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NEW PERFECT COLORINGS OF INFINITE CIRCULANT GRAPHS WITH CONTINUOUS SETS OF DISTANCES

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Abstract. This paper presents series of perfect colorings of circulant graphs with continuous set of n distances, which disproves the hypothesis [2] that period of such a coloring is orbital or has length at most $2n + 2$.

Keywords: perfect coloring, equitable partition, circulant graph, Cayley graph.

Introduction

The Cayley graph of an infinite cyclic group, which has a starting segment of the set of natural numbers as its generating set is called a circulant graph with a continuous set of distances and is denoted by $C_\infty(n)$. A vertex coloring is said to be perfect, if every pair of similarly colored vertices has similar sets of colors in their neighbourhoods.

The results of research of perfect colorings of circulant graphs are presented in papers [1, 3, 4, 5, 6]. In [6], a study of continuous perfect 2-colorings of circulant graphs is carried out and all parameters of such colorings are presented. In [5], an infinite series of non-continuous 2-colorings of such graphs with new parameters is considered.

It is known, that every perfect coloring of an infinite circulant graph is periodic [6]. We will denote a period of such coloring as a string in square brackets. Topics of periodicity and periodizability of colorings of infinite graphs are studied in [7, 8, 9].

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Perfect colorings (specifically distance-regular ones) of infinite Cayley graphs were also considered in [10, 11, 12, 13, 14, 15]. A 2-dimensional square grid is an object of specific interest, since there exists a natural homomorphism into an arbitrary circulant graph with n distances. This means, for instance, that one can use a perfect coloring of a circulant graph with n distances to construct such a perfect coloring of an n-dimensional square grid, that will have similar parameters.

Perfect 2-colorings of a graph $C_\infty(n)$ were studied in [1]. In [2], one can find the hypothesis stating, that all possible perfect colorings into an arbitrary number of colors are either orbital or perfect periods of lengts $2n, 2n+1$ and $2n+2$. In [3], a description of all perfect colorings of that graph with distances 1 and 2 with an arbitrary finite number of colors is obtained. This result provides a positive answer to the hypothesis in the case of two distances.

In our paper we show that this hypothesis is wrong for the case $n = 3m+1$. Recall that a vertex coloring of a graph is called orbital if all monochromatic sets of vertices of that coloring are orbits of some subgroup of an automorphism group of that graph.

Main result

For the rest of the paper by G_n we denote a graph $C_\infty(n)$, where $n = 3m+1$. This graph is infinite with integers as its vertices. The edges are connecting all pairs of vertices i and j, such that $|i-j| \le n$.

Let $t \in Z$. Consider the following coloring of vertices of G_n:

$$\phi(i) = \begin{cases} \text{a}, & i = 1 \text{ or } 2 \bmod 6 \\ \text{b}, & i = 4 \text{ or } 5 \bmod 6 \\ t, & i = 3t \text{ or } 3t+6m+3 \end{cases}$$

For example, for $m = 1$ a period of this coloring will have length 18 and will look as follows: [0 a a 1 b b 2 a a 0 b b 1 a a 2 b b]. We will call the colors, denoted by letters and numbers "lettered" colors and "numbered" colors respectively.

Theorem 1. *The coloring $\phi(i)$ is a perfect coloring of the graph $C_\infty(n)$ with the period $4n+2$.*

Proof. We split the proof into two parts: first we show that this coloring is indeed perfect, then we confirm that it has period $4n+2$. We will require some auxiliary statements. The first three lemmas are more general, while the other concern specific properties of the coloring ϕ.

Lemma 1. *For every "numbered" color, every sphere of radius 1 in the graph G_n contains exactly one vertice colored by that color.*

Proof. This statement is trivial, because by merging colors a and b we obtain a coloring of period $2n+1$, which is equal to the volume of the ball. Note that if all vertices in that period are colored differently, then each color is a so-called 1-perfect code, and these codes can be merged with each other to obtain new perfect colorings. □

Lemma 2. *Consider a set of colors of an arbitrary graph G. Pick two of its non-intersecting subsets A and B, such that by merging colors from A one obtains a perfect coloring, and a perfect coloring is also obtained by merging colors from B. Then the original coloring is also perfect.*

Proof. For any two colors x and y there exists a merge (one of the two possible merges), which uniquely determines how many vertices of color y are in the neighbourhood of a vertex of color x. Hence the original coloring satisfies a definition of a perfect coloring. $\square$

Now consider an arbitrary coloring of a graph G_n. Define a period of a color as a period of its characteristic function (which is equal to 1 on the vertices of that color and equal to 0 on other vertices).

Lemma 3. *A minimal period of a perfect coloring in an infinite circulant graph is equal to the least common multiple of periods of all the colors of that coloring.*

Proof. This statement is well known from theory of formal languages and requires only some elementary calculations to check. $\square$

The following statement holds:

Lemma 4. *The coloring $\phi(i)$ possesses the following properties:*

1. By merging "numbered" colors one obtains an orbital perfect coloring with the period of length 6: [a a 0 b b 0]
2. For every "numbered" color, every sphere of radius 1 contains exactly one vertice colored by that color.
3. A coloring, obtained by merging two colors is a perfect coloring of period $2n+1$.

Proof. All properties can be verified directly. $\square$

Now we can move on to the proof of the theorem. Lemma 2, and properties 1 and 3 of Lemma 4 imply that the coloring is perfect. By using Lemma 1, we obtain that the period of the coloring $\phi(i)$ is equal to $4n+2$ (as the least common multiple of an odd number $2n+1$ and even number 6). Thus, the theorem is proved. $\square$

Remark. *Note that by merging all "numbered" colors except one, we obtain a perfect coloring with 4 colors of period $4n+2$. This construction does not work for the number of colors less than 4, since such colorings have a smaller period.*

Conclusion

The first example of a coloring with a non-standard period was found by computer search and contained 4 colors. Note, that the question of existence of non-standard colorings for three colors and the number of distances, different from $3m+1$ is still open.

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