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ON VARIETY $\mathcal{N}$ OF NORMAL VALUED m-GROUPS

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Abstract. Recall that an m-group is a pair $(G,_*)$, where G is an ℓ-group and $_*$ is a decreasing order two automorphism of G. An m-group can be regarded as an algebraic system of signature m and it is obvious that the m-groups form a variety in this signature. The set M of varieties of all m-groups is a partially ordered set with respect to the set-theoretic inclusion. Moreover, M is a lattice with respect to the naturally defined operations of intersection and union of varieties of m-groups. In this article we study the characteristics of a variety $\mathcal{N}$ of normal valued m-groups which is defined by the identity $|x||y| \wedge |y|^2|x|^2 = |x||y|$. We will prove that $\mathcal{N}$ is an idempotent of M and $\mathcal{N} = \bigvee_{n\in\mathbb{N}} \mathcal{A}^n$, where $\mathcal{A}$ is the variety of all abelian m-groups.

Keywords: m-group, variety, normal valued m-group.

1. Introduction

Recall that an m-group is an algebraic system G in a signature

$$m = \langle \cdot,\ ^{-1},\ e,\ \vee,\ \wedge,\ _*\rangle$$

such that $\langle G, \cdot,\ ^{-1},\ e,\ \vee,\ \wedge\rangle$ is a lattice-ordered group (ℓ-group) and the unary operation $_*$ can be interpreted as an automorphism of order two of a group $\langle G, \cdot,\ ^{-1},\ e\rangle$ and an antiautomorphism of a lattice $\langle G,\ \vee,\ \wedge\rangle$, i.e. $_*$ is a bijective map from G to itself, which satisfies the following relations

$$(xy)_* = x_*y_*,\ \ (x_*)_* = x,\ \ (x \vee y)_* = x_* \wedge y_*,\ \ (x \wedge y)_* = x_* \vee y_*.$$

From now on we will call $_*$ a reversing automorphism of order two of a ℓ-group G and by $(G,\ _*)$. we will denote an m-group G with a fixed reversing automorphism

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$_*$. As always, an m-ideal of a m-group $(G, {}_*)$ is any convex normal m-subgroup of that group.

The class of all m-groups, denoted by $\mathcal{M}$, forms a variety of signature m. A set M of all varieties of m-groups is partially ordered by set-theoretic inclusion. Furthermore, M is a lattice with respect to naturally defined intersection and union of varieties of m-groups. Also, M is a semigroup with respect to the product of varietes of m-groups, i.e. if $\mathcal{U}, \mathcal{V}$ are m-group varieties then an m-group $(G, {}_*)$ lies in their product $\mathcal{U} \cdot \mathcal{V}$ iff it contains an m-ideal $(H, {}_*)$, such that $(H, {}_*) \in \mathcal{U}$ and $(G/H, {}_*) \in \mathcal{V}$.

The goal of this paper is to study the properties of the variety $\mathcal{N}$ of all normal valued m-groups which satisfy the following identity

$$|x||y| \wedge |y|^2|x|^2 = |x||y|, \qquad (*)$$

where $|x| = x \vee x^{-1}$ is an absolute value of x. It is clear that $|x| \geq e$. In [1], it had been shown by Kopytov and Rachuneck, that $\mathcal{N}$ is the biggest element of the lattice M. Hence, either $\mathcal{N}^2 = \mathcal{M}$, or $\mathcal{N}^2 = \mathcal{N}$, i.e. $\mathcal{N}$ is an idempotent of a semigroup of m-group varieties M. We will show (Proposition 2.1) that the second statement is the correct one. By $\mathcal{A}$ denote the variety of abelian m-groups. From Proposition 2.1, using induction, one can derive that $\bigvee\limits_{n\in\mathbb{N}} \mathcal{A}^n \subseteq \mathcal{N}$. We will also show that $\bigvee\limits_{n\in\mathbb{N}} \mathcal{A}^n = \mathcal{N}$ (Theorem 2.2). Note that this result gives a positive answer to the question 3.12 from the Erlagol notebook [2]. We actively use representation theory of m-groups and techniques, associated with wreath products of said groups. Theorem 2.2 is proved by the first author and Proposition 2.1 is proved by the second author.

Almost all notions of group theory and lattice ordered groups used throughout the paper are similar to those used in [3] and [4].

2. Main results

The notion of an m-group as an algebraic system is relatively new and was introduced by M. Giraudet and J. Rachunek in [5], when they were studying order properties of monotonic permutation groups of linearly ordered sets. In fact, m-group theory can be viewed as the theory of special monotonic permutation groups. Indeed, let Ω denote some (infinite) linearly ordered set. Denote by $Aut(\Omega)$ and $Mon(\Omega)$ groups (with composition as an operation) of all monotonically increasing (order-preserving (???)) and monotonic permutations of elements in Ω respectively. It is clear that $Aut(\Omega) \leq Mon(\Omega)$, moreover, with respect to pointwise order, $Aut(\Omega)$ is an ℓ-group. It is clear that any meaningful and interesting results can take place only when $Aut(\Omega) \neq Mon(\Omega)$. In that case, there always exists a decreasing (reversing) automorphism of order 2 $a \in Mon(\Omega) \setminus Aut(\Omega)$. This allows us to define an m-group structure on $Aut(\Omega)$ by setting $g_* = aga$ for $g \in Aut(\Omega)$. A faithful representation by order-preserving permutations of an m-group $(G, {}_*)$ is an m-isomorphism between $(G, {}_*)$ and $(Aut(\Omega),{}_*)$. We will denote such representations by (G, Ω, a). (какой факт???). Let o be the point from Ω which remains fixed under action a. Note, that there exist representations, which have a fixed point and those, that do not. However, if a fixed point does exist, it is unique. Let $L = \{\omega \in \Omega \mid (\omega)a > \omega\}, R = \{(\ell)a \mid \ell \in L\}$. By $A \overleftarrow{\bigcup} B$ denote a lexicographic union of linearly ordered sets A and B, i.e.$A \overleftarrow{\bigcup} B$ is a linearly ordered set in which $a < b$ for all $a \in A$ and $b \in B$. It can be shown that $\Omega = L \overleftarrow{\bigcup} \{o\}^{\varepsilon} \overleftarrow{\bigcup} R,$, where $\varepsilon = 1$, if

o exists and $\varepsilon = 0$ otherwise. Since Ω is an infinite and a linearly ordered set R is order anti-isomorphic to L, we get that both of these sets are infinite. Furthermore, they are convex i.e., for example, for all $\omega \in \Omega$ the inequality $\ell < \omega < \ell'$ implies $\omega \in L$, only if $\ell, \ell' \in L$.

The representation (G, Ω, a), where $\Omega = \overleftarrow{L \bigcup \{o\}^{\varepsilon}} \overleftarrow{\bigcup R}$, is said to be *proper*, if for all $g \in G$ and any point $\ell \in L$ it is true that $(\ell)g \in L$. Note that in terms of ℓ-groups, if a representation is proper, linearly ordered sets L, $\{o\}^{\varepsilon}$ и R are called G-invariant.

In order to prove that $\mathcal{N}$ is an idempotent of a group M of m-group varieties, consider a group $B(\mathbb{R})$ of all order-preserving permutations of a naturally linearly ordered set $\mathbb{R}$ or real numbers with a bounded domain, i.e. for all $g \in B(\mathbb{R})$ there exist real numbers $z < t$, such that $(x)g = x$ for $x \notin [z, t]$. It is clear that the function $(x)a$, defined by the rule $(x)a = -x$, is a reversing automorphism of $\mathbb{R}$ of order two, moreover for any $g \in B(\mathbb{R})$ it is true that $aga \in B(\mathbb{R})$. Hence, we can consider an m-group $(B(\mathbb{R}), \mathbb{R}, a)$. We will show that this m-group is not normal valued. For that, consider the following elements

$$(x)g = \begin{cases} x, & \text{if } x \le 1, \\ x^2, & \text{if } 1 < x \le 2, \\ x + 2, & \text{if } 2 < x \le 1000, \\ \frac{1}{2}x + 502, & \text{if } 1000 < x \le 1004, \\ x, & \text{if } x > 1004, \end{cases}$$

$$(x)f = \begin{cases} x, & \text{if } x \le 0, \\ 2x, & \text{if } 0 < x \le 1 \\ x + 1, & \text{if } 1 < x \le 1000 \\ \frac{1}{2}x + 501, & \text{if } 1000 < x \le 1002 \\ x, & \text{if } x > 1002. \end{cases}$$

Let $\frac{1+\sqrt{5}}{4} < x \le 1$. Then $1 < \frac{1+\sqrt{5}}{2} < 2x \le 2$ and therefore $(x)g^2f^2 = (x)f^2 = (2x)f = 2x + 1, (x)fg = (2x)g = 4x^2$. Hence $(x)fg > (x)g^2f^2$ on the interval $(\frac{1+\sqrt{5}}{4}, 1]$. This claim proves our statement. It is known (see, for instance, [4]) that $B(\mathbb{R})$ is a *simple* group and therefore it is also an m-simple group, i.e. it does not contain non-trivial m-ideals. Therefore an m-group $(B(\mathbb{R}), \mathbb{R}, a) \notin \mathcal{N}^2$. We have just proved

Proposition 2.1. *A variety $\mathcal{N}$ of all normal-valued m-groups is an idempotent of a semigroup M of m-group varieties.*

A representation (G, Ω, a) is said to be *m-transitive* if for all $\omega, \omega' \in \Omega$, except maybe for o, there exists such $x \in G_* = gr.(G, a)$, that $(\omega)x = \omega'$.

A non-unit m-group $(G,_*)$ is said to be *subdirectly m-irreducible*, if it contains a smallest non-trivial m-ideal. It is known (see, for instance, [6], page 115), that every algebraic system is a subdirect product of subdirectly irreducible algebraic systems. Since it was proved in [7] that a subdirectly m-irreducible m-group has a faithful m-transitive representation, it is enough to only consider m-transitive representations, when studying m-group varieties.

Consider an m-transitive representation (G, Ω, a). An equivalence relation Θ, defined on Ω is called an *m-equivalence* if it is convex and $\omega\Theta\omega' \Leftrightarrow (\omega)x\Theta(\omega')x$ for every $x \in G_*$. Obviously, the set $\mathcal{K}$ of all m-equivalences, defined on Ω is non-empty and, moreover, is partially ordered by $\preceq$, which is defined by the rule $\Theta_1 \preceq \Theta_2 \Leftrightarrow w\Theta_1 w' \Rightarrow w\Theta_2 w'$.

We will call the following m-equivalences *trivial*: A)an equivalence with all classes containing only one element; B) an equivalence with all classes containing only two elements; C) an equivalence with three (or two) classes $L, \{o\}, R$ (L, R); D) an equivalence with Ω as its only class. We say that a representation (G, Ω, a) is *m-primitive* if it does not admit non-trivial m-equivalences.

From the description of m-primitive representations of normal-valued m-groups, which was obtained in [8] (Theorem 2.1), it follows that these representations are in fact abelian m-groups.

We say that a pair of m-equivalences $\Theta_1 \preceq \Theta_2$ is a covering pair, if these equivalences are different and, with respect to $\preceq$, there is no other equivalence between them. Every covering pair allows us to construct a *primitive* m-transitive representation (H, Λ, b), given a representation (G, Ω, a). We will call this new representation *a primitive component* (corresponding to a covering pair $\Theta_1 \preceq \Theta_2$). More information on constructing primitive components can be found in [9].

We will now recall the generalised wreath product construction, which was introduced in [9], as it plays a key role in the proof ot Theorem 2.2. Let K be some linearly ordered index set and let $\{(H_k, \Lambda_k, b_k) \mid k \in K\}$ be a set of representations, where $\Lambda_k = L_k \overleftarrow{\bigcup} \{\mathbf{o}_k\}^{\varepsilon_k} \overleftarrow{\bigcup} R_k$. On the set $\widehat{\Lambda} = \prod_{k \in K} \Lambda_k$ define an authomorphism b of order two as follows: $(\lambda)b = (...(\lambda_k)b_k...)$, where $\lambda = (...\lambda_k...) \in \widehat{\Lambda}$. For every $k \in K$ define equivalences $\equiv^k, \equiv_k$ on $\widehat{\Lambda}$ using the rule $\lambda \equiv^k \lambda' \iff \lambda_\kappa = \lambda'_\kappa$ for all $\kappa > k$ and $\lambda \equiv_k \lambda' \iff \lambda_\kappa = \lambda'_\kappa$ for all $\kappa \geq k$. It is clear, that b preserves these equivalences for every $k \in K$.

Define $o = (...o_k...)$ as follows: $o_k = \mathbf{o}_k$, if $\varepsilon_k = 1$ and $o_k = \ell_k$ for some arbitrary, but *fixed* point $\ell_k \in L_k$, if $\varepsilon_k = 0$. As per usual notation, $supp(\lambda) = \{k \in K \mid \lambda_k \neq o_k\}$ is a domain of λ. By $\widetilde{\Lambda}$ denote a set of all elements of $\widehat{\Lambda}$, whose domain is well ordered in a descending order, i.e. every subset of $\widetilde{\Lambda}$ has a greatest element. For different $\lambda, \lambda' \in \widetilde{\Lambda}$ there exists $\alpha \in K$, such that $\lambda_\alpha \neq \lambda'_\alpha$ and $\lambda_\beta = \lambda'_\beta$ for all $\beta > \alpha$. It is clear that for $\gamma < \alpha$ it is true that $\lambda \not\equiv^\gamma \lambda'$ and $\lambda \not\equiv_\gamma \lambda'$; $\lambda \equiv^\alpha \lambda'$, but $\lambda \not\equiv_\alpha \lambda'$; finally, for $\beta > \alpha$ it is true that $\lambda \equiv^\beta \lambda'$ and $\lambda \equiv_\alpha \lambda'$. We will keep referring to the element of K, possesing this property as α (possibly with a subscript). Now one can naturally introduce a linear order relation on the set $\widetilde{\Lambda}$ as follows:

$$\lambda < \lambda' \iff \lambda_\alpha < \lambda'_\alpha.$$

It is clear that $o \in \widetilde{\Lambda}$ and, as was stated above, *all* points of $\widetilde{\Lambda}$ are pairwise equivalent for some suitable $k \in K$. Therefore a construction of a wreath product can be split into two cases: 1)$(o)b \in \widetilde{\Lambda}$, 2)$(o)b \notin \widetilde{\Lambda}$.

Consider the first case. Due to the above agruments $\Lambda = \widetilde{\Lambda} = (\widetilde{\Lambda})b$ is a linearly ordered set and, moreover, b is its reversing automorphism of order 2.

By $\widehat{W}$ denote a set of all elements of $Aut\Lambda$, that preserve all equivalences introduced above for every $k \in K$. Given an element $D \in \widehat{W}$ and a point $\lambda = (\ldots\lambda_\iota\ldots\lambda_k\ldots\lambda_\kappa\ldots) \in \Lambda$, we can define a map $d_{k,\lambda} : \Lambda_k \to \Lambda_k$ as follows. Let $\sigma \in \Lambda_k$. Then $\lambda_\sigma = (\ldots\lambda_\iota\ldots\sigma\ldots\lambda_\kappa\ldots) \in \Lambda$ and $\lambda_\sigma \equiv^k \lambda$. Therefore, $(\lambda_\sigma)D \equiv^k (\lambda)D$ and we assume that $(\sigma)d_{k,\lambda} = ((\lambda_\sigma)D)_k$. It can be shown that $d_{k,\lambda} \in Aut\Lambda_k$. Now elements of $\widehat{W}$ can be viewed as $K \times \Lambda$-"matrices", i.e. $D = (d_{k,\lambda})$. By W denote a set of all matrices $D = (d_{k,\lambda})$, such that $d_{k,\lambda} \in H_k$ for all $k \in K$ and $\lambda \in \Lambda$. Note that from the definition of the element $D = (d_{k,\lambda})$, it follows that $d_{\kappa,\lambda} = d_{\kappa,\lambda'}$ for all $\kappa \geq k$, if $\lambda \equiv_k \lambda', k \in K$. Now lattice operations can be naturally defined on W, thus making it an ℓ-group and allowing us to consider a representation (W, Λ, b), which we will call a generalised wreath product of the linearly ordered set of m-groups of permutations $\{(H_k, \Lambda_k, b_k) \mid k \in K\}$.

Consider the second case, i.e. $(o)b \notin \widetilde{\Lambda}$. Then linearly ordered sets $\widetilde{\Lambda}$ and $(\widetilde{\Lambda})b$ have no common elements and thus we can construct a linearly ordered set $\Lambda = \widetilde{\Lambda}\overleftarrow{\cup}\{o\}^{\varepsilon}\overleftarrow{\cup}(\widetilde{\Lambda})b$, where $\varepsilon = 0$ or $\varepsilon = 1$ with the reversing automorphism b of order two. Note again that no point of the "left"side of $\widetilde{\Lambda}$ cannot be equivalent to any point of the "right"side $(\widetilde{\Lambda})b$ for any $k \in K$. From this point, a generalised wreath product W is defined in a way, similar to the one used in the first case.

The following lemma specifies the structure of the generalised wreath product in the second case.

Lemma 2.1. *A generalised wreath product*(W, Λ, b)*, where* $\Lambda = \widetilde{\Lambda}\overleftarrow{\cup}\{o\}^{\varepsilon}\overleftarrow{\cup}(\widetilde{\Lambda})b$ *is proper.*

Proof. Assume that this representation is not proper and therefore there exist $\lambda \in \widetilde{\Lambda}$ and $g \in W$, such that $(\lambda)g \notin \widetilde{\Lambda}$. Let $(\lambda)g = o$. Since $(\lambda)g < (\lambda)g^2$, it can be seen that $(\lambda)g^2 \in (\widetilde{\Lambda})b$. Hence, we can assume, by substituting g with g^2 that there exists $g \in G$ such that $(\lambda)g \in (\widetilde{\Lambda})b$. Therefore, $(\lambda)g < (\lambda)g^2 < (\lambda)g^3 \in (\widetilde{\Lambda})b$. Since $(\lambda)g^2 \in (\widetilde{\Lambda})b$, there exists $\lambda' \in \widetilde{\Lambda}$, such that $(\lambda)g^2 = (\lambda')b$. From this we obtain that $(\lambda)g < ((\lambda')b)g \in (\widetilde{\Lambda})b$. From definiton of $\widetilde{\Lambda}$ and b it can be derived that $(\lambda)g \equiv_\alpha ((\lambda')b)g$ for a suitable $\alpha \in K$. Since $g \in W$, we get that $(\lambda)g \equiv_\alpha ((\lambda')b)g$ implies $\lambda \equiv_\alpha (\lambda')b$, which, in this case, is impossible. □

Note that, if the index set K is finite, we obtain a construction of a "general"wreath product, introduced in [5]. Also it can be seen that if the wreathed m-groups (G, Ω, a) and (H, Λ, b) belong to varietes $\mathcal{V}$ and $\mathcal{U}$ respectively, then their wreath product $GWrH \in \mathcal{VU}$.

The following theorem had been proved in [9] (Theorem 3.2.).

Theorem 2.1. *Every m-transitive permutation group (G, Ω, a) can be embedded in a generalised wreath product of its primitive components.*

Therefore every m-transitive normal-valued m group (G, Ω, a) can be viewed as an m-subgroup of the generalised wreath product (W, Λ, b) of its *abelian* primitive components. Hence, every m-group identity $w(\overline{x}) = e$, containing variables $\overline{x} = (x_1, \ldots, x_n)$, which does not hold in (G, Ω, a), also does not hold in (W, Λ, b). Now, for proving that $\bigvee\limits_{n \in \mathbb{N}} \mathcal{A}^n = \mathcal{N}$ it suffices to show that the generalised wreath product only needs to include a *finite* number of groups.

Every m-group word $w(\overline{x})$ of variables $\overline{x} = (x_1, \ldots, x_n)$ can be viewed as an ℓ-group word of variables $\overline{z} = (x_1, \ldots, x_n, x_{1_*}, \ldots, x_{n_*})$. Therefore,

$$w(\overline{z}) = \bigvee_{i \in I} \bigwedge_{j \in J} w_{ij}(\overline{z}),$$

where I, J are finite index sets and $w_{ij}(\overline{z}) = z_{ij1}^{\varepsilon_{ij1}} \cdot \ldots \cdot z_{ijt}^{\varepsilon_{ijt}}$ is a group word of these variables, $\varepsilon_{ijs} = \pm 1, z_{ijs} \in \overline{z}, 1 \leq s \leq t$. For every s, by $w_{ij}^{(s)}(\overline{z})$ denote a starting segment $w_{ij}(\overline{z})$ of length s, i.e. $w_{ij}^{(s)}(\overline{z}) = z_{ij1}^{\varepsilon_{ij1}} \cdot \ldots \cdot z_{ijs}^{\varepsilon_{ijs}}$. We will consider every word $w_{ij}^{(0)}(\overline{z})$ empty.

Assume that the identity $w(\overline{z}) = e$ does not hold in (W, Λ, b). Therefore there exists $\overline{h} = (h_1, \ldots, h_n; h_{1_*}, \ldots, h_{n_*}) \in W^{2n}$ и $\lambda_0 \in \Lambda$, such that $(\lambda_0)w(\overline{h}) \neq \lambda_0$. We will also need "a first half"of $\overline{h}$, i.e. a set $H = \{h_1, \ldots, h_n\}$.

Consider a case, when $\Lambda = \widetilde{\Lambda} = (\widetilde{\Lambda})b$ (see the first case of the definition of the generalised wreath product). Let $\Lambda_0 = \{\lambda_0, (\lambda_0)w_{ij}^{(s)}(\overline{h})\} \| \ i \in I, j \in J\}$ and s takes all possible values. It is clear, that Λ_0 is finite and contains at least two distinct points: $\lambda_0, (\lambda_0)w(\overline{h})$. For all different $\lambda', \lambda'' \in \Lambda_0$ there exists $\alpha \in K$, such that $\lambda'_\alpha \neq \lambda''_\alpha$ and $\lambda'_\beta = \lambda''_\beta$ for all $\beta > \alpha$. By K_0 denote the set of all such α. It can be seen that K_0 is finite since Λ_0 is finite and that K_0 contains at least one index because $|\Lambda_0| \geq 2$. For convenience we write $K_0 : \alpha_{min} < \ldots < \alpha < \ldots < \alpha_{max}$. By φ denote a projection map of a linearly ordered Λ on a lexicographically ordered set $\Gamma = \overleftarrow{\prod\limits_{\alpha \in K_0}} \Lambda_\alpha$, i.e. for a point $\lambda = (\ldots, \lambda_{\alpha_{min}}, \ldots, \lambda_{\alpha_{max}} \ldots) \in \Lambda$ we set $(\lambda)\varphi = (\lambda_{\alpha_{min}}, \ldots, \lambda_{\alpha_{max}}) \in \Gamma$. By definition of Λ_0, it is being projected on Γ bijectively and with preservation of order. Let $W_0 = WrH_\alpha$ be a wreath product of *abelian* m-groups of permutations $(H_\alpha, \Lambda_\alpha, b_\alpha)$, where $\alpha \in K_0$. Therefore $W_0 \in \mathcal{A}^m$, where m is the order of the set K_0. It is clear that W_0, as a group of order-preserving permutations acts on Γ, i.e. we can consider a representation (W_0, Γ, b_0), where b_0 is a reversing automorphism of order two on Γ, defined as a restriction of b on the considered set. It can be checked that for every $\lambda \in \Lambda$ it is true that $((\lambda)b)\varphi = ((\lambda)\varphi)b_0$.

We now construct a map $\tau : H \to W_0$ as follows: if $h = (h_{k,\lambda}) \in H$, then $(h)\tau = (f_{\alpha,(\lambda)\varphi})$, where

$$f_{\alpha,(\lambda)\varphi} = \begin{cases} h_{\alpha,\lambda'}, \text{if there exist } \lambda', \lambda'' \in \Lambda_0, \text{ such that } (\lambda')h \equiv^\alpha \lambda'', \ (\lambda')\varphi \equiv^\alpha (\lambda)\varphi, \\ e \text{ otherwise.} \end{cases}$$

We will first show that this definition of an element $(h)\tau = (f_{\alpha,(\lambda)\varphi})$ is correct. Indeed, let $\lambda_1, \lambda_2 \in \Lambda_0$, $(\lambda_1)h \equiv^\alpha \lambda_2$ and $(\lambda_1)\varphi \equiv^\alpha (\lambda)\varphi$. Then $(\lambda_1)\varphi \equiv^\alpha (\lambda)\varphi \equiv^\alpha (\lambda')\varphi$. Since $\lambda_1, \lambda' \in \Lambda_0$, we get that $(\lambda_1)\varphi \equiv^\alpha (\lambda')\varphi$ implies $\lambda_1 \equiv^\alpha \lambda'$ and therefore $h_{\alpha,\lambda'} = h_{\alpha,\lambda_1}$.

Consider a point $\lambda_{ij(s-1)} = (\lambda_0)w_{ij}^{(s-1)}(\overline{h}) \in \Lambda_0$, such that

$$(\lambda_{ij(s-1)})h = (\lambda_0)w_{ij}^{(s)}(\overline{h}) = \lambda_{ijs} \in \Lambda_0.$$

Note that, depending on the structure of the word, h_* can also be used as h. Now, from definitions and properties of maps φ and τ, it follows that $((\lambda_{ij(s-1)})h)\varphi = ((\lambda_{ij(s-1)}))\varphi(h)\tau$. Since φ is an order-preserving embedding of a linearly ordered set Λ_0 into a linearly ordered set Γ and we only consider unions and intersections in these sets, we get that $((\lambda_0)\varphi)(w(\overline{h}))\tau \neq (\lambda_0)\varphi$. Therefore in the considered case, our statement holds.

Now we will study the second case from the definition of the generalised wreath product. Then the generalised wreath product (W, Λ, b), where $\Lambda = \widetilde{\Lambda} \overleftarrow{\cup} \{o\}^\varepsilon \overleftarrow{\cup} (\widetilde{\Lambda})b$ is proper due to Lemma 2.1. Again, assume that there exists

$$\overline{h} = (h_1, \ldots, h_n; h_{1*}, \ldots, h_{n*}) \in W^{2n}$$

and $\lambda_0 \in \Lambda$, such that $(\lambda_0)w(\overline{h}) \neq \lambda_0$. Since $\{o\}^\varepsilon$ is W-invariant, we get that $\lambda_0 \neq \{o\}^\varepsilon$ and, therefore, λ_0 belongs to, for instance $\widetilde{\Lambda}$. But then we get that the whole set $\Lambda_0 = \{\lambda_0, (\lambda_0)w_{ij}^{(s)}(\overline{h})\} \| \ i \in I, j \in J\}$ is contained inside $\widetilde{\Lambda}$ and it only remains for us to repeat the steps, used when considering the first case. Therefore, we have proved

Theorem 2.2. *It is true that* $\mathcal{N} = \bigvee_{n \in \mathbb{N}} \mathcal{A}^n$, *where* $\mathcal{A}$ *is a variety of abelian* m*-groups.*

It is well known (see, for instance, [10]), that a variety $\mathcal{N}_\ell$ of all normal-valued ℓ-groups is the only idempotent of a semigroup L of lattice-ordered group varieties. Therefore, Theorem 2.2 can be considered as an m-group analogue of that statement. However the question of a complete description of idempotents of M remains open, since this semigroup contains at least one non-trivial idempotent – a variety $\mathcal{J} = \bigvee_{n \in \mathbb{N}} \mathcal{I}^n$, where $\mathcal{I}$ is an m-group variety, which is given by the identity $xx_* = e$. This is noted in [5]. This situation can be explained by the fact, that $\mathcal{I}$ is the least non-trivial element of the lattice M ([5],Theorem 3.1).

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