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ABOUT CONVERGENCE OF DIFFERENCE SCHEMES FOR A THIRD-ORDER PSEUDO-PARABOLIC EQUATION WITH NONLOCAL BOUNDARY VALUE CONDITION

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Abstract. A nonlocal boundary value problem for a third-order pseudo-parabolic equation with variable coefficients is considered. For solving this problem, a priori estimates in the differential and difference forms are obtained. The obtained a priori estimates imply the uniqueness and stability of the solution on a layer with respect to the initial data and the right-hand side and the convergence of the solution of the difference problem to the solution of the differential problem.

Keywords: boundary value problem, a nonlocal boundary value problem, a nonlocal condition, a third-order pseudo-parabolic equation, difference schemes, stability and convergence of difference schemes, a priori estimates, energy inequality method.

1. Introduction

Many issues of fluid filtration in porous media, heat transfer in a heterogeneous environment, moisture transfer in soils lead to differential equations for a pseudo-parabolic equation with variable coefficients [1]—[5].

A boundary value problems for parabolic equations with nonlocal condition arise in the study of particle diffusion in turbulent plasma, heat propagation in a thin heated rod, if the law of change in the total amount of rod heat is given. The first works for parabolic equations with nonclassical (integral) boundary conditions include, likely, the works of L.I. Kamynin [6] and F.A. Chudnovsky [7]. After the appearance of the work of A.V. Bitsadze and A.A. Samarskii [8], the attention

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of mathematicians increasingly began to be attracted by nonlocal boundary value problems of mathematical physics. Various classes of nonlocal boundary value problems were studied in the works of N.I. Ionkin [9], [10], V.A. Il'in, E.I. Moiseev [11], N.I. Ionkin, E.I. Moiseev [12], D.G. Gordeziani [13], A.M. Nakhushev [14], A.P. Soldatov, M.Kh. Shkhanukov [15] and etc.

A.F. Chudnovsky in work [7] drew attention to an insufficiently critical approach to the formulation of the boundary conditions for the moisture transfer equation

$$\frac{\partial w}{\partial t}=\frac{\partial}{\partial x}\left(D(w)\frac{\partial w}{\partial x}\right),\ 0<x<\ell,\ 0<t\leq T, \tag{1}$$

where $D(w)$ — diffusivity coefficient, w — moisture in fractions of a unit, x — depth.

For equation (1) A.F. Chudnovsky formulated a problem with the nonlocal condition:

$$D\frac{\partial w}{\partial x}\Big|_{x=0}=\int\limits_0^\alpha w dx, \tag{2}$$

$$\frac{\partial w}{\partial x}\Big|_{x=\ell}=0, \tag{3}$$

$$w(x,0)=\varphi(x),\ 0\leq x\leq \ell. \tag{4}$$

Nonlocal condition (2) means that the moisture flux through the surface $x=0$ is equal to the moisture content in the active soil layer from 0 to α, condition (3) means isolation in the sense of moisture exchange between the soil layer $x=\ell$ and its lower layers, and in the initial moment is set to the depth variation of moisture (4).

Note that work [16] is devoted to the study of locally one-dimensional schemes for the heat equation with a nonlocal condition of type (3) on the boundary. By the method of energy inequalities, an a priori estimate for the constructed locally one-dimensional scheme is obtained, its stability and convergence are proved.

Numerical methods for solving pseudo-parabolic equations of the third order are discussed in the works of M.Kh. Beshtokov [17] — [19]. In these papers, boundary value problems are considered for loaded pseudo-parabolic equations of the third order. To solve the problems posed, a priori estimates are obtained in differential and difference interpretations.

Difference methods for solving local and nonlocal boundary value problems for pseudoparabolic equations were considered in [20] — [22].

Papers [23] — [25] are devoted to difference methods for solving a fractional-order differential diffusion equation with Robin boundary value conditions in a multidimensional domain. Note that with an increase in the order of approximation of Robin's boundary value conditions on solutions of the fractional-order diffusion equation, we obtain a difference problems with nonlocal boundary conditions [26].

To solve the grid equations obtained by the difference approximation of differential equations with a nonlocal condition, the bordering method should be used ([27], p. 187).

2. Problem statement

In the rectangle $\overline{Q}_T \equiv \{(x,t) : 0 \leq x \leq \ell,\ 0 \leq t \leq T\}$ consider the problem with the nonlocal condition

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[k(x,t)\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial t}\frac{\partial}{\partial x}\left[k(x,t)\frac{\partial u}{\partial x}\right] + f(x,t), \tag{5}$$

$$\begin{cases} k\dfrac{\partial u}{\partial x} + \dfrac{\partial}{\partial t}\left(k\dfrac{\partial u}{\partial x}\right) = \beta_1(t)u + \displaystyle\int_0^\ell u dx - \mu_1(t), \text{ for } x = 0, \\ -\left[k\dfrac{\partial u}{\partial x} + \dfrac{\partial}{\partial t}\left(k\dfrac{\partial u}{\partial x}\right)\right] = \beta_2(t)u - \mu_2(t), \text{ for } x = \ell, \end{cases} \tag{6}$$

$$u(x,0) = u_0(x). \tag{7}$$

The coefficients of problem (5) — (7) satisfy the following conditions:

$$0 < c_1 \leq k(x,t) \leq c_2,\ |k_t(x,t)|, |\beta_2|, |\beta_1| \leq c_3. \tag{8}$$

Henceforward, it is assumed that problem (5) — (7) has a solution having the necessary derivatives. It is also assumed that the coefficients of Eq. (5) and boundary conditions (6) and (7) satisfy the necessary smoothness conditions ensuring the required order of approximation of the difference scheme. Also, in the course of the presentation, we will use positive constants $M_i, i = 1, 2, \ldots$, depending on the input data of problem (5) — (7).

Equation (5) is called the modified equation of moisture transfer in soils and soils.

3. A priori estimate for a differential problem

Theorem 1. *Let conditions (8) be satisfied. Then the solution of the differential problem (5) — (7) satisfies a priori estimate*

$$\|u\|^2_{W_2^1(0,\ell)} \leq M(t)\left(\int_0^t F(\tau)d\tau + \|u_0\|_0^2 + \|u_{0x}\|_0^2\right), \tag{9}$$

where

$F(t) = \int_0^t \left(\|f\|_0^2 + \mu_1^2(\tau) + \mu_2^2(\tau)\right) d\tau + \|u_0\|_0^2 + \|u_0'\|_0^2$, *$M(t)$ depends on the input data of problem (5) — (7).*

Proof. Suppose that there exists a solution to the problem (5) — (7) in the rectangle $\overline{Q}_T$. To obtain a priori estimate for the solution of problem (5) — (7), we use the method of energy inequalities. For this, let us multiply Eq. (5) scalarly by u:

$$(u_t, u) = ((ku_x)_x, u) + ((ku_x)_{xt}, u) + (f, u), \tag{10}$$

where

$$(u, v) = \int_0^\ell uvdx,\ \|u\|_0^2 = (u, u).$$

Now We transform the terms included in identity (10):

$$(u_t, u) = \frac{1}{2}\frac{\partial}{\partial t}\|u\|_0^2,$$

$$((ku_x)_x, u) = ku_x u\Big|_0^\ell - \int_0^\ell ku_x^2 dx,$$

$$((ku_x)_{xt}, u) = \int_0^\ell (ku_x)_{xt} u\ dx = (ku_x)_t\, u\Big|_0^\ell - \int_0^\ell (ku_x)_t\, u_x\ dx =$$

$$= (ku_x)_t\, u\Big|_0^\ell - \int_0^\ell \left(k_t u_x^2 + ku_x u_{xt}\right) dx =$$

$$= (ku_x)_t\, u\Big|_0^\ell - \frac{1}{2}\frac{\partial}{\partial t}\int_0^\ell ku_x^2 dx - \frac{1}{2}\int_0^\ell k_t u_x^2 dx,$$

$$(f, u) \le \frac{1}{2}\|f\|_0^2 + \frac{1}{2}\|u\|_0^2.$$

Substituting the obtained expressions into equality (10), then then

$$\frac{1}{2}\frac{\partial}{\partial t}\|u\|_0^2 + \frac{1}{2}\frac{\partial}{\partial t}\int_0^\ell ku_x^2 dx + \int_0^\ell ku_x^2 dx \le$$

$$\le (ku_x)_t\, u\Big|_0^\ell + ku_x u\Big|_0^\ell - \frac{1}{2}\int_0^\ell k_t u_x^2 dx + \frac{1}{2}\|f\|_0^2 + \frac{1}{2}\|u\|_0^2. \tag{11}$$

Using the boundary conditions (6), from the last inequality we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\|u\|_0^2 + \frac{c_1}{2}\frac{\partial}{\partial t}\|u_x\|_0^2 + \int_0^\ell ku_x^2 dx \le -\frac{1}{2}\int_0^\ell k_t u_x^2 dx -$$

$$-\beta_2(t)u^2(\ell,t) + \mu_2(t)u(\ell,t) - u(0,t)\int_0^\ell u dx - \beta_1(t)u^2(0,t) + \mu_1(t)u(0,t) +$$

$$+\frac{1}{2}\|f\|_0^2 + \frac{1}{2}\|u\|_0^2.$$

Hence,

$$\frac{\partial}{\partial t}\|u\|_0^2 + c_1\frac{\partial}{\partial t}\|u_x\|_0^2 + 2c_1\|u_x\|_0^2 \le c_2\|u_x\|_0^2 -$$

$$-2u(0,t)\int_0^\ell u dx + 2c_3\left(u^2(\ell,t) + u^2(0,t)\right) + \mu_2^2(t) + u^2(\ell,t) + \mu_1^2(t) + u^2(0,t) + \|f\|_0^2 + \|u\|_0^2.$$

We apply the embedding theorem [28] to the terms $u^2(l,t)$ and $u^2(0,t)$. Then we get

$$\frac{\partial}{\partial t}\|u\|_0^2 + c_1\frac{\partial}{\partial t}\|u_x\|_0^2 + 2c_1\|u_x\|_0^2 \le -2u(0,t)\int_0^\ell u dx + c_2\|u_x\|_0^2 +$$

$$+4c_3\varepsilon\|u_x\|_0^2 + 4c_3c_\varepsilon\|u\|_0^2 + \mu_2^2(t) + \mu_1^2(t) + 2\varepsilon\|u_x\|_0^2 + 2c_\varepsilon\|u\|_0^2 + \|f\|_0^2 + \|u\|_0^2. \tag{12}$$

Let us estimate the term containing the integral:

$$-2u(0,t)\int\limits_0^\ell udx \le \left(\int\limits_0^\ell udx\right)^2 + u^2(0,t) \le$$

$$\le \ell\int\limits_0^\ell u^2dx + \varepsilon\|u_x\|_0^2 + c_\varepsilon\|u\|_0^2 = (\ell + c_\varepsilon)\,\|u\|_0^2 + \varepsilon\|u_x\|_0^2.$$

We substitute the obtained result into inequality (12). We get

$$\frac{\partial}{\partial t}\|u\|_0^2 + c_1\frac{\partial}{\partial t}\|u_x\|_0^2 \le M_1\|u_x\|_0^2 + M_2\|u\|_0^2 + \mu_2^2(t) + \mu_1^2(t) + \|f\|_0^2$$

where $M_1 = 4c_3\varepsilon + 3\varepsilon + c_2 - 2c_1$, $M_2 = 4c_3c_\varepsilon + 3c_\varepsilon + \ell + 1$.

Let us integrate the resulting inequality over τ in the range from 0 to t:

$$\|u\|_0^2 + c_1\|u_x\|_0^2 \le M_3\left[\int\limits_0^t\|u\|_0^2d\tau + \int\limits_0^t\|u_x\|_0^2d\tau\right] + \int\limits_0^t\left(\|f\|_0^2 + \mu_1^2(\tau) + \mu_2^2(\tau)\right)d\tau +$$

$$+\|u_0\|_0^2 + \|u_0'\|_0^2$$

or

$$\|u\|_0^2 + \|u_x\|_0^2 \le M_4\int\limits_0^t\left(\|u\|_0^2 + \|u_x\|_0^2\right)d\tau + F(t),$$

where $F(t) = \int\limits_0^t\left(\|f\|_0^2 + \mu_1^2(\tau) + \mu_2^2(\tau)\right)d\tau + \|u_0\|_0^2 + \|u_0'\|_0^2$, M_4 — is a known positive constant.

Applying Gronwall's lemma [?], to the last inequality, we obtain the estimate

$$\|u\|^2_{W_2^1(0,\ell)} \le M(t)\left(\int\limits_0^t F(\tau)d\tau + \|u_0\|_0^2 + \|u_0'\|_0^2\right). \tag{13}$$

A priori estimate (13) implies the uniqueness of the solution to problem (5) — (7), as well as the continuous dependence of the solution to the problem on the input data in the norm $\|u\|_{W_2^1(0,\ell)} = \|u\|_0^2 + \|u_x\|_0^2$. □

4. The difference scheme

On the segment $[0,\ell]$ we introduce a grid $\bar\omega_h$ with step $h = \frac{\ell}{N}$:

$$\bar\omega_h = \{x_i = i\hbar : i = 0,1,\ldots,N\},$$

$$\hbar = \begin{cases} h, & i = 1,2,\ldots,N-1,\\ \dfrac{h}{2}, & i = 0,N.\end{cases}$$

On the segment $[0,T]$ we also introduce a uniform grid $\bar\omega_\tau$ with step $\tau = \dfrac{T}{j_0}$:

$$\bar\omega_\tau = \{t_j = j\tau : j = 0,1,\ldots,j_0\}.$$

Then $\bar\omega_{h\tau} = \bar\omega_h\times\overline{\omega}_\tau = \{(x_i,t_j), x\in\bar\omega_h, t\in\bar\omega_\tau\}$ — grid in rectangle $\overline{Q}_T$.

Equation (5) is approximated by a two-layer purely implicit scheme on the interval $[t_{j-1}, t_j]$, then we obtain the difference equation

$$y_{\bar{t}} = \Lambda y + (a y_{\bar{x}})_{x\bar{t}} + \varphi, \qquad (14)$$
$$\Lambda y = (a y_{\bar{x}})_x,$$

where the coefficients a_i are grid functions that are selected from the conditions of the second order of approximation on a uniform grid. We will use the following approximation of the coefficient $k(x,t)$ [29]:

$$a_i = k_{i-\frac{1}{2}} = k(x_i - \frac{h}{2}, t), \ i = 1, 2, \ldots, N.$$

The difference analog for boundary conditions (6) has the form:

$$\begin{cases} a_1 y_{x,0} + (a_1 y_{x,0})_{\bar{t}} = \beta_1 y_0 + \dfrac{1}{0.5h} \displaystyle\sum_{i=1}^{N} y_i \hbar - \mu_1, \ x = 0, \\ -\, a_N y_{\bar{x},N} - (a_N y_{\bar{x},N})_{\bar{t}} = \beta_2 y_N - \mu_2, \ x = \ell. \end{cases} \qquad (15)$$

Conditions (15) are of the order of approximation $O(h)$. Increasing in a known way the order of approximation to $O(h^2)$ on solutions of equation (5), we have:

$$a_1 y_{x,0} = k y_0' + \frac{h}{2}(k y_0')' + O(h^2),$$

$$(a_1 y_{x,0})_{\bar{t}} = \frac{a_1 y_{x,0} - \check{a}_1 \check{y}_{x,0}}{\tau} = \frac{1}{\tau}\left(k y_0' + \frac{h}{2}(k y_0')' + O(h^2) - \check{k}\check{y}_0' + \frac{h}{2}(\check{k}\check{y}_0')' \right),$$

where

$$y = y_i^j = y(x_i, t_j), \ \check{y} = y_i^{j-1}, \ y_{\bar{t}} = \frac{y^j - y^{j-1}}{\tau}, \ y_t = \frac{y^{j+1} - y^j}{\tau},$$
$$y_{\bar{x}} = \frac{y_i - y_{i-1}}{h}, \ y_x = \frac{y_{i+1} - y_i}{h}.$$

Hence

$$k y_0' = a_1 y_{x,0} - 0.5h (k y_0')' + O(h^2),$$
$$(k y_0')_{\bar{t}} = (a_1 y_{x,0})_{\bar{t}} - 0.5h (k y_0')'_{\bar{t}} + O(h^2).$$

Thus,

$$k y_0' + (k y_0')_{\bar{t}} = a_1 y_{x,0} + (a_1 y_{x,0})_{\bar{t}} - 0.5h (y_{\bar{t},0} - f_0) + O(h^2).$$

So,

$$a_1 y_{x,0} + (a_1 y_{x,0})_{\bar{t}} - 0.5h(y_{\bar{t},0} - f_0) = \beta_1 y_0 + \frac{1}{0.5h}\sum_{i=1}^{N} y_i \hbar - \mu_1 + O(h^2). \qquad (16)$$

We discard the value of the order of smallness $O(h^2)$, then in (15) the boundary condition at $x = 0$ takes the form:

$$y_{\bar{t},0} = \frac{a_1 y_{x,0} + (a_1 y_{x,0})_{\bar{t}} - \beta_1 y_0}{0.5h} - \frac{1}{0.5h}\sum_{i=1}^{N} y_i \hbar + \bar{\mu}_1, \ x = 0,$$

where

$$\bar{\mu}_1 = \frac{\mu_1}{0.5h} + f_0.$$

Similarly, for $x = \ell$ we obtain

$$y_{\bar{t},N} = -\frac{a_N y_{\bar{x},N} + (a_N y_{\bar{x},N})_{\bar{t}} + \beta_2 y_N}{0.5h} + \bar{\mu}_2,$$

where

$$\bar{\mu}_2 = \frac{\mu_2}{0.5h} + f_N.$$

Thus, to the differential problem (5) — (7) on grid $\bar{\omega}_{h\tau}$ we associate a purely implicit difference scheme:

(17)
$$y_{\bar{t}} = \bar{\Lambda} y + \Phi,$$

(18)
$$y(x,0) = u_0(x),$$

where

$$\bar{\Lambda} y = \begin{cases} \dfrac{a_1 y_{x,0} + (a_1 y_{x,0})_{\bar{t}} - \beta_1 y_0}{0.5h} - \dfrac{1}{0.5h} \sum\limits_{i=1}^{N} y_i \hbar, & \text{for } x = 0, \\ (a y_{\bar{x}})_x + (a y_{\bar{x}})_{x\bar{t}}, & \text{for } x \in \omega_h, \\ -\dfrac{a_N y_{\bar{x},N} + (a_N y_{\bar{x},N})_{\bar{t}} + \beta_2 y_N}{0.5h}, & \text{for } x = \ell, \end{cases}$$

$$\Phi = \begin{cases} \bar{\mu}_1, & \text{for } x = 0, \\ \varphi, & \text{for } x \in \omega_h, \\ \bar{\mu}_2, & \text{for } x = \ell. \end{cases}$$

Under the assumption that problem (5) — (7) has a solution having the necessary derivatives, and also the coefficients of Eq. (5) and boundary conditions (6), (7) satisfy the necessary smoothness conditions, the difference scheme (17) — (18) has an approximation order $O(\hbar^2 + \tau)$, according to [29].

5. Stability and convergence of the difference scheme

Since the maximum principle has not been established for nonlocal boundary value problems, we will obtain an a priori estimate for the difference problem (17) — (18) using the method of energy inequalities.

We introduce the scalar product and the norm

$$[u,v] = \sum_{i=0}^{N} u_i v_i \hbar, \ (u,v] = \sum_{i=1}^{N} u_i v_i \hbar, \ \|u]|_0^2 = \sum_{i=1}^{N} u_i^2 \hbar = (1, u^2],$$

$$\hbar = \begin{cases} h, & i = 1, 2, \ldots, N-1, \\ \dfrac{h}{2}, & i = 0, N. \end{cases}$$

Let us multiply equation (17) scalarly by y:

(19)
$$[y_{\bar{t}}, y] - [\bar{\Lambda} y, y] = [\Phi, y].$$

We will transform each term of the identity (19):

$$[y_{\bar{t}}, y] = \sum_{i=0}^{N} y_{\bar{t},i} y_i \hbar = \sum_{i=0}^{N} \frac{y_i - \check{y}_i}{\tau} y_i \hbar = \frac{1}{\tau} \sum_{i=0}^{N} (y_i^2 - y_i \check{y}_i) \hbar = \frac{1}{\tau} |[y]|_0^2 -$$

$$- \frac{1}{\tau} \sum_{i=0}^{N} \check{y}_i^2 \hbar + \frac{1}{\tau} \sum_{i=0}^{N} (\check{y}_i^2 - \check{y}_i y_i) \hbar = \left(|[y]|_0^2\right)_{\bar{t}} + \frac{1}{\tau} \sum_{i=0}^{N} \left[\frac{y_i^2 - 2 y_i \check{y}_i + \check{y}_i^2}{\tau^2} \tau^2 \hbar + (y_i \check{y}_i - y_i^2) \hbar \right] =$$

$$= \left(|[y]|_0^2\right)_{\bar{t}} + \tau |[y_{\bar{t}}]|_0^2 - [y_{\bar{t}}, y].$$

Hence we get

(20)
$$[y_{\bar{t}}, y] = \frac{1}{2} \left(|[y]|_0^2\right)_{\bar{t}} + \frac{\tau}{2} |[y_{\bar{t}}]|_0^2.$$

$$[\bar{\Lambda}y, y] = \sum_{i=0}^{N} \bar{\Lambda}y_i \cdot y_i\hbar = \sum_{i=1}^{N-1} \left[(ay_{\bar{x}})_{x,i} + ay_{\bar{x}})_{x\bar{t},i}\right] y_i h +$$
$$+ \frac{a_1 y_{x,0} + (a_1 y_{x,0})_{\bar{t}} - \beta_1 y_0}{0.5h} \cdot y_0 \cdot 0.5h + \frac{1}{0.5h} \sum_{i=0}^{N} y_i\hbar \cdot y_0 \cdot 0.5h +$$
$$+ \frac{-a_N y_{\bar{x},N} - (a_N y_{\bar{x},N})_{\bar{t}} - \beta_2 y_N}{0.5h} \cdot y_N \cdot 0.5h =$$
$$= \sum_{i=1}^{N-1} \left(\frac{a_{i+1} y_{\bar{x},i+1} - a_i y_{\bar{x},i}}{h} \cdot y_i h + \frac{(ay_{\bar{x}})_{\bar{t},i+1} - (ay_{\bar{x}})_{\bar{t},i}}{h} \cdot y_i h \right) +$$
$$+ a_1 y_{\bar{x},1}\ y_0 + (a_1 y_{\bar{x},1})_{\bar{t}}\ y_0 - \beta_1 y_0^2 + \sum_{i=0}^{N} y_i\hbar \cdot y_0 - a_N y_{\bar{x},N}\ y_N - (ay_{\bar{x}})_{\bar{t},N}\ y_N -$$
$$- \beta_2 y_N^2 = \sum_{i=2}^{N} a_i y_{\bar{x},i} y_{i-1} - \sum_{i=1}^{N-1} a_i y_{\bar{x},i} y_i + \sum_{i=2}^{N} (ay_{\bar{x}})_{\bar{t},i} y_{i-1} - \sum_{i=1}^{N-1} (ay_{\bar{x}})_{\bar{t},i} y_i +$$
$$+ a_1 y_{\bar{x},1}\ y_0 - a_N y_{\bar{x},N}\ y_N + (ay_{\bar{x}})_{\bar{t},1}\ y_0 - (ay_{\bar{x}})_{\bar{t},N}\ y_N -$$
$$- \beta_1 y_0^2 - \beta_2 y_N^2 + \sum_{i=0}^{N} y_i\hbar \cdot y_0 = \sum_{i=1}^{N} a_i y_{\bar{x},i} y_{i-1} - \sum_{i=1}^{N} a_i y_{\bar{x},i} y_i + \sum_{i=1}^{N} (ay_{\bar{x}})_{\bar{t},i} y_{i-1} -$$
$$- \sum_{i=1}^{N} (ay_{\bar{x}})_{\bar{t},i} y_i - \beta_1 y_0^2 - \beta_2 y_N^2 + \sum_{i=0}^{N} y_i\hbar \cdot y_0 =$$
$$= - \sum_{i=1}^{N} a_i (y_{\bar{x},i})^2 h - \sum_{i=1}^{N} (ay_{\bar{x}})_{t,i} y_{\bar{x},i} h - \beta_1 y_0^2 - \beta_2 y_N^2 + \sum_{i=1}^{N} y_i\hbar \cdot y_0 =$$
$$= - \left(a, (y_{\bar{x}})^2\right] - ((ay_{\bar{x}})_{\bar{t}}, y_{\bar{x}}] - \beta_1 y_0^2 - \beta_2 y_N^2 + \sum_{i=0}^{N} y_i\hbar \cdot y_0. \tag{21}$$

$$[\Phi, y] = \sum_{i=0}^{N} \Phi_i y_i \hbar = \sum_{i=1}^{N-1} \varphi_i y_i h + \bar{\mu}_1 y_0 \cdot 0.5h + \bar{\mu}_2 y_N \cdot 0.5h =$$
$$= \sum_{i=1}^{N-1} \varphi_i y_i h + \left(\frac{\mu_1}{0.5h} + f_0 \right) y_0 \cdot 0.5h + \left(\frac{\mu_2}{0.5h} + f_N \right) y_N \cdot 0.5h =$$
$$= \sum_{i=1}^{N-1} \varphi_i y_i h + \mu_1 y_0 + 0.5h y_0 \varphi_0 + \mu_2 y_N + 0.5h y_N \varphi_N =$$
$$= \sum_{i=0}^{N} \varphi_i y_i \hbar + \mu_1 y_0 + \mu_2 y_N = [\varphi, y] + \mu_1 y_0 + \mu_2 y_N. \tag{22}$$

Substituting (20), (21) and (22) into identity (19), we obtain

$$\left(|[y]|_0^2\right)_{\bar{t}} + \tau |[y_{\bar{t}}]|_0^2 + 2\left(a, (y_{\bar{x}})^2\right] + 2\left((ay_{\bar{x}})_{\bar{t}}, y_{\bar{x}}\right] + 2\beta_1 y_0^2 + 2\beta_2 y_N^2 -$$
$$-2 \sum_{i=0}^{N} y_i\hbar \cdot y_0 = 2[\varphi, y] + 2\mu_1 y_0 + 2\mu_2 y_N. \tag{23}$$

Transform separately the amount

$$\left(a,(y_{\bar{x}})^2\right]+((ay_{\bar{x}})_{\bar{t}},y_{\bar{x}}]=\sum_{i=1}^{N}a(y_{\bar{x}})^2h+\sum_{i=1}^{N}(ay_{\bar{x}})_{\bar{t}}y_{\bar{x}}h=$$

$$=\sum_{i=1}^{N}a(y_{\bar{x}})^2h+\sum_{i=1}^{N}(a_{\bar{t}}y_{\bar{x}}^2+ay_{\bar{x}\bar{t}}y_{\bar{x}})h=$$

$$=\sum_{i=1}^{N}a(y_{\bar{x}})^2h+\sum_{i=1}^{N}a_{\bar{t}}(y_{\bar{x}})^2h+\sum_{i=1}^{N}ay_{\bar{x}\bar{t}}y_{\bar{x}}h. \tag{24}$$

Let us estimate the last term in (24):

$$\sum_{i=1}^{N}ay_{\bar{x}\bar{t}}y_{\bar{x}}h\geq c_1\sum_{i=1}^{N}y_{\bar{x}\bar{t}}y_{\bar{x}}h=c_1\sum_{i=1}^{N}\frac{y_{\bar{x}}-\check{y}_{\bar{x}}}{\tau}y_{\bar{x}}h=$$

$$=\frac{c_1}{2}\sum_{i=1}^{N}\left(\frac{y_{\bar{x}}^2-2y_{\bar{x}}\check{y}_{\bar{x}}+\check{y}_{\bar{x}}^2}{\tau^2}\ \tau+\frac{y_{\bar{x}}^2-\check{y}_{\bar{x}}^2}{\tau}\right)h=\frac{c_1}{2}\tau\sum_{i=1}^{N}\left(\frac{y_{\bar{x}}-\check{y}_{\bar{x}}}{\tau}\right)^2h+$$

$$+\frac{c_1}{2}\sum_{i=1}^{N}\frac{y_{\bar{x}}^2-\check{y}_{\bar{x}}^2}{\tau}h=\frac{c_1}{2}\tau\sum_{i=1}^{N}(y_{\bar{x}\bar{t}})^2h+\frac{c_1}{2}\cdot\frac{\|y_{\bar{x}}]|_0^2-\|\check{y}_{\bar{x}}]|_0^2}{\tau}=$$

$$=\frac{c_1}{2}\tau\|y_{\bar{x}\bar{t}}]|_0^2+\frac{c_1}{2}\left(\|y_{\bar{x}}]|_0^2\right)_{\bar{t}}.$$

In this way,

$$\left(a,(y_{\bar{x}})^2\right]+((ay_{\bar{x}})_{\bar{t}},y_{\bar{x}}]\geq\sum_{i=1}^{N}a(y_{\bar{x}})^2h+\sum_{i=1}^{N}a_{\bar{t}}(y_{\bar{x}})^2h+$$

$$+\frac{c_1}{2}\tau\|y_{\bar{x}\bar{t}}]|_0^2+\frac{c_1}{2}\left(\|y_{\bar{x}}]|_0^2\right)_{\bar{t}}. \tag{25}$$

Substituting (25) into equality (23), we obtain:

$$\left(|[y]|_0^2\right)_{\bar{t}}+\tau|[y_{\bar{t}}]|_0^2+c_1\tau\|y_{\bar{x}\bar{t}}]|_0^2+c_1\left(\|y_{\bar{x}}]|_0^2\right)_{\bar{t}}\leq$$

$$\leq-2\sum_{i=1}^{N}a_{\bar{t}}(y_{\bar{x}})^2h-2\sum_{i=1}^{N}a(y_{\bar{x}})^2h-2\beta_1y_0^2-2\beta_2y_N^2+2\sum_{i=0}^{N}y_i\hbar\cdot y_0+$$

$$+2[\varphi,y]+2\mu_1y_0+2\mu_2y_N\leq2(c_2+c_3)\|y_{\bar{x}}]|_0^2+2\sum_{i=0}^{N}y_i\hbar\cdot y_0-$$

$$-2\beta_1y_0^2-2\beta_2y_N^2+2[\varphi,y]+2\mu_1y_0+2\mu_2y_N. \tag{26}$$

Since

$$2[\varphi,y]\leq\|\varphi]|_0^2+\|y]|_0^2,$$

$$-2\beta_1y_0^2-2\beta_2y_N^2\leq4c_3(\varepsilon\|y_{\bar{x}}]|_0^2+c_\varepsilon\|y]|_0^2),$$

$$2\mu_1y_0+2\mu_2y_N\leq\mu_1^2+y_0^2+\mu_2^2+y_N^2\leq2(\varepsilon\|y_{\bar{x}}]|_0^2+c_\varepsilon\|y]|_0^2)+\mu_1^2+\mu_2^2,$$

the inequality (26) takes the form

$$\left(|[y]|_0^2\right)_{\bar{t}}+\tau|[y_{\bar{t}}]|_0^2+c_1\tau\|y_{\bar{x}\bar{t}}]|_0^2+c_1\left(\|y_{\bar{x}}]|_0^2\right)_{\bar{t}}\leq2(c_2+c_3)\|y_{\bar{x}}]|_0^2+\|\varphi]|_0^2+\|y]|_0^2+$$

$$+(4c_3\varepsilon+2\varepsilon)\|y_{\bar{x}}]|_0^2+(4c_3c_\varepsilon+2c_\varepsilon)\|y]|_0^2+\mu_1^2+\mu_2^2+2\sum_{i=0}^{N}y_i\hbar\cdot y_0. \tag{27}$$

Let's estimate the sum

$$2\sum_{i=0}^{N}y_i\hbar\cdot y_0\le\sum_{i=0}^{N}(y_i^2+y_0^2)\hbar\le 2\sum_{i=0}^{N}(\varepsilon\|y_{\bar{x}}]|_0^2+c_\varepsilon\|y]|_0^2)\hbar=$$
$$=2hN(\varepsilon\|y_{\bar{x}}]|_0^2+c_\varepsilon\|y]|_0^2).$$

Substituting this result into inequality (27), we obtain:

$$\left(|[y]|_0^2\right)_{\bar{t}}+\tau|[y_{\bar{t}}]|_0^2+c_1\tau\|y_{\bar{x}\bar{t}}]|_0^2+c_1\left(\|y_{\bar{x}}]|_0^2\right)_{\bar{t}}\le c_4|[y_{\bar{x}}]|_0^2+c_5|[y]|_0^2+$$
$$+|[\varphi]|_0^2+\mu_1^2+\mu_2^2,$$

where

$$c_4=2c_2+2c_3+4c_3\varepsilon+2\varepsilon+2Nh\varepsilon,$$
$$c_5=1+4c_3c_\varepsilon+2c_\varepsilon+2Nh\varepsilon.$$

Hence

$$\|y^j]|_0^2-\|y^{j-1}]|_0^2+c_1\|y_{\bar{x}}^j]|_0^2-c_1\|y_{\bar{x}}^{j-1}]|_0^2\le$$
$$\le M_1\left(\|y^j]|_0^2+\|y_{\bar{x}}^j]|_0^2\right)\tau+\left(\|\varphi^j]|_0^2+\mu_1^2(t_j)+\mu_2^2(t_j)\right)\tau.$$

Summing up the last inequality over all j' from 1 to $j+1$, we obtain:

$$\|y^{j+1}]|_0^2+\|y_{\bar{x}}^{j+1}]|_0^2\le M_2\sum_{j'=1}^{j+1}\left(\|y^{j'}]|_0^2+\|y_{\bar{x}}^{j'}]|_0^2\right)\tau+$$
$$+M_3\left(\|u_0]|_0^2+\|u_{0x}]|_0^2+\sum_{j'=1}^{j+1}\left(\|\varphi^{j'}]|_0^2+\mu_1^2(t_j)+\mu_2^2(t_j)\right)\tau\right).$$

The following inequality holds:

$$\|y^{j+1}]|_0^2+\|y_{\bar{x}}^{j+1}]|_0^2\le \nu_1\sum_{j'=1}^{j}\left(\|y^{j'}]|_0^2+\|y_{\bar{x}}^{j'}]|_0^2\right)\tau+$$
$$\nu_2\left(\|u_0]|_0^2+\|u_{0x}]|_0^2+\sum_{j'=1}^{j+1}\left(\|\varphi^{j'}]|_0^2+\mu_1^2(t_j)+\mu_2^2(t_j)\right)\tau\right),$$

where ν_1,ν_2 — known positive constants.

Based on Lemma 4 (см. [30], с.171) we obtain the following estimate:

$$\|y^{j+1}]|_0^2+\|y_{\bar{x}}^{j+1}]|_0^2\le$$
$$\le M(t)\left(\|u_0]|_0^2+\|u_{0x}]|_0^2+\sum_{j'=1}^{j+1}\left(\|\varphi^{j'}]|_0^2+\mu_1^2(t_j)+\mu_2^2(t_j)\right)\tau\right). \tag{28}$$

Theorem 2. *Let conditions (8) be satisfied. Then there are such h_0,τ_0, which for $h\le h_0,\tau\le\tau_0$ for the solution of the difference problem (17) — (18) a priori estimate (28) is valid, which implies the uniqueness and stability of the solution to the difference problem (17) — (18) with respect to the initial data and the right-hand side.*

Let $u(x,t)$ be a solution of the problem (5) — (7), $y = y_i^j = y(x_i^j)$ — be a solution of the difference problem (17) — (18). Let us denote the error by $z_i^j = y_i^j - u_i^j$. Substituting $y = z + u$ into (17) — (18), we obtain the problem for the error z:

$$z_{\bar{t}} = \bar{\Lambda} z + \Psi, \tag{29}$$

$$z(x,0) = 0, \tag{30}$$

where

$$\bar{\Lambda} z = \begin{cases} \dfrac{a_1 z_{x,0} + (a_1 z_{x,0})_{\bar{t}} - \beta_1 z_0}{0.5h} - \dfrac{1}{0.5h}\sum\limits_{i=1}^{N} z_i \hbar, & \text{for } x = 0, \\ (a z_{\bar{x}})_x + (a z_{\bar{x}})_{x\bar{t}}\,, & \text{for } x \in \omega_h, \\ -\dfrac{a_N z_{\bar{x},N} + (a_N z_{\bar{x},N})_{\bar{t}} + \beta_2 z_N}{0.5h}, & \text{for } x = \ell, \end{cases}$$

$$\Phi = \begin{cases} \bar{\psi}_-, & \text{for } x = 0, \\ \psi, & \text{for } x \in \omega_h, \\ \bar{\psi}_+, & \text{for } x = \ell, \end{cases}$$

$\psi = O(h^2 + \tau),\ \psi_- = O(h^2 + \tau),\ \psi_+ = O(h^2 + \tau)$ are the errors of approximation of the differential problem (5) — (7) by the difference scheme (17) — (18) in the class of solutions of the problem (5) — (7).

Applying a priori estimate (28) to the solution of problem (29) — (30), we obtain:

$$\|z^{j+1}]|_0^2 + \|z_{\bar{x}}^{j+1}]|_0^2 \le M \sum_{j'=1}^{j+1} \left(\|\psi^{j'}]|_0^2 + \psi_-^2(t_j) + \psi_+^2(t_j) \right) \tau, \tag{31}$$

where M — is a positive constant independent of h h and τ.

A priori estimate (31) implies the convergence of the solution of difference problem (17) — (18) to the solution of differential problem (5) — (7) in the norm $\|z^{j+1}]|_1^2 = \|z^{j+1}]|_0^2 + \|z_{\bar{x}}^{j+1}]|_0^2$ with the rate $O(\hbar^2 + \tau)$.

Note that similar results can be obtained in the case of the following nonlocal boundary value problem:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[k(x,t)\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial t}\frac{\partial}{\partial x}\left[k(x,t)\frac{\partial u}{\partial x}\right] + \int\limits_0^\ell u dx + f(x,t),$$

$$\begin{cases} \quad k\dfrac{\partial u}{\partial x} + \dfrac{\partial}{\partial t}\left(k\dfrac{\partial u}{\partial x}\right) = \beta_1(t)u - \mu_1(t), \text{ for } x = 0, \\ -\left[k\dfrac{\partial u}{\partial x} + \dfrac{\partial}{\partial t}\left(k\dfrac{\partial u}{\partial x}\right)\right] = \beta_2(t)u - \mu_2(t), \text{ for } x = \ell, \end{cases}$$

$$u(x,0) = u_0(x).$$

References

[1] E.S. Dzekcer, *Equation of motion of underground water with a free surface in multilayer media*, Sov. Phys., Dokl., **20** (1975), 24–26. Zbl 0331.76056

[2] L.I. Rubinshtein, *On heat propagation in heterogeneous media*, Izv. Akad. Nauk SSSR, Ser. Geogr., **12**:1 (1948), 27–45.

[3] T.W. Ting, *A cooling process according to two-temperature theory of heat conduction*, J. Math. Anal. Appl., **45**:1 (1974), 23–31. Zbl 0272.35039

[4] M. Hallaire, *L'eau et la production vegetable*, Inst. National de la Recherche Agronomique, **9** (1964).

[5] A.F. Chudnovskii, *Thermal physics of soils*, Nauka, Moskow, 1976.

[6] L.I. Kamynin, *A boundary value problem in the theory of heat conduction with a nonclassical boundary condition*, U.S.S.R. Comput. Math. Math. Phys., **4**:6 (1964), 33–59. Zbl 0206.39801

[7] A.F. Chudnovskii *Nekotorye korrektivy v postanovke i reshenii zadach teplo i vlagoperenosa v pochve*, Sb. trudov po agrofizike, 1969, 41–54.

[8] A.V. Bitsadze, A.A. Samarskii, *On some simple generalizations of linear elliptic boundary problems*, Sov. Math., Dokl., **10** (1969), 398–400. Zbl 0187.35501

[9] N.I. Ionkin, *The solution of a certain boundary value problem of the theory of heat conduction with a nonclassical boundary condition*, Differ. Uravn., **13**:2 (1977), 294–304. Zbl 0349.35040

[10] N.I. Ionkin, *Uniform convergence of the difference scheme for one nonstationary nonlocal boundary-value problem*, Comput. Math. Model., **2**:3 (1991), 223–228. Zbl 0799.65095

[11] V.A. Il'in, E.I. Moiseev, *A nonlocal boundary value problem for the Sturm-Liouville operator in the differential and difference treatments*, Sov. Math., Dokl., **34** (1987), 507–511. Zbl 0643.34016

[12] N.I. Ionkin, E.I. Moiseev, *A problem for a heat equation with two-point boundary conditions*, Differ. Uravn., **15**:7 (1979), 1284–1295. Zbl 0415.35032

[13] D.G. Gordeziani, *On the methods of solution for one class of non-local boundary value problems*, Izdatel'stvo Tbilisskogo Universiteta, Tbilisi, 1981. Zbl 0464.35037

[14] A.M. Nakhushev, *A nonlocal problem and the Goursat problem for a loaded equation of hyperbolic type, and their applications to the prediction of ground moisture*, Sov. Math., Dokl., **19** (1978), 1243–1247. Zbl 0433.35043

[15] A.P. Soldatov, M.Kh. Shkhanukov, *Boundary value problems with general nonlocal Samarskij condition for pseudoparabolic equations of higher order*, Sov. Math., Dokl., **36**:3 (1988), 507–511. Zbl 0701.35092

[16] A.K. Bazzaev, D.K. Gutnova, M.Kh. Shkhanukov-Lafishev, *Locally one-dimensional scheme for parabolic equation with a nonlocal condition*, Zh. Vychisl. Mat. Mat. Fiz., **52**:6 (2012), 1048–1057. Zbl 1274.35032

[17] M.Kh. Beshtokov, *Difference method for solving a nonlocal boundary value problem for a degenerating third-order pseudo-parabolic equation with variable coefficients*, Comput. Math. Math. Phys., **56**:10 (2016), 1763–1777. Zbl 1358.65054

[18] M.Kh. Beshtokov, *Differential and difference boundary value problem for loaded third-order pseudo-parabolic differential equations and difference methods for their numerical solution*, Comput. Math. Math. Phys., **57**:12 (2017), 1973–1993. Zbl 1393.65007

[19] M.Kh. Beshtokov, V.Z. Kanchukoyev, F.A. Erzhibova, *A boundary value problem for a degenerate moisture transfer equation with a condition of the third kind*, Vladikavkaz. Mat. Zh., **19**:4 (2017), 13-–26. Zbl 1452.65150

[20] M.Kh. Beshtokov, *Riemann method for solving non-local boundary value problems for the third order pseudoparabolic equations*, Vestn. Samar. Gos. Tekh. Univ., Ser. Fiz.-Mat. Nauki, **2013**:4(33) (2013), 15–24. Zbl 1413.35270

[21] M.Kh. Beshtokov, *Finite difference method for a nonlocal boundary value problem for a third-order pseudoparabolic equation*, Differ. Equ., **49**:9 (2013), 1134–1141. Zbl 1282.65099

[22] M.KH. Beshtokov, *A numerical method for solving one nonlocal boundary value problem for a third-order hyperbolic equation*, Comput. Math. Math. Phys., **54**:9 (2014), 1441–1458. Zbl 1327.65157

[23] A.K. Bazzaev, M.Kh. Shkhanukov-Lafishev, Locally one-dimensional scheme for fractional diffusion equations with Robin boundary conditions, Comput. Math. Math. Phys., **50**:7 (2010), 1141–1149. Zbl 1224.65198

[24] A.K. Bazzaev, *The third boundary problem for general parabolic differential equation of fractional order in multidimensional field*, Vestn. VGU, Ser. Fiz. Mat., **2010**:2 (2010), 5–14. Zbl 1325.35256

[25] A.K. Bazzaev, A.V. Mambetova, M.Kh. Shkhanukov-Lafishev, *Locally one-dimensional scheme for the heat equation of fractional order with concentrated heat capacity*, Zh. Vychisl. Mat. Mat. Fiz., **52**:9 (2012), 1656–1665. Zbl 1274.35154

[26] A.K. Bazzaev, M.Kh. Shkhanukov-Lafishev, *Locally one-dimensional scheme for fractional diffusion equations with Robin boundary conditions*, Comput. Math. Math. Phys., **50**:7 (2010), 1141-–1149. Zbl 1224.65198

[27] D.K. Faddeev, V.N. Faddeeva, *Numerical methods of linear algebra*, Fizmatgiz, Moscow, 1960. Zbl 0094.11005

[28] O.A. Ladyzhenskaya, *The boundary value problems of mathematical physics*, Springer-Verlag, New York etc., 1985. Zbl 0588.35003

[29] A.A. Samarskii *The Theory of difference schemes*, Marcel Dekker, New York, 2001. Zbl 0971.65076

[30] A.A. Samarskii, A.V. Gulin, *Stability of finite difference schemes*, Nauka, Moscow, 1973. Zbl 0304.65003

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