) \geq m-2$. Prove statement (ii). Let $e$ be an edge of the graph $H_m$. Then $m \geq 3$. If the edge $e$ is of the form $v_iv_j$ (the case $e = v'_iv'_j$ is similar), then $e$ belongs to the shortest path $(v_i, v_j, v'_j)$ of length 2. In the case when $e$ has the form $v_i v'_j$ (the case $e = v'_iv_j$ is similar), the edge $e$ belongs to the shortest path $(v'_i, v_i, v_j)$ of length 2. There remains the case when $m$ is odd and $e = v_0v_i$ (similarly $e = v_0v'_i$), where $v_0$ is a vertex adjacent to all other vertices. Then $H_m$ contains the shortest path $(v_i, v_0, v'_i)$ of length 2. Hence, in all cases $H_m$ does not contain a shuttlecock on the edge $e$.

Let $P = (x, x', y', y)$ be a 4-vertex simple path with endpoints $x, y$, $G_1$ and $G_2$ are arbitrary graphs, and $z$ is a new vertex, moreover, the sets $V(G_1), V(G_2), \{x, x', y', y\}, \{z\}$ are pairwise disjoint. Join each vertex of the graph $G_1$ by edges with vertices $x, x', y', z$ and connect each vertex of the graph $G_2$ by edges with vertices $x', y', y, z$. Also, join each vertex of the graph $G_1$ by edges with all vertices of the graph $G_2$. The resulting graph denote by $F(G_1 + G_2)$ (see Fig. 3).

![Fig. 3. Graph $F(G_1 + G_2)$](image)

**Lemma 8** (graph $F(G_1 + G_2)$). Let graphs $G_1$, $G_2$ do not contain shuttlecocks and $\delta(G_1) \geq p - 1$, $\delta(G_2) \geq p - 1$, $p \geq 1$. Then $F(G_1 + G_2) \in \mathcal{F}_{n,3,p}(x, y)$, where $n = |V(G_1)| + |V(G_2)| + 5$.

**Proof.** Denote the graph $F(G_1 + G_2)$ by $F$. It is directly verified that the graph $F$ has diameter 3, $z$ is its pole and $x, y$ is the unique pair of diametral vertices.

Let us show that $F$ does not contain shuttlecocks. Let $e \in E(F)$. If $e \in E(G_1)$, then the edge $e$ in the graph $G_1$ belongs to some shortest path of length 2, as far as $G_1$ does not contain shuttlecocks. Since this shortest path is also the shortest in the graph $F$, the graph $F$ does not contain a shuttlecock on the edge $e$. Now let $e \notin E(G_1) \cup E(G_2)$. Then in the graph $F$ the edge $e$ belongs to one of the following shortest paths of length 2: $(z, v_1, u_1), (z, v_2, u_2), (x, x', y'), (x', y', y), (v_1, v_2, y)$, where $v_1 \in V(G_1), v_2 \in V(G_2), u_1 \in \{x, x', y'\}$ and $u_2 \in \{x', y', y\}$. Hence, $F$ does not contain a shuttlecock on the edge $e$.  

---

**Fig. 3. Graph $F(G_1 + G_2)$**
Check the property of spheres. Let \( v_1 \in V(G_1) \) and \( v_2 \in V(G_2) \cup \{x',y',z\} \). Then the following inequalities hold

\[
|S^F_1(x) \cap S^F_1(v_2)| \geq |V(G_1)| \geq \delta(G_1) + 1 \geq p,
\]

\[
|S^F_1(x) \cap S^F_1(v_1)| \geq |S^G_1(v_1) \cup \{x'\}| \geq p.
\]

The same property of spheres for the diametral vertex \( y \) is established similarly. Let \( v, v'_i \in V(G_i) \cup \{x',y',z\} \), \( w_i \in V(G_i) \) and \( v \neq v'_i \), \( i = 1, 2 \). Then the following inequalities hold

\[
|S^F_1(v_i) \cap S^F_1(v'_j)| \geq |V(G_j)| \geq p, \text{ where } j \neq i,
\]

\[
|S^F_1(w_1) \cap S^F_1(w_2)| \geq |S^G_1(w_1) \cup \{z\}| \geq p.
\]

\[\square\]

Directly from Lemmas 7 and 8 we obtain the following consequences (here, for all specified values of the parameters, we define a graph \( F \in \mathcal{F}_{n,3,\phi}(x,y) \) such that \( |S^F_1(y)| = m + 1 \).

**Corollary 7** \((1 = p = m \leq n - 5 - p)\). For all \( n \geq 8 \) the graph \( F(H_{n-6} + K_1) \) belongs to the class \( \mathcal{F}_{n,3,1}(x,y) \) and \( |S^F_1(H_{n-6} + K_1)(y)| = 2 \).

**Corollary 8** \((1 = p < m = n - 5 - p)\). For all \( n \geq 8 \) the graph \( F(H_{n-6} + K_1) \) belongs to the class \( \mathcal{F}_{n,3,1}(y,x) \) and \( |S^F_1(H_{n-6} + K_1)(x)| = n - 5 \).

**Corollary 9** \((1 \leq p < m < n - 5 - p)\). Let \( p \geq 1 \). Then for all \( n \geq 2p + 7 \) and any integer \( m \) such that \( p < m < n - 5 - p \), the graph \( F(H_{n-5-m} + H_m) \) belongs to the class \( \mathcal{F}_{n,3,\phi}(x,y) \) and \( |S^F_1(H_{n-5-m} + H_m)(y)| = m + 1 \).

For arbitrary graphs \( G_1 \) and \( G_2 \) such that \( |V(G_1)| \geq |V(G_2)| \), define a graph \( F(G_1 \oplus G_2) \) in the following way. Consider an arbitrary subset \( W \subseteq V(G_1) \) of the cardinality \( |V(G_2)| \) and bijection \( \varphi : W \to V(G_2) \). Put \( F(G_1 \oplus G_2) = F(G_1 + G_2) \setminus \{e \mid e \text{ is an edge with endpoints } w \in W \text{ and } \varphi(w) \in V(G_2)\} \) (see Fig. 4).
Lemma 9 (graph \( F(G_1 \oplus G_2) \)). Let \( G_1 \) and \( G_2 \) be graphs such that \( |V(G_1)| \geq |V(G_2)| \geq 2 \), \( G_1 \) does not contain shuttlecocks and \( \delta(G_1) \geq p - 1 \), \( \delta(G_2) \geq p - 1 \), \( p \geq 1 \). Then \( F(G_1 \oplus G_2) \in \mathcal{F}_{n,3,p}(x,y) \), where \( n = |V(G_1)| + |V(G_2)| + 5 \).

Proof. Denote the graph \( F(G_1 \oplus G_2) \) by \( F \). Directly it is verified that the graph \( F \) has diameter 3, \( z \) is its pole, and \( x,y \) is the unique pair of diametral vertices.

Show that \( F \) does not contain shuttlecocks. Let \( e \in E(F) \). The case \( e \notin E(G_2) \) is considered in the same way as in Lemma 8. Now let \( e \) be an edge of the graph \( G_2 \) with endpoints \( v_2, v'_2 \in V(G_2) \). Then \( F \) contains the shortest path \( (v_2, v'_2, \varphi^{-1}(v_2)) \) of length 2. Hence, \( F \) does not contain a shuttlecock on the edge \( e \).

Check the property of spheres. Let \( u \in \{x', y', z\} \) and \( v_i \in V(G_i), i = 1, 2 \). Then the following inequalities hold

\[
\begin{align*}
|S_F^i(x) \cap S_F^i(u)| &\geq |V(G_i)| \geq p, \\
|S_F^i(x) \cap S_F^i(v_1)| &\geq |S_G^i(v_1) \cup \{x\}| \geq p, \\
|S_F^i(x) \cap S_F^i(v_2)| &\geq |(V(G_1) \setminus \{\varphi^{-1}(v_2)\}) \cup \{x\}| \geq p, \\
|S_F^i(y) \cap S_F^i(u)| &\geq |V(G_2)| \geq p, \\
|S_F^i(y) \cap S_F^i(v_2)| &\geq |S_G^i(v_2) \cup \{y\}| \geq p, \\
|S_F^i(y) \cap S_F^i(v_1)| &\geq |(V(G_2) \setminus \{\varphi^{-1}(v_2)\}) \cup \{y\}| \geq p, \\
V_2 &\subseteq \{\varphi(v_1)\} \quad \text{if } v_1 \in W, \\
&\varnothing \quad \text{else.}
\end{align*}
\]

Let \( u, u' \in \{x', y', z\}, u \neq u' \) and \( v_i, v'_i \in V(G_i), v_i \neq v'_i, i = 1, 2 \). Reckoning the condition \( |V(G_i)| \geq 2 \), we obtain the following inequalities

\[
\begin{align*}
|S_F^i(v_1) \cap S_F^i(v'_1)| &\geq |(V(G_j) \setminus V_j) \cup \{x', y'\}| \geq p, \\
|S_F^i(v_1) \cap S_F^i(u)| &\geq |S_G^j(v_1) \cup (V(G_j) \setminus V'_j)| \geq \delta(G_j) + 1 \geq p, \\
|S_F^i(v_1) \cap S_F^i(v_2)| &\geq |S_G^j(v_1) \setminus \{\varphi^{-1}(v_2)\} \cup \{x', y'\}| \geq \delta(G_j) + 1 \geq p, \\
|S_F^j(u) \cap S_F^j(u')| &\geq |V(G_i)| \geq p, \quad \text{where } j \neq i \text{ and}
\end{align*}
\]

\[
V_j = \begin{cases} 
\{\varphi^{-1}(v_1), \varphi^{-1}(v'_1)\} & \text{if } i = 2, \\
\{\varphi(v_1), \varphi(v'_1)\} & \text{if } i = 1 \text{ and } v_i, v'_i \in W, \\
\{\varphi(v_1)\} & \text{if } i = 1 \text{ and } v_i \in W, v'_i \notin W, \\
\varnothing & \text{else;}
\end{cases}
\]

\[
V'_j = \begin{cases} 
\{\varphi^{-1}(v_1)\} & \text{if } i = 2, \\
\{\varphi(v_1)\} & \text{if } i = 1 \text{ and } v_i \in W, \\
\varnothing & \text{else.}
\end{cases}
\]

Directly from Lemmas 7 and 9 we obtain the following consequences (here, for all specified values of the parameters, we define a graph \( F \in \mathcal{F}_{n,3,p}(x,y) \) such that \( |S_F^i(y)| = m + 1 \).

Corollary 10 (\( 2 \leq p = m \leq n - 5 - p \)). Let \( p \geq 2 \). Then for all \( n \geq 2p+6 \) the graph \( F(H_{n-5-p} \oplus K_p) \) belongs to the class \( \mathcal{F}_{n,3,p}(x,y) \) and \( |S_F^i(H_{n-5-p} \oplus K_p)(y)| = p + 1 \).
Corollary 11 \( (2 \leq p < m = n - 5 - p) \). Let \( p \geq 2 \). Then for all \( n \geq 2p + 6 \) the graph \( F(H_{n-5-p} \circ K_p) \) belongs to the class \( \mathcal{F}_{n,3,p}(y,x) \) and \( |S^F_1(H_{n-5-p} \circ K_p)(x)| = n - 4 - p \).

Theorem 5 (spectrum \( \mathcal{S}_p(\mathcal{F}_{n,k,p}) \)). Let \( k \geq 3 \) and \( p \geq 1 \). Then for every \( n \geq 2p + k + 4 \) the following equalities hold

(i) \( \mathcal{S}_p(\mathcal{F}_{n,k=3,p}) = \{n-2\} \);
(ii) \( \mathcal{S}_p(\mathcal{F}_{n,k=4,p}) = \{1 + p, n - 5 - p\} \);
(iii) \( \mathcal{S}_p(\mathcal{F}_{n,k=5,p}) = \{2 + p, n - 5 - p\} \cup \{n-4\} \);
(iv) \( \mathcal{S}_p(\mathcal{F}_{n,k,p}) = \{1\} \cup \{1 + p, n - k - 1 - p\} \) for every even \( k \geq 6 \);
(v) \( \mathcal{S}_p(\mathcal{F}_{n,k,p}) = \{2\} \cup \{2 + p, n - k - p\} \cup \{n-k+1\} \) for every odd \( k \geq 7 \).

Proof. From Lemma 6 we obtain the inclusion of the spectrum \( \mathcal{S}_p(\mathcal{F}_{n,k,p}) \) into the set of values of the center cardinalities indicated in the statement of the theorem. Show that any such value is realized as the cardinality of the center of a suitable graph from the class \( \mathcal{F}_{n,k,p} \) for all \( n \geq 2p + k + 4 \).

Let \( k \geq 4 \) be even. By Lemmas 5 and 6, for every \( k \geq 4 \) and \( n \geq 2p + k + 4 \), it is required to construct graphs \( G(u, s, F) \) such that \( s = \frac{k-2}{2} - 2, F \in \mathcal{F}_{n-k+3,3,p}(u_s, u_{s+1}) \), \( |S^F_1(u_{s+1})| = m+1 \) and \( m \) can take any value satisfying the inequalities \( p \leq m \leq n - k - 2 - p \). Equivalently, for every \( n \geq 2p + 7 \) and \( m, p \leq m \leq n - 5 - p \), it is required to construct a graph \( F \in \mathcal{F}_{n,3,p}(x,y) \) such that \( |S^F_1(y)| = m + 1 \). For \( m \) the following cases are possible:

1. \( p = m = n - 5 - p \) (Corollary 7);
2. \( p = m = n - 5 - p \) (Corollary 8);
3. \( 1 \leq p < m < n - 5 - p \) (Corollary 9);
4. \( 2 \leq p = m \leq n - 5 - p \) (Corollary 10);
5. \( 2 \leq p < m = n - 5 - p \) (Corollary 11).

For these cases, in the above corollaries it is shown the existence of the required graph \( F \). Further, for every \( k \geq 6 \) and each graph \( F \in \mathcal{F}_{n-k+3,3,p}(u_s, u_{s+1}) \) (for example, for \( F(H_{p+1} + H_{n-k-3} \circ K_p) \) if \( n \geq 2p + k + 4 \)), we have \( |C(G)| = 1 \) by Lemma 6, where \( G = G(u, 0, F) \in \mathcal{F}_{n,k,p} \). Thus, for any even diameter \( k \geq 4 \), each value of the center cardinality indicated in the statement of the theorem is realized in graphs of the class \( \mathcal{F}_{n,k,p} \) for all \( n \geq 2p + k + 4 \).

Now let \( k \geq 3 \) be odd. Note that for each graph \( F \in \mathcal{F}_{n-k+3,3,p}(u_s, u_{s+1}) \) (for example, for \( F(H_{p+1} + H_{n-k-3} \circ K_p) \) if \( n \geq 2p + k + 4 \)), we have \( |C(G)| = n - k + 1 \) by Lemma 6, where \( G = G(u, \frac{k-2}{2}, F) \in \mathcal{F}_{n,k,p} \).

By Lemmas 5 and 6, it is required for every \( k \geq 5 \) and \( n \geq 2p + k + 4 \) to construct graphs \( G(u, s, F) \) such that \( s = \frac{k-5}{2}, F \in \mathcal{F}_{n-k+3,3,p}(u_s, u_{s+1}) \), \( |B^F_1(u_{s+1})| = m \) and \( m \) can take any value satisfying the inequalities \( 2 + p \leq m \leq n - k - p \). Equivalently, for every \( n \geq 2p + 7 \) and \( m, p \leq m \leq n - 5 - p \), it is required to construct a graph \( F \in \mathcal{F}_{n,3,p}(x,y) \) such that \( |S^F_1(y)| = m + 1 \). The existence of such graph was proved above when considering the case of even \( k \).

Further, for every \( k \geq 7, n \geq 2p + k + 4 \) and each graph \( F \in \mathcal{F}_{n-k+3,3,p}(u_s, u_{s+1}) \), we have \( |C(G)| = 2 \) by Lemma 6, where \( G = G(u, \frac{k-2}{2}, F) \in \mathcal{F}_{n,k,p} \). Thus, for any odd diameter \( k \geq 7 \) the spectrum \( \mathcal{S}_p(\mathcal{F}_{n,k,p}) \) has the required form. \( \square \)

Theorem 6 (spectrum of almost all graphs from \( \mathcal{F}_{n,d=k} \)). Let \( k \geq 1 \) and \( p \geq 1 \) be fixed integer constants. Then

(i) \( |C(G)| = n \) for almost all \( n \)-vertex graphs \( G \) of diameter \( k = 1, 2 \);
Hence, by Lemmas 1(i) and 2, obtain inclusions hold

\[ \text{Proof. In Section 3, we noticed that } C(G) = V(G) \text{ for almost all graphs } G \text{ of diameter } k = 1, 2. \]

Further, directly from Corollary 1 and Theorem 5, we obtain the possible values of the center cardinality of almost all n-vertex graphs of fixed diameter \( k \geq 3 \) for the cases (ii)-(v). Find the asymptotic fractions of the indicated classes of graphs. By \( C_{n,k,i} \) we denote the class of all graphs from \( J_{n,d} \) with an \( i \)-vertex center.

Let \( k \geq 4 \) be even and \( s^* = \frac{k}{2} - 2 \). Note that, by Lemmas 2 and 6, the following inclusions hold

\[ F_{n,k,p} \setminus F_{n,k,p}^* \subseteq C_{n,k,1} \subseteq J_{n,d} \setminus F_{n,k,p}^*. \]  

Hence, by Lemmas 1(i) and 2, obtain

\[ |F_{n,k,p}| \left( 1 - \frac{\sigma(s^*)}{k-2} \right) \leq |C_{n,k,1}| \leq |J_{n,d} - F_{n,k,p}|. \]  

Thus, from (16), Lemma 2 and Corollary 1 as \( n \to \infty \) we conclude

\[ \frac{|C_{n,k,1}|}{|J_{n,d} - F_{n,k,p}|} \to 1 - \frac{\sigma(s^*)}{k-2} = \frac{k-4}{k-2}. \]  

Now, let \( k \geq 3 \) be odd and \( s^* = \frac{k-3}{2} \). By Lemmas 2, 6 and Corollary 6, the following inclusions hold

\[ F_{n,k,p}^* \subseteq C_{n,k,n-k+1} \subseteq J_{n,d} \setminus (F_{n,k,p} \setminus F_{n,k,p}^*). \]  

Hence, by Lemmas 1(i) and 2, obtain

\[ \frac{\sigma(s^*)}{k-2} |F_{n,k,p}| \leq |C_{n,k,n-k+1}| \leq |J_{n,d} - F_{n,k,p}| \left( 1 - \frac{\sigma(s^*)}{k-2} \right). \]  

Thus, from (18), Lemma 2 and Corollary 1 as \( n \to \infty \) we conclude

\[ \frac{|C_{n,k,n-k+1}|}{|J_{n,d} - F_{n,k,p}|} \to \frac{\sigma(s^*)}{k-2} = \frac{1}{k-2}. \]  

Finally, let \( k \geq 7 \) be odd and \( s^{**} = \frac{k-5}{2} \). By Lemmas 2 and 6, the following inclusions hold

\[ F_{n,k,p} \setminus (F_{n,k,p}^* \cup F_{n,k,p}^{**}) \subseteq C_{n,k,2} \subseteq J_{n,d} \setminus (F_{n,k,p}^* \cup F_{n,k,p}^{**}). \]  

Hence, by Lemmas 1(i) and 2, obtain

\[ |F_{n,k,p}| \left( 1 - \frac{\sigma(s^*) + \sigma(s^{**})}{k-2} \right) \leq |C_{n,k,2}| \leq |J_{n,d} - F_{n,k,p}| \left( 1 - \frac{\sigma(s^*) + \sigma(s^{**})}{k-2} \right). \]
Thus, from (20), Lemma 2 and Corollary 1 as \( n \to \infty \) we conclude

\[
\frac{|C_{n,k,2}|}{|J_{n,d=k}|} \to 1 - \frac{\sigma(s^*) + \sigma(s^{**})}{k - 2} = \frac{k - 5}{k - 2}.
\]

\( \square \)

Theorem 6 implies a number of properties of the centers of almost all graphs of
fixed diameter \( k \). For example, there are almost no graphs with a trivial center of
diameter \( k = 2,4 \) and odd diameter \( k \), while for any even \( k \geq 6 \) this is not true.
Similarly, there are almost no graphs with a 2-vertex center of diameter \( k = 1,3,5 \)
and even diameter \( k \), however, for every odd \( k \geq 7 \) this does not hold. Unexpected is
the jump of the center cardinality outside the interval of consecutive integer values
both from above from \( n - k - p \) to \( n - k + 1 \) for odd \( k \geq 5 \), and from below from
\( 1 + p \) to 1 for even \( k \geq 6 \) and from \( 2 + p \) to 2 for odd \( k \geq 7 \).

In the following corollaries we find typical graphs for graphs classes corresponding
to the cardinality cases of the center in Theorem 6.

**Corollary 12.** Let \( k \geq 4 \) be an even integer, \( p \geq 1 \) and \( s^* = \frac{k}{2} - 2 \). Then

(i) \( F_{n,k,p}^{s*} \) is a class of typical graphs of the class of \( n \)-vertex graphs of fixed
diameter \( k \geq 4 \) with a nontrivial center;

(ii) \( J_{n,d=k} \setminus F_{n,k,p}^{s*} \) is a class of typical graphs of the class of \( n \)-vertex graphs of
fixed diameter \( k \geq 6 \) with a trivial center.

**Proof.** Note that \( F_{n,k,p} \setminus F_{n,k,p}^{s*} \) is a subclass of \( C_{n,k,1} \) and \( F_{n,k,p}^{s*} \) is a subclass of
\( J_{n,d=k} \setminus C_{n,k,1} \) by (15) and Lemma 1(i). Reckoning Lemma 2, Corollary 1 and
Theorem 6, for \( n \to \infty \) we obtain

\[
\frac{|F_{n,k,p}^{s*}|}{|J_{n,d=k} \setminus C_{n,k,1}|} = \frac{2}{k - 2} \frac{|F_{n,k,p}|}{|J_{n,d=k}|} \left(1 - \frac{|C_{n,k,1}|}{|J_{n,d=k}|}\right)^{-1} \to 1,
\]

\[
\frac{|F_{n,k,p} \setminus F_{n,k,p}^{s*}|}{|C_{n,k,1}|} = \frac{k - 4}{k - 2} \frac{|F_{n,k,p}|}{|J_{n,d=k}|} \frac{|J_{n,d=k}|}{|C_{n,k,1}|} \to 1.
\]

\( \square \)

**Corollary 13.** Let \( k \geq 3 \) be an odd integer, \( p \geq 1 \) and \( s^* = \frac{k - 3}{2} \), \( s^{**} = \frac{k - 5}{2} \). Then

(i) \( F_{n,k,p}^{s*} \) is a class of typical graphs of the class of \( n \)-vertex graphs of diameter
\( k \) with an \((n - k + 1)\)-vertex center;

(ii) \( F_{n,k,p} \setminus F_{n,k,p}^{s*} \) is a class of typical graphs of the class of \( n \)-vertex graphs of
diameter 5, whose center cardinality is not equal to \( n - 4 \);

(iii) \( F_{n,k,p} \setminus (F_{n,k,p}^{s*} \cup F_{n,k,p}^{s**}) \) is a class of typical graphs of the class of \( n \)-vertex
graphs of fixed diameter \( k \geq 7 \) with a 2-vertex center;

(iv) \( F_{n,k,p}^{s**} \) is a class of typical graphs of the class of \( n \)-vertex graphs of fixed
diameter \( k \geq 7 \), whose the center cardinality is not equal to 2 and \( n - k + 1 \).

**Proof.** By (17), (19) and Lemmas 1(i), 2, we have \( F_{n,k,p}^{s*} \subseteq C_{n,k,n-k+1} \), \( F_{n,k,p} \setminus
F_{n,k,p}^{s*} \subseteq J_{n,d=5} \setminus C_{n,5,n-4} \) and \( F_{n,k,p} \setminus (F_{n,k,p}^{s*} \cup F_{n,k,p}^{s**}) \subseteq C_{n,k,2} \), \( F_{n,k,p}^{s**} \subseteq J_{n,d=k} \setminus
(C_{n,k,2} \cup C_{n,k,n-k+1}) \) for \( k \geq 7 \). Reckoning Lemma 2, Corollary 1 and Theorem 6,
we obtain

\[
\frac{|F_{n,k,p}^{s*}|}{|C_{n,k,n-k+1}|} = \frac{1}{k - 2} \frac{|F_{n,k,p}|}{|J_{n,d=k}|} \frac{|J_{n,d=k}|}{|C_{n,k,n-k+1}|}.
\]
In view of Corollary 1, the obtained properties of the centers are also valid for graphs of the classes \( J_n, d \geq k \) and \( J_n, d \geq k \).

**Corollary 14.** For every fixed \( k \geq 2 \), almost all \( n \)-vertex graphs of each of the following classes \( J_n, d \geq k \), \( J_n, d \geq k \) are connected, have diameter \( k \), and their center satisfies the properties stated in Theorem 6.

**References**


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