ERGODIC THEOREMS IN BANACH IDEALS OF COMPACT OPERATORS

A.N. AZIZOV, V.I. CHILIN

ABSTRACT. Let $\mathcal{H}$ be an infinite-dimensional Hilbert space, and let $\mathcal{B}(\mathcal{H})$ $(\mathcal{K}(\mathcal{H}))$ be the $C^*$-algebra of all bounded (compact) linear operators in $\mathcal{H}$. Let $(E, \| \cdot \|_E)$ be a fully symmetric sequence space. If $\{s_n(x)\}_{n=1}^{\infty}$ are the singular values of $x \in \mathcal{K}(\mathcal{H})$, let $\mathcal{C}_E = \{x \in \mathcal{K}(\mathcal{H}) : \{s_n(x)\} \in E\}$ with $\|x\|_{\mathcal{C}_E} = \|\{s_n(x)\}\|_E$, $x \in \mathcal{C}_E$, be the Banach ideal of compact operators generated by $E$. We show that the averages $A_n(T)(x) = \frac{1}{n+1} \sum_{k=0}^{n} T^k(x)$ converge uniformly in $\mathcal{C}_E$ for any Dunford-Schwartz operator $T$ and $x \in \mathcal{C}_E$. Besides, if $0 \leq x \in \mathcal{B}(\mathcal{H}) \setminus \mathcal{K}(\mathcal{H})$, there exists a Dunford-Schwartz operator $T$ such that the sequence $\{A_n(T)(x)\}$ does not converge uniformly. We also show that the averages $A_n(T)$ converge strongly in $(\mathcal{C}_E, \| \cdot \|_{\mathcal{C}_E})$ if and only if $E$ is separable and $E \neq l^1$ as sets.

Keywords: symmetric sequence space, Banach ideal of compact operators, Dunford-Schwartz operator, individual ergodic theorem, mean ergodic theorem.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ be the $C^*$-algebra of all bounded linear operators in a complex Hilbert space $\mathcal{H}$, equipped with the uniform norm $\| \cdot \|_{\infty}$. The study of noncommutative individual ergodic theorems in the space of measurable operators affiliated with a semifinite von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ equipped with a faithful normal semifinite trace $\tau$ was initiated by F. Yeadon. In [23], as a corollary of a noncommutative maximal ergodic inequality in $L^1 = L^1(\mathcal{M}, \tau)$, the following individual ergodic theorem was established.

Azizov, A.N., Chilin, V.I., Ergodic theorems in Banach ideals of compact operators.
© 2021 Azizov A.N., Chilin V.I.
Received February, 26, 2021, published May, 21, 2021.
**Theorem 1.** Let $T : L^1 \to L^1$ be a positive $L^1 - L^\infty$-contraction. Then for any $x \in L^1$ there exists $\hat{x} \in L^1$ such that the averages

$$A_n(T)(x) = \frac{1}{n+1} \sum_{k=0}^{n} T^k(x)$$

converge to $\hat{x}$ bilaterally almost uniformly (in Egorov’s sense), that is, given $\varepsilon > 0$, there exists a projection $e \in \mathcal{M}$ such that $\tau(1-e) < \varepsilon$ and

$$\|e(A_n(T)(x) - \hat{x})\|_\infty \to 0 \quad \text{as} \quad n \to \infty,$$

where $1$ is the unit of $\mathcal{M}$.

The study of individual ergodic theorems beyond $L^1(\mathcal{M}, \tau)$ started much later with another fundamental paper by M. Junge and Q. Xu [13], where, among other results, individual ergodic theorem was extended to the case with a positive Dunford-Schwartz operator acting in the space $L^p(\mathcal{M}, \tau)$, $1 < p < \infty$. In [3] ([4]), utilizing the approach of [16], an individual ergodic theorem was proved for a positive Dunford-Schwartz operator in a noncommutative Lorentz (respectively, Orlicz) space.

Let $\mathcal{H}$ be a complex infinite-dimensional Hilbert space. Let $E \subset c_0$ be a fully symmetric sequence space. Denote by $\mathcal{C}_E$ the Banach ideal of compact operators in $\mathcal{H}$ associated with $E$. In Section 3 of the article, we obtain the following individual Dunford-Schwartz-type ergodic theorem.

**Theorem 2.** (i) Given a Dunford-Schwartz operator $T : \mathcal{C}_E \to \mathcal{C}_E$ and $x \in \mathcal{C}_E$, there exists $\hat{x} \in \mathcal{C}_E$ such that $\|A_n(T)(x) - \hat{x}\|_\infty \to 0$ as $n \to \infty$;

(ii) If $0 \leq x \in B(\mathcal{H}) \setminus K(\mathcal{H})$, then there exists a Dunford-Schwartz operator $T : B(\mathcal{H}) \to B(\mathcal{H})$ such that the averages $A_n(T)(x)$ do not converge uniformly. 

Noncommutative mean ergodic theorem can be stated as follows: if $T$ is an $L^1 - L^\infty$-contraction and $1 < p < \infty$, then the averages $A_n(T)$ converge strongly in $L^p = L^p(\mathcal{M}, \tau)$, that is, given $x \in L^p$, there exists $\hat{x} \in L^p$ such that $\|A_n(T)(x) - \hat{x}\|_p \to 0$ as $n \to \infty$. If $p = 1$ and $\tau(1) = \infty$, this is not true in general. As a consequence, if $\tau(1) = \infty$, mean ergodic theorem may not hold in some noncommutative symmetric spaces. In Yeadan’s paper [24], the following mean ergodic theorem was established.

**Theorem 3.** Let $E = (E(\mathcal{M}, \tau), \|\cdot\|_E)$ be a noncommutative fully symmetric space such that

(i) $L^1(\mathcal{M}, \tau) \cap \mathcal{M}$ is dense in $E$;

(ii) $\|e_n\|_E \to 0$ for any sequence of projections $\{e_n\} \subset L^1(\mathcal{M}, \tau) \cap \mathcal{M}$ with $e_n \downarrow 0$;

(iii) $\|e_n\|_E/\tau(e_n) \to 0$ for any increasing sequence of projections $\{e_n\} \subset \mathcal{M}$, $0 < \tau(e_n) < \infty$, with $\tau(e_n) \to \infty$.

Then for any $x \in E$ and a positive $L^1 - L^\infty$-contraction $T : E \to E$ there exists $\hat{x} \in E$ such that $\|A_n(T)(x) - \hat{x}\|_E \to 0$.

In [3], the mean ergodic theorem was established for a noncommutative symmetric space $E(\mathcal{M}, \tau)$ associated with a fully symmetric function space with nontrivial Boyd indices and order continuous norm.

In Section 4, we give the following criterion for the validity of the mean ergodic theorem in a Banach ideal of compact operators in $\mathcal{H}$.

**Theorem 4.** The following conditions are equivalent:
For any Dunford-Schwartz operator \( T : C_E \rightarrow C_E \) the averages \( A_n(T) \) converge strongly in \( C_E \):

(ii). \( (E, \| \cdot \|_E) \) is separable and \( E \neq l^1 \) as sets.

Commutative counterparts of Theorems 2 and 4 were established in [2].

In the end of the article, we give applications of Theorems 2 and 4 to the well-studied Orlicz and Lorentz ideals of compact operators. We note that our noncommutative versions of ergodic theorems are true for any Dunford-Schwarz operators without the assumption that these operators are positive.

2. Preliminaries

2.1. Symmetric sequence spaces. Let \( l^\infty \) (respectively, \( c_0 \)) be the Banach space of bounded (respectively, converging to zero) sequences \( \{\xi_n\}_{n=1}^\infty \) of complex numbers equipped with the uniform norm \( \|\{\xi_n\}\|_\infty = \sup_{n \in \mathbb{N}} |\xi_n| \), where \( \mathbb{N} \) is the set of natural numbers. If \( 2^\mathbb{N} \) is the \( \sigma \)-algebra of all subsets of \( \mathbb{N} \) and \( \mu(\{n\}) = 1 \) for each \( n \in \mathbb{N} \), then \( (\mathbb{N}, 2^\mathbb{N}, \mu) \) is a \( \sigma \)-finite measure space such that \( L^\infty(\mathbb{N}, 2^\mathbb{N}, \mu) = l^\infty \) and

\[
L^1(\mathbb{N}, 2^\mathbb{N}, \mu) = l^1 = \left\{ \{\xi_n\}_{n=1}^\infty \subset C : \|\{\xi_n\}\|_1 = \sum_{n=1}^\infty |\xi_n| < \infty \right\} \subset l^\infty,
\]

where \( C \) is the field of complex numbers.

For any subset \( E \subset l^\infty \) we denote \( E_h = \{\{\xi_n\}_{n=1}^\infty \in E : \xi_n \in \mathbb{R} \text{ for each } n\} \), where \( \mathbb{R} \) is the field of real numbers. It is known that \( (l^\infty_h, \| \cdot \|_\infty) \) and \( ((c_0)_{h}, \| \cdot \|_\infty) \) are Banach lattices with respect to the natural partial order

\[
\{\xi_n\} \leq \{\eta_n\} \iff \xi_n \leq \eta_n \text{ for all } n \in \mathbb{N}.
\]

If \( \xi = \{\xi_n\}_{n=1}^\infty \in l^\infty \), then the non-increasing rearrangement \( \xi^* : (0, \infty) \to (0, \infty) \) of \( \xi \) is defined by

\[
\xi^*(t) = \inf \{\lambda : \mu(\{|\xi| > \lambda\} \leq t), \ t > 0, \}
\]

(see, for example, [1, Ch. 2, Definition 1.5]). As such, the non-increasing rearrangement of a sequence \( \{\xi_n\}_{n=1}^\infty \in l^\infty \) can be identified with the sequence \( \xi^* = \{\xi^*_n\}_{n=1}^\infty \), where

\[
\xi^*_n = \inf \left\{\sup_{n \in F} \xi_n : F \subset \mathbb{N}, |F| < n \right\}.
\]

If \( \{\xi_n\} \in c_0 \), then \( \xi^*_n \downarrow 0 \); in this case there exists a bijection \( \pi : \mathbb{N} \to \mathbb{N} \) such that \( |\xi_n(\pi(n))| = \xi^*_n, \ n \in \mathbb{N} \).

Hardy-Littlewood-Polya partial order in the space \( l^\infty \) is defined as follows:

\[
\xi = \{\xi_n\} \prec \prec \eta = \{\eta_n\} \iff \sum_{n=1}^m \xi_n^* \leq \sum_{n=1}^m \eta_n^* \text{ for all } m \in \mathbb{N}.
\]

A non-zero linear subspace \( E \subset l^\infty \) with a Banach norm \( \| \cdot \|_E \) is called a symmetric (fully symmetric) sequence space if

\[
\eta \in E, \ \xi \in l^\infty, \ \xi^* \prec \prec \eta^* \iff \xi \in E \text{ and } \|\xi\|_E \leq \|\eta\|_E.
\]

Every fully symmetric sequence space is a symmetric sequence space. The converse is not true in general. At the same time, any separable symmetric sequence space is a fully symmetric space.

If \( (E, \| \cdot \|_E) \) is a symmetric sequence space, then

\[
\|\xi\|_E = \|\xi\|_E = \|\xi^*\|_E \text{ for all } \xi \in E.
\]
Besides, \((E_h, \| \cdot \|_E)\) is a Banach lattice with respect to the partial order induced from \(l^\infty\).

Immediate examples of fully symmetric sequence spaces are \((l^\infty, \| \cdot \|_\infty)\), \((c_0, \| \cdot \|_\infty)\) and the Banach spaces

\[
\ell^p = \left\{ \xi = (\xi_n)_{n=1}^\infty \in l^\infty : \|\xi\|_p = \left( \sum_{n=1}^\infty |\xi_n|^p \right)^{1/p} < \infty \right\}, \quad 1 \leq p < \infty.
\]

For any symmetric sequence space \((E, \| \cdot \|_E)\) the following continuous embeddings hold [1, Ch. 2, §6, Theorem 6.6]: \((l^1, \| \cdot \|_1) \subset (E, \| \cdot \|_E) \subset (l^\infty, \| \cdot \|_\infty)\). Besides, \(\|\xi\|_E \leq \|\xi\|_1\) for all \(\xi \in l^1\) and \(\|\xi\|_\infty \leq \|\xi\|_E\) for all \(\xi \in E\).

If there is \(\xi \in E \setminus c_0\), then \(\xi^* \geq \alpha 1\) for some \(\alpha > 0\), where \(1 = \{1, 1, \ldots\}\). Consequently, \(1 \in E\) and \(E = l^\infty\). Therefore, either \(E \subset c_0\) or \(E = l^\infty\).

2.2. Symmetric operator spaces. Now, let \((\mathcal{H}, (\cdot, \cdot))\) be an infinite-dimensional Hilbert space over \(\mathbb{C}\), and let \((B(\mathcal{H}), \| \cdot \|_{\infty})\) be the C*-algebra of all bounded linear operators in \(\mathcal{H}\). Denote by \(K(\mathcal{H}) (F(\mathcal{H}))\) the two-sided ideal of compact (respectively, finite rank) linear operators in \(B(\mathcal{H})\). It is well known that, for any proper two-sided ideal \(\mathcal{I} \subset B(\mathcal{H})\), we have \(F(\mathcal{H}) \subset \mathcal{I}\), and if \(\mathcal{H}\) is separable, then \(\mathcal{I} \subset K(\mathcal{H})\) (see, for example, [19, Proposition 2.1]). At the same time, if \(\mathcal{H}\) is a non-separable Hilbert space, then there exists a proper two-sided ideal \(\mathcal{I} \subset B(\mathcal{H})\) such that \(K(\mathcal{H}) \not\subseteq \mathcal{I}\).

Denote \(B_h(\mathcal{H}) = \{x \in B(\mathcal{H}) : x = x^*\}\), \(B_+(\mathcal{H}) = \{x \in B_h(\mathcal{H}) : x \geq 0\}\), and let \(\tau : B_+(\mathcal{H}) \to [0, \infty]\) be the canonical trace on \(B(\mathcal{H})\), that is,

\[
\tau(x) = \sum_{j \in J} (x\varphi_j, \varphi_j), \quad x \in B_+(\mathcal{H}),
\]

where \(\{\varphi_j\}_{j \in J}\) is an orthonormal basis in \(\mathcal{H}\) (see, for example, [20, Ch. 7, E. 7.5]).

Let \(\mathcal{P}(\mathcal{H}) = \{e \in B(\mathcal{H}) : e = e^2 = e^\perp\}\) be the lattice of projectors in \(B(\mathcal{H})\). If \(1\) is the identity of \(B(\mathcal{H})\) and \(e \in \mathcal{P}(\mathcal{H})\), we will write \(e^\perp = 1 - e\).

Let \(x \in B(\mathcal{H})\), and let \(\{e_\lambda(|x|)\}_{\lambda \geq 0}\) be the spectral family of projections for the absolute value \(|x| = (x^*x)^{1/2}\) of \(x\), that is, \(e_\lambda(|x|) = \{|x| \leq \lambda\}\). If \(t > 0\), then the \(t\)-th generalized singular number of \(x\), or the non-increasing rearrangement of \(x\), is defined as

\[
\mu_t(x) = \inf\{\lambda > 0 : \tau(e_\lambda(|x|)^{\perp}) \leq t\}
\]

(see [11]).

A non-zero linear subspace \(X \subset B(\mathcal{H})\) with a Banach norm \(\| \cdot \|_X\) is called symmetric (fully symmetric) if the conditions

\[
x \in X, \; y \in B(\mathcal{H}), \; \mu_t(y) \leq \mu_t(x) \quad \text{for all} \quad t > 0
\]

(respectively,

\[
x \in X, \; y \in B(\mathcal{H}), \; \int_0^s \mu_t(y)dt \leq \int_0^s \mu_t(x)dt \quad \text{for all} \quad s > 0 \quad (\text{writing} \; y \prec \prec x)
\]

imply that \(y \in X\) and \(\|y\|_X \leq \|x\|_X\).

The spaces \((B(\mathcal{H}), \| \cdot \|_{\infty})\) and \((K(\mathcal{H}), \| \cdot \|_{\infty})\) as well as the classical Banach two-sided ideals

\[
C^p = \{x \in K(\mathcal{H}) : \|x\|_p = \tau(|x|^p)^{1/p} < \infty\}, \quad 1 \leq p < \infty,
\]
are examples of fully symmetric spaces.

It should be noted that for every symmetric space \((X, \| \cdot \|_X) \subset \mathcal{B}(\mathcal{H})\) and all \(x \in X, a, b \in \mathcal{B}(\mathcal{H})\),

\[
\|x\|_X = \|x\|_X = \|x\|_X, \quad axb \in X, \quad \text{and} \quad \|AXB\|_X \leq \|a\|_\infty \|b\|_\infty \|x\|_X.
\]

**Remark 1.** If \(X \subset \mathcal{B}(\mathcal{H})\) is a symmetric space and there exists a projection \(e \in \mathcal{P}(\mathcal{H}) \cap \mathcal{X}\) such that \(\tau(e) = \infty\), that is, \(\dim(e(\mathcal{H})) = \infty\), then \(\nu_t(e) = \nu_t(1) = 1\) for every \(t \in (0, \infty)\). Consequently, \(1 \in X\) and \(X = \mathcal{B}(\mathcal{H})\). If \(X \not\subset \mathcal{B}(\mathcal{H})\) and \(x \in X\), then \(e_{\lambda}(|x|)^{-\lambda} = \{|x| > \lambda\}\) is a finite-dimensional projection, that is, \(\dim(e_{\lambda}(|x|)^{-\lambda}(\mathcal{H})) < \infty\) for all \(\lambda > 0\). This means that \(x \in \mathcal{K}(\mathcal{H})\), hence \(X \subset \mathcal{K}(\mathcal{H})\).

Therefore, either \(X = \mathcal{B}(\mathcal{H})\) or \(X \subset \mathcal{K}(\mathcal{H})\).

Thus, if \(\mathcal{H}\) is non-separable, then there exists a proper two-sided ideal \(\mathcal{I} \subset \mathcal{B}(\mathcal{H})\) such that \(\mathcal{K}(\mathcal{H}) \subset \mathcal{I}\) and \((\mathcal{I}, \| \cdot \|_\infty)\) is a Banach space which is not a symmetric subspace of \(\mathcal{B}(\mathcal{H})\).

If \(x \in \mathcal{K}(\mathcal{H})\), then \(|x| = \sum_{n=1}^{m(x)} s_n(x) p_n\) (if \(m(x) = \infty\), the series converges uniformly), where \(\{s_n(x)\}_{n=1}^{m(x)}\) is the set of singular values of \(x\), that is, the set of eigenvalues of the compact operator \(|x|\) in the decreasing order, and \(p_n\) is the projection onto the eigenspace corresponding to \(s_n(x)\). Consequently, the non-increasing rearrangement \(\nu_t(x) \in \mathcal{K}(\mathcal{H})\) can be identified with the sequence \(\{s_n(x)\}_{n=1}^\infty\). It is clear that \(s_n(x) = 0\) for all \(n > m(x)\).

**2.3 Duality between symmetric sequence and operator spaces.** Let \((X, \| \cdot \|_X) \subset \mathcal{K}(\mathcal{H})\) be a symmetric space. Fix an orthonormal basis \(\{|x_j\}_{j=1}^\infty\) in \(\mathcal{H}\) and choose a countable subset \(\{|x_j\}_{j=1}^\infty\). Let \(p_n\) be the one-dimensional projection on the subspace \(C : x_j, n \subset \mathcal{H}\). It is clear that the set

\[
E(x) = \left\{ x : \|x\|_X = \sum_{n=1}^\infty \xi_n p_n \in X \right\}
\]

(the series converges uniformly), is a symmetric sequence space with respect to the norm \(\|x\|_{E(X)} = \|x\|_X\). Consequently, each symmetric subspace \((X, \| \cdot \|_X) \subset \mathcal{K}(\mathcal{H})\) uniquely generates a symmetric sequence space \((E(x), \| \cdot \|_{E(x)}) \subset c_0\). The converse is also true: every symmetric sequence space \((E, \| \cdot \|_E) \subset c_0\) uniquely generates a symmetric space \((C_E, \| \cdot \|_{C_E}) \subset \mathcal{K}(\mathcal{H})\) by the following rule (see, for example, [17, Ch.3, Section 3.5]):

\[
C_E = \{ x \in \mathcal{K}(\mathcal{H}) : \{s_n(x)\} \in E \}, \quad \|x\|_{C_E} = \|\{s_n(x)\}\|_E.
\]

In addition,

\[
E(C_E) = E, \quad \| \cdot \|_{E(C_E)} = \| \cdot \|_E, \quad C_E(C_E) = C_E, \quad \| \cdot \|_{C_E(C_E)} = \| \cdot \|_{C_E}.
\]

We will call the pair \((C_E, \| \cdot \|_{C_E})\) a **Banach ideal of compact operators** (cf. [12, Ch.3.3]). It is known that \((C_p, \| \cdot \|_p) = (C_{lp}, \| \cdot \|_{lp})\) for all \(1 \leq p < \infty\) and \((\mathcal{K}(\mathcal{H}), \| \cdot \|_\infty) = (C_{\text{ca}}, \| \cdot \|_{\text{ca}})\).

Hardy-Littlewood-Polya partial order in the Banach ideal \(\mathcal{K}(\mathcal{H})\) is defined by

\[
x \preceq y, \quad x, y \in \mathcal{K}(\mathcal{H}) \iff \{s_n(x)\} \preceq \{s_n(y)\}.
\]

We say that a Banach ideal \((C_E, \| \cdot \|_{C_E})\) is **fully symmetric** if conditions \(y \in C_E, x \in \mathcal{K}(\mathcal{H}), x \preceq y\) entail that \(x \in C_E\) and \(\|x\|_{C_E} \leq \|y\|_{C_E}\). It is clear that \((C_E, \| \cdot \|_{C_E})\)
is a fully symmetric ideal if and only if \((E, \| \cdot \|_E)\) is a fully symmetric sequence space.

Examples of fully symmetric ideals include \((\mathcal{K}(\mathcal{H}), \| \cdot \|_{\infty})\) as well as the Banach ideals \((C^p, \| \cdot \|_p)\) for all \(1 \leq p < \infty\). It is clear that \(C^1 \subset C_E \subset \mathcal{K}(\mathcal{H})\) for every symmetric sequence space \(E \subset c_0\) with \(\| x \|_{C_E} \leq \| x \|_1\) and \(\| y \|_{\infty} \leq \| y \|_{C_E}\) for all \(x \in C^1\) and \(y \in C_E\).

**Remark 2.** If \(x, y, y_k \in \mathcal{K}(\mathcal{H})\) are such that \(y_k \prec \prec x\) for all \(k \in \mathbb{N}\) and \(\| y_k - y \|_{\infty} \to 0\) as \(k \to \infty\), then \(y \prec \prec x\).

Indeed, since \(y_k \prec \prec x\), it follows that \(\sum_{n=1}^{m} s_n(y_k) \leq \sum_{n=1}^{m} s_n(x)\) for all \(m, k \in \mathbb{N}\). By [12, Ch.H, § 2, Sec. 3, Corollary 2.3], \(|s_n(y_k) - s_n(y)| \leq \| y_k - y \|_{\infty} \to 0\), hence \(\sum_{n=1}^{m} s_n(y_k) \to \sum_{n=1}^{m} s_n(y)\) as \(k \to \infty\) for every \(m \in \mathbb{N}\). Therefore

\[
\sum_{n=1}^{m} s_n(y) = \lim_{k \to \infty} \sum_{n=1}^{m} s_n(y_k) \leq \sum_{n=1}^{m} s_n(x)
\]

for all \(m\).

2.4. **Dunford-Schwartz operators and conditional expectation.** A linear operator \(T : B(\mathcal{H}) \to B(\mathcal{H})\) is called a Dunford-Schwartz operator if

\[
\| T(x) \|_1 \leq \| x \|_1\text{ for all } x \in C^1\text{ and } \| T(x) \|_{\infty} \leq \| x \|_{\infty}\text{ for all } x \in B(\mathcal{H}).
\]

In what follows, we will write \(T \in DS\) to indicate that \(T\) is a Dunford-Schwartz operator.

Any fully symmetric ideal \(C_E\) is an exact interpolation space in the Banach pair \((C^1, B(\mathcal{H}))\) (see [7, Theorem 2.4]), in particular, \(T(C_E) \subset C_E\) and \(\| T \|_{C_E \to C_E} \leq 1\) for all \(T \in DS\). Hence \(T(\mathcal{K}(\mathcal{H})) \subset \mathcal{K}(\mathcal{H})\), and the restriction of \(T\) on \(\mathcal{K}(\mathcal{H})\) is a linear contraction (also denoted by \(T\)). We note that if \(T \in DS\), then \(A_n(T) \in DS\); also, \(T(x) \prec \prec x\) and \(A_n(T)(x) \prec \prec x\) for any \(x \in \mathcal{K}(\mathcal{H})\) and \(n \in \mathbb{N}\).

We need the following Theorem on the existence of conditional expectation from \(B(\mathcal{H})\) into von Neumann subalgebra \(\mathcal{N} \subset B(\mathcal{H})\) (see, for example, [21], [22]).

**Theorem 5.** Let \(\mathcal{N}\) be a von Neumann subalgebra in \(B(\mathcal{H})\) such that the restriction of the canonical trace \(\tau\) on \(\mathcal{N}\) is a semifinite trace. Then there exists a unique positive linear map \(U : B(\mathcal{H}) \to \mathcal{N}\) (conditional expectation on \(\mathcal{N}\)), having the following properties:

\begin{enumerate}[(i)]
  \item \(\tau(x) = \tau(U(x))\) for all \(x \in C^1\);
  \item \(\tau(U(x)) = x\) for all \(x \in \mathcal{N}\);
  \item \(\| U \|_{B(\mathcal{H}) \to \mathcal{N}} = 1\).
\end{enumerate}

Moreover, the conditional expectation \(U\) is projection of norm one from \((L^p(\mathcal{N}, \tau), \| \cdot \|_p) \to (L^p(\mathcal{N}, \tau), \| \cdot \|_p), 1 \leq p < \infty\).

Thus, the conditional expectation \(U : B(\mathcal{H}) \to \mathcal{N} \subset B(\mathcal{H})\) is a positive Dunford-Schwartz operator.

3. **Individual ergodic theorem in fully symmetric ideals of compact operators**

Let \(\mathcal{H}, \tau : B_+(\mathcal{H}) \to [0, \infty]\), and \(C^1\) be as above. Below we give a proof of Theorem 2 (i).
Proof. Since $T(C^2) \subset C^2$, $\|T\|_{C^2 \rightarrow C^2} \leq 1$ and the Banach space $C^2$ is reflexive, by the mean ergodic theorem [6, Ch. VIII, §5, Corollary 4], the sequence $\{A_n(T)(x)\}$ converges strongly in $C^2$, that is, for every $x \in C^2$ there exists $\hat{x} \in C^2$ such that $\|A_n(T)(x) - \hat{x}\|_2 \rightarrow 0$. As $\|\xi\|_\infty \leq \|\xi\|_2$ for all $\xi \in l^2$, it follows that $\|x\|_\infty \leq \|x\|_2$ for all $x \in C^2$. Consequently,
$$\|A_n(T)(x) - \hat{x}\|_\infty \rightarrow 0 \quad \text{for every } x \in C^2.$$ 

Let now $x \in K(H)$ and $\varepsilon > 0$. Then there exists $x_\varepsilon \in F(H) \subset C^2$ such that $\|x - x_\varepsilon\|_\infty < \varepsilon/4$. Since the sequence $A_n(T)(x_\varepsilon)$ converges uniformly, there exists $N = N(\varepsilon)$ such that
$$\|A_m(T)(x_\varepsilon) - A_n(T)(x_\varepsilon)\|_\infty < \frac{\varepsilon}{2} \quad \text{whenever } m, n \geq N.$$

Therefore,
$$\|A_m(T)(x) - A_n(T)(x)\|_\infty \leq \|A_m(T)(x - x_\varepsilon) - A_n(T)(x - x_\varepsilon)\|_\infty$$ 
$$+ \|A_m(T)(x_\varepsilon) - A_n(T)(x_\varepsilon)\|_\infty \leq 2\|x - x_\varepsilon\|_\infty + \frac{\varepsilon}{2} < \varepsilon.$$ 

for all $m, n \geq N$. Thus, since $C_E \subset K(H)$ and the space $(K(H), \|\cdot\|_\infty)$ is complete, it follows that for any $x \in C_E$ there exists $\hat{x} \in K(H)$ such that $\|A_n(T)(x) - \hat{x}\|_\infty \rightarrow 0$. Using Remark 2, we obtain that $\hat{x} \prec \prec x$, hence $\hat{x} \in C_E$.

Now we give a proof of the part (ii) of Theorem 2. We begin with a Dunford-Schwartz operator acting in $(l^\infty, \|\cdot\|_\infty)$, that is, when a linear operator $T : l^\infty \rightarrow l^\infty$ is such that $\|T(\xi)\|_1 \leq \|\xi\|_1$ for all $\xi \in l^1$ and $\|T(\xi)\|_\infty \leq \|\xi\|_\infty$ for all $\xi \in l^\infty$ (writing $T \in DS$). The following Theorem is a commutative version of Theorem 2 (ii) (proof see in [2, Theorem 3.3]).

**Theorem 6.** If $\xi \in l^\infty \setminus c_0$, then there exists $T \in DS$ such that the averages $A_n(T)(\xi)$ do not converge coordinate-wise, hence uniformly.

Assume first that $(H, (\cdot, \cdot))$ is a separable infinite-dimensional complex Hilbert space. Fix an orthonormal basis $\{\varphi_n\}_{n \in \mathbb{N}}$ in $H$. Let $p_n$ be the one-dimensional projection on the linear subspace $C \cdot \varphi_n \subset H$. It is clear that $p_mp_n = 0$ for all $m, n \in \mathbb{N}$, $n \neq m$.

For any $\xi = \{\xi_n\}_{n=1}^\infty \in l^\infty$ and $h = \sum_{n=1}^\infty (h, \varphi_n)\varphi_n \in H$ we set
$$x_\xi(h) = \sum_{n=1}^\infty \xi_n(h, \varphi_n)\varphi_n = \sum_{n=1}^\infty \xi_n p_n(h).$$

It is clear that $x_\xi \in B(H)$ and $x_\xi = (wo) - \sum_{n=1}^\infty \xi_np_n$, where $(wo)$ stands for the weak operator topology. In addition,
$$N = \{x_\xi \in B(H) : \xi = \{\xi_n\}_{n=1}^\infty \in l^\infty\}$$

is the smallest commutative von Neumann subalgebra in $B(H)$ containing all projections $p_n$. Besides, the restriction of the trace $\tau$ on $N$ is a semifinite trace.

Define the linear map $\Phi : (N, \|\cdot\|_\infty) \rightarrow (l^\infty, \|\cdot\|_\infty)$ by setting $\Phi(x_\xi) = \xi$. By definition of $\Phi$, we have $\Phi(N) = l^\infty$. Using [17, Ch. 1, §1.1, E. 1.1.11], we see that $\|x_\xi\|_\infty = \|\xi\|_\infty = \|\Phi(x_\xi)\|_\infty$, that is, $\Phi$ is a linear surjective isometry. Since $\xi = \{\xi_n\}_{n=1}^\infty \geq 0$ whenever $x_\xi \in N_+$, the map $\Phi$ is positive. Therefore, $\Phi$ is a positive linear surjective isometry.
If \((E, \| \cdot \|_E) \subset c_0\) is a symmetric sequence space and \(N_E = N \cap C_E\), then for any \(x_\xi = \sum_{n=1}^{\infty} \xi_n p_n \in N_E\) we have that \(\{ s_n(x_\xi) \}_{n=1}^{\infty} = \{ \xi_n^* \} \in E\), hence \(\{ \xi_n \} \in \mathcal{E}\).

In addition, \(\| x_\xi \|_{c_E} = \| \{ \xi_n^* \} \|_E = \| \{ \xi_n \} \|_E\). Consequently, the restriction \(\Phi|_{c_E} : (N_E, \| \cdot \|_{c_E}) \to (E, \| \cdot \|_E)\) is a positive linear surjective isometry (we denote this restriction also by \(\Phi\)).

Below we give a proof of Theorem 2 (ii).

**Proof.** Let \(0 \leq x \in \mathcal{B}(\mathcal{H}) \setminus \mathcal{K}(\mathcal{H})\). Assume first that \(\mathcal{H}\) is separable. Since \(x \notin \mathcal{K}(\mathcal{H})\), it follows that there exists a spectral projection \(e_\lambda(|x|)\), \(\lambda > 0\), such that \(\tau(e_\lambda(|x|)^+) = \infty\). Choose an orthonormal basis \(\{ \phi_n \}_{n=1}^{\infty} \) in \(\mathcal{H}\) such that \(e_\lambda(|x|) \geq p_n\) for some sequence \(\{ p_n \}_{n=1}^{\infty}\), where \(p_n\) is the one-dimensional projection on the subspace \(\mathbb{C} \cdot \phi_n \subset \mathcal{H}\).

Let \(\mathcal{N} = \{ x_\xi \in \mathcal{B}(\mathcal{H}) : \xi = \{ \xi_n \}_{n=1}^{\infty} \in l^\infty \} \) be the smallest commutative von Neumann subalgebra in \(\mathcal{B}(\mathcal{H})\) containing all projections \(p_n\). Since the restriction of the trace \(\tau\) on \(\mathcal{N}\) is a semifinite trace, it follows by Theorem 5 that there exists a conditional expectation \(U : \mathcal{B}(\mathcal{H}) \to \mathcal{N}\) such that

\[
0 \leq y = U(x) \geq U(\lambda e_\lambda(|x|)) = \lambda U(p_n), \quad \text{for all } \lambda > 0, i \in \mathbb{N}.
\]

Consequently, \(y \notin \mathcal{K}(\mathcal{H})\) and \(y = x_\xi \in \mathcal{N}\), where \(0 \leq \xi = \{ \xi_n \}_{n=1}^{\infty} \in l^\infty \setminus c_0\). Besides, by definition of \(\Phi\), we have \(\Phi(y) = \xi\).

Next, by Theorem 6, there exists an operator \(S : l^\infty \to l^\infty\), \(S \in DS\), such that the sequence \(\{ A_n(S)(\xi) \}_{n=1}^{\infty}\) does not converge uniformly. Consider the operator

\[
T = \Phi^{-1} S \Phi U : \mathcal{B}(\mathcal{H}) \to \mathcal{N} \subset \mathcal{B}(\mathcal{H}).
\]

It is clear that \(T \in DS\). Since \(U : \mathcal{B}(\mathcal{H}) \to \mathcal{N}\) is a conditional expectation and \(y = U(x)\), it follows that \(U(y) = y, U \Phi^{-1} = \Phi^{-1}\), and \(T^k(y) = \Phi^{-1} S^k \Phi(y)\) for each \(k \in \mathbb{N}\).

Since \(\Phi^{-1}\) is an isometry and

\[
A_n(T)(y) = \frac{1}{n+1} \sum_{k=0}^{n} T^k(y) = \Phi^{-1} \left( \frac{1}{n+1} \sum_{k=0}^{n} S^k \Phi(y) \right) = \Phi^{-1} (A_n(S)(\xi)),
\]

for all \(n \in \mathbb{N}\), it follows that the sequence \(\{ A_n(T)(y) \}_{n=1}^{\infty}\) does not converge uniformly.

Now, as above, \(y = U(x) \in \mathcal{N}\) entails \(T^k(x) = \Phi^{-1} S^k \Phi(y) = T^k(y)\) for all \(k \in \mathbb{N}\). Therefore, we have

\[
A_n(T)(x) - A_n(T)(y) = \frac{1}{n+1} (x - y),
\]

and it follows that the sequence \(\{ A_n(T)(x) \}_{n=1}^{\infty}\) also does not converge uniformly.

Let now \(\mathcal{H}\) be non-separable, and let \(0 \leq x \in \mathcal{B}(\mathcal{H}) \setminus \mathcal{K}(\mathcal{H})\). Since \(x \notin \mathcal{K}(\mathcal{H})\), it follows that there exists a spectral projection \(e_\lambda(|x|)\), \(\lambda > 0\), such that \(\tau(e_\lambda(|x|)^+) = \infty\). Choose an orthonormal basis \(\{ \varphi_j \}_{j \in J} \) in \(\mathcal{H}\) such that \(e_\lambda(|x|)^+ \geq p_j\), for some sequence \(\{ p_j \}_{n=1}^{\infty}\), where \(p_j\) is the one-dimensional projection on the subspace \(\mathbb{C} \cdot \varphi_j \subset \mathcal{H}\). If \(p = \sup_{n \in \mathbb{N}} p_n\), then \(\mathcal{H}_0 = p(\mathcal{H})\) is a separable infinite-dimensional Hilbert subspace in \(\mathcal{H}\) such that \(\mathcal{K}(\mathcal{H}_0) = pK(\mathcal{H}).p\).

Since \(z = p_x p \in B_+(\mathcal{H}_0)\) and \(z \geq \lambda p e_\lambda(|x|)^+ p \geq \lambda p\), it follows that \(z \in B_+(\mathcal{H}_0) \setminus \mathcal{K}(\mathcal{H}_0)\). In view of the above, there exists a Dunford-Schwartz operator
$D_0 : B(H_0) \to B(H_0)$ such that the sequence $\{A_n(D_0)(z)\}_{n=1}^\infty$ does not converge uniformly.

It is clear that $D(y) = D_0(pyp), y \in B(H)$, is a Dunford-Schwartz operator in $B(H)$ such that $D^k(x) = D_0^k(z)$ for each $k \in \mathbb{N}$. Then

$$A_n(D)(x) - A_n(D_0)(z) = \frac{1}{n+1}(x - z),$$

and we conclude that the sequence $\{A_n(D)(x)\}_{n=1}^\infty$ does not converge uniformly. \hfill \Box

Note that the commutative version of Theorem 2 (ii) for symmetric spaces of measurable functions was obtained in [5].

4. Mean ergodic theorem in fully symmetric ideals of compact operators

In this section, our goal is to prove Theorem 4. So, let $(E, \| \cdot \|_E) \subset c_0$ be a fully symmetric sequence space, and let $(C_E, \| \cdot \|_{C_E})$ be a fully symmetric ideal generated by $(E, \| \cdot \|_E)$. Let us show that the mean ergodic theorem, generally speaking, is not true in $(C_E, \| \cdot \|_{C_E})$, in the cases when $E = l^1$ as sets, or when $(E, \| \cdot \|_E)$ is non-separable space.

**Proposition 1.** There exists $T \in DS$ such that the averages $A_n(T)$ do not converge strongly in $(C^1, \| \cdot \|_1)$.

**Proof.** Let $S : l^\infty \to l^\infty$ be the Dunford-Schwartz operator defined by

$$S(\{\xi_n\}_{n=1}^\infty) = \{0, \xi_1, \xi_2, \ldots\}, \quad \{\xi_n\}_{n=1}^\infty \in l^\infty.$$

If $\xi = \{1, 0, 0, \ldots\} \in l^1$, then

$$\left\|A_{2n-1}(S)(\xi) - A_{n-1}(S)(\xi)\right\|_1 = \left\|\frac{1}{2n}\{1,1,\ldots,1,0,0,\ldots\} - \frac{1}{n}\{1,1,\ldots,1,0,0,\ldots\}\right\|_1 = 1.$$

Consequently, the sequence $\{A_n(S)(\xi)\}$ does not converge in the norm $\| \cdot \|_1$.

Let $p_n, \quad p = \sup_{n \in \mathbb{N}} p_n, \quad H_0 = p(H), \quad N(H_0) = \left\{x_\xi = (w_0) - \sum_{n=1}^\infty \xi_n p_n \in B(H_0) : \xi = \{\xi_n\}_{n=1}^\infty \in l^\infty\right\},$ 

$\Phi : N(H_0) \to l^\infty$ and $U : B(H_0) \to N(H_0)$ be the same as in the proof of Theorem 2 (ii). Then

$$T = \Phi^{-1}S\Phi : B(H_0) \to N(H_0) \subset B(H_0)$$

is a positive Dunford-Schwartz operator. In addition, for $\xi = \{1, 0, 0, \ldots\} \in l^1$ and $x_\xi = \Phi^{-1}(\xi)$ we have that $x_\xi \in N(H_0) \cap C^1$ and $U(x_\xi) = x_\xi$ (see proof of Theorem 2 (ii)). Consequently,

$$T(x_\xi) = \Phi^{-1}S\Phi(x_\xi) = \Phi^{-1}S\Phi(x_\xi).$$

Now, repeating the proof of Theorem 2 (ii), we conclude that the averages

$$\{A_n(T)(x_\xi)\}$$

do not converge in the norm $\| \cdot \|_1$. \hfill \Box
Proposition 2. If \((E, \| \cdot \|_E) \subset c_0\) is non-separable fully symmetric sequence space, then there exists \(T \in DS\) such that the averages \(A_n(T)\) do not converge strongly in \((C_E, \| \cdot \|_{C_E})\).

Proof. If \((E, \| \cdot \|_E) \subset c_0\) is a non-separable fully symmetric sequence space, then there exists \(\xi = \{\xi_n\}_{n=1}^{\infty} = \{\xi_n\}_{n=1}^{\infty} \in E\) such that \(\xi_n \downarrow 0\) and

\[
\|\{0, 0, \ldots, 0, \xi_{n+1}, \ldots\}\|_E \downarrow \alpha > 0.
\]

Let the operator \(S \in DS\) be defined as in the proof of Proposition 1. Then \(S^k(\xi) = \{0, 0, \ldots, 0, \xi, \xi_2, \ldots\}\) and

\[
\sum_{k=0}^{n} S^k(\xi) = \{\eta_m^{(n)}\}_{m=1}^{\infty},
\]

where

\[
\eta_m^{(n)} = \xi_1 + \xi_2 + \ldots + \xi_m \quad \text{for} \quad 1 \leq m \leq n + 1
\]

and

\[
\eta_m^{(n)} = \xi_{m-n} + \xi_{m-n+1} + \ldots + \xi_m \quad \text{for} \quad m > n + 1.
\]

Since \(\xi_n \downarrow 0\), given \(1 \leq m \leq n + 1\), we have

\[
0 \leq \frac{1}{n+1} \eta_m^{(n)} \leq \frac{1}{n+1} \sum_{k=1}^{n+1} \xi_k \to 0 \quad \text{as} \quad n \to \infty,
\]

implying that \(A_n(S)(\xi) \to 0\) coordinate-wise.

Assume that there exists \(\hat{\xi} \in E\) such that \(\|A_n(S)(\xi) - \hat{\xi}\|_E \to 0\). Then we have \(\|A_n(S)(\xi) - \hat{\xi}\|_E \to 0\); in particular, \(A_n(S)(\xi) \to 0\) coordinate-wise, hence \(\hat{\xi} = 0\).

On the other hand, as \(\xi_n \downarrow 0\), we obtain

\[
A_n(S)(\xi) = \left\{ \frac{\xi_1}{n+1}, \frac{\xi_1 + \xi_2}{n+1}, \ldots, \frac{\xi_1 + \xi_2 + \ldots + \xi_{n+1}}{n+1}, \frac{\xi_2 + \xi_3 + \ldots + \xi_{n+2}}{n+1}, \ldots, \frac{\xi_{m-n} + \xi_{m-n+1} + \ldots + \xi_m}{n+1}, \ldots \right\}
\]

\[
\geq \{0, 0, \ldots, 0, \xi_{n+2}, \ldots\}.
\]

Therefore, in view of (1), \(\|A_n(S)(\xi)\|_E \geq \alpha\), implying that the sequence \(\{A_n(S)(\xi)\}\) does not converge in the norm \(\| \cdot \|_E\).

Now, if we define the Dunford-Schwartz operator \(T \in DS\) as in the proof of Proposition 1, then repeating its proof for \(x = \Phi^{-1}(\xi)\), we conclude that the sequence \(\{A_n(T)(x)\}\) does not converge in \((C_E, \| \cdot \|_{C_E})\). \(\square\)

Fix \(T \in DS\). By Theorem 2 (i), for every \(x \in K(H)\) there exists \(\tilde{x} \in K(H)\) such that \(\|A_n(T)(x) - \tilde{x}\|_\infty \to 0\) as \(n \to \infty\). Therefore, one can define a linear operator \(P_T : K(H) \to K(H)\) by setting \(P_T(x) = \tilde{x}\). Then we have

\[
\|P_T(x)\|_\infty = \lim_{n \to \infty} \|A_n(T)(x)\|_\infty \leq \|x\|_\infty,
\]

Besides, since the unit ball in \((C^1, \| \cdot \|_1)\) is closed in measure topology \([8, Proposition 3.3]\) and \(\|A_n(T)(x)\|_1 \leq \|x\|_1\) for all \(x \in C^1\), it follows that \(\|P_T(x)\|_1 \leq \|x\|_1\), \(x \in C^1\). Consequently, \(\|P_T\|_{C^1 \to C^1} \leq 1\), and, according to \([3, Proposition 1.1]\), there exists
a unique operator \( \hat{P} \in DS \) such that \( \hat{P}(x) = P_T(x) \) whenever \( x \in \mathcal{K}(\mathcal{H}) \). In what follows, we denote \( \hat{P} \) by \( P_T \).

**Lemma 1.** If \( T \in DS \) and \( x \in \mathcal{K}(\mathcal{H}) \), then

\[
P_T T(x) = P_T(x) = T P_T(x).
\]

**Proof.** We have

\[
\|(I - T) A_n(T)(x)\|_\infty = \left\| \frac{(I - T^{n+1})(x)}{n + 1} \right\|_\infty \to 0 \quad \text{as} \quad n \to \infty.
\]

On the other hand,

\[
T A_n(T)(x) = \frac{1}{n + 1} \sum_{k=0}^{n} T^k(Tx) \to P_T(Tx),
\]

implying that

\[
(I - T) A_n(T)(x) = A_n(T)(x) - T A_n(T)(x) \to P_T(x) - P_T T(x),
\]

hence \( P_T T(x) = P_T(x) \).

Now, as \( \|A_n(T)(x) - P_T(x)\|_\infty \to 0 \), we have \( \|T(A_n(T)(x)) - T(P_T(x))\|_\infty \to 0 \) as \( n \to \infty \), and the result follows. \( \square \)

**Corollary 1.** If \( T \in DS \) and \( x \in \mathcal{K}(\mathcal{H}) \), then

\[
T^k(P_T(x)) = P_T(x) \quad \text{for all} \quad k \in \mathbb{N}, \quad \text{and} \quad P_T^2(x) = P_T(x).
\]

We need the following property of separable symmetric sequence spaces [9, Proposition 2.2].

**Proposition 3.** Let \((E, \| \cdot \|_E)\) be a separable symmetric sequence space and \( E \neq l^1 \) as sets. If \( \mathcal{C}_E \ni y_n \prec x \in \mathcal{C}_E \) for every \( n \in \mathbb{N} \) and \( \|y_n\|_\infty \to 0 \) as \( n \to \infty \), then \( \|y_n\|_{\mathcal{C}_E} \to 0 \) as \( n \to \infty \).

Now we can finalize the proof of Theorem 4:

**Proof.** (i) \( \Rightarrow \) (ii): Proposition 2 implies that \( E \) is separable. If \( E = l^1 \) as sets, then the norms \( \| \cdot \|_E \) and \( \| \cdot \|_1 \) are equivalent [18, Part II, Ch. 6, §6.1]. Therefore, in view of Proposition 1, we would have that item (i) in Theorem 4 is not true.

(ii) \( \Rightarrow \) (i): Let \((E, \| \cdot \|_E)\) be separable, \( E \neq l^1 \) as sets, and let \( T \in DS \). If \( x \in \mathcal{C}_E \) and \( y = x - P_T(x) \), then \( P_T(y) = 0 \), which, by Theorem 2 (i), implies \( \|A_n(T)(y)\|_\infty \to 0 \). Since \( E \) is a separable symmetric sequence space, \( E \neq l^1 \) as sets, and \( A_n(T)(y) \prec y \in \mathcal{C}_E \), it follows from Proposition 3 that

\[
\|A_n(T)(y)\|_{\mathcal{C}_E} \to 0.
\]

Since \( P_T(z) \prec z \) for all \( z \in \mathcal{K}(\mathcal{H}) \), it follows that \( A_n(T)(P_T(x)) \prec P_T(x) \prec x \), hence \( A_n(T)(P_T(x)) - P_T(x) \prec 2x \). Next, as \( A_n(T)(P_T(x)) \to P_T(x) \), Proposition 3 entails

\[
\|A_n(T)(P_T(x)) - P_T(x)\|_{\mathcal{C}_E} \to 0.
\]

Now, utilizing (2) and (3), we obtain

\[
\|A_n(T)(x) - P_T(x)\|_{\mathcal{C}_E} = \|A_n(T)(x) - A_n(T)(P_T(x)) + A_n(T)(P_T(x)) - P_T(x)\|_{\mathcal{C}_E} \leq \|A_n(T)(y)\|_{\mathcal{C}_E} + \|A_n(T)(P_T(x)) - P_T(x)\|_{\mathcal{C}_E} \to 0
\]

as \( n \to \infty \). \( \square \)
Now we give applications of Theorems 2 and 4 to Orlicz and Lorentz ideals of compact operators.

1. Let $\Phi$ be an Orlicz function, that is, $\Phi : [0, \infty) \to [0, \infty)$ is convex, continuous at 0, $\Phi(0) = 0$ and $\Phi(u) > 0$ if $u > 0$ (see, for example, [10, Ch. 2, §2.1], [15, Ch. 4]). Let

$$l^\Phi(\mathbb{N}) = \left\{ \xi = (\xi_n)_{n=1}^{\infty} \in l^\infty : \sum_{n=1}^{\infty} \Phi \left( \frac{\xi_n}{a} \right) < \infty \text{ for some } a > 0 \right\}$$

be the Orlicz sequence space, and let

$$\|\xi\| = \inf \left\{ a > 0 : \sum_{n=1}^{\infty} \Phi \left( \frac{\xi_n}{a} \right) \leq 1 \right\}$$

be the Luxemburg norm in $l^\Phi(\mathbb{N})$. It is well-known that $(l^\Phi(\mathbb{N}), \|\cdot\|)$ is a fully symmetric sequence space.

Since $\Phi(u) > 0$, $u > 0$, it follows that $\sum_{n=1}^{\infty} \Phi(a^{-1}) = \infty$ for each $a > 0$, hence \(1 = \{1, 1, \ldots\} \notin l^\Phi(\mathbb{N})\) and $l^\Phi(\mathbb{N}) \subset c_0$. Therefore, we can define Orlicz ideal of compact operators

$$C^\Phi = C_{l^\Phi(\mathbb{N})}, \quad \|x\| = \|x\|_{l^\Phi(\mathbb{N})}, \quad x \in C^\Phi.$$

By Theorem 2 (i) we obtain that given Dunford-Schwartz operator $T$ and $x \in C^\Phi$, there exists $\hat{x} \in C^\Phi$ such that $\|A_n(T)(x) - \hat{x}\| = 0$ as $n \to \infty$ (cf. Theorem 3.2 [4]).

It is said that an Orlicz function $\Phi$ satisfies $(\Delta_2)$-condition at 0 if there exist $u_0 \in (0, \infty)$ and $k > 0$ such that $\Phi(2u) < k \Phi(u)$ for all $0 < u < u_0$. It is well known that an Orlicz function $\Phi$ satisfies $(\Delta_2)$-condition at 0 if and only if $(l^\Phi(\mathbb{N}), \|\cdot\|)$ is separable (see [10, Ch. 2, §2.1, Theorem 2.1.7], [15, Ch. 4, Proposition 4.a.4]). In addition, $l^\Phi(\mathbb{N}) = l^1$ as sets, if and only if $\limsup_{u \to 0} \frac{\Phi(u)}{u} > 0$ (see [15, Ch. 4, Proposition 4.a.5], [18, Ch. 16, §16.2]).

Thus, using Theorem 4, we obtain that the averages $A_n(T)$ converge strongly in $C^\Phi$ for any Dunford-Schwartz operator $T$ if and only if $\Phi$ satisfies $(\Delta_2)$-condition at 0 and $\lim_{u \to 0} \frac{\Phi(u)}{u} = 0$.

2. Let $\psi$ be a concave function on $[0, \infty)$ with $\psi(0) = 0$ and $\psi(t) > 0$ for all $t > 0$, and let

$$\Lambda_{\psi}(\mathbb{N}) = \left\{ \xi = (\xi_n)_{n=1}^{\infty} \in l^\infty : \|\xi\|_\psi = \sum_{n=1}^{\infty} \xi_n^\psi(\psi(n) - \psi(n-1)) < \infty \right\},$$

the Lorentz sequence space. The pair $(\Lambda_{\psi}(\mathbb{N}), \|\cdot\|_\psi)$ is a fully symmetric sequence space (see, for example, [14, Ch.II, §5], [18, Part III, Ch. 9, §9.1]). Besides, if $\psi(\infty) = \infty$, then $1 \notin \Lambda_{\psi}(\mathbb{N})$ and $\Lambda_{\psi}(\mathbb{N}) \subset c_0$. In this case we can define Lorentz ideal of compact operators

$$C_{\psi} = C_{\Lambda_{\psi}(\mathbb{N})}, \quad \|x\|_\psi = \|x\|_{\Lambda_{\psi}(\mathbb{N})}, \quad x \in C_{\psi},$$

for which is true Theorem 2 (i).

It is well known that $(\Lambda_{\psi}(\mathbb{N}), \|\cdot\|_\psi)$ is separable if and only if $\psi(0) = 0$ and $\psi(\infty) = \infty$ (see, for example, [14, Ch.II, §5, Lemma 5.1], [18, Ch.9, §9.3, Theorem 9.3.1]). In addition, $\lim_{t \to \infty} \frac{\psi(t)}{t} > 0$ if and only if the norms $\|\cdot\|_\psi$ and $\|\cdot\|_1$
are equivalent on $\Lambda_\psi(N)$, that is, $\Lambda_\psi(N) = l^1$ as sets. Therefore, by Theorem 4, we obtain that the averages $A_n(T)$ converge strongly in $C_\psi$ for any Dunford-Schwartz operator $T$ if and only if $\psi(+0) = 0$, $\psi(\infty) = \infty$ and $\lim_{t \to \infty} \frac{\psi(t)}{t} = 0$.

References


Azizkhon Nodirovich Azizov
National University of Uzbekistan,
4, Universitet str.,
Tashkent, 100174, Uzbekistan
Email address: azizov.07@mail.ru

Vladimir Ivanovich Chilin
National University of Uzbekistan,
4, Universitet str.,
Tashkent, 100174, Uzbekistan
Email address: vladimirchil@gmail.com, VladimirChilin@micros.uz