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BOUNDARY VALUE PROBLEMS WITH CONJUGATION CONDITIONS FOR QUASI-PARABOLIC EQUATIONS OF THE THIRD ORDER WITH A DISCONTINUOUS SIGN–VARIABLE COEFFICIENT

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Abstract. The aim of this work is to study the solvability in Sobolev spaces of boundary value problems for third order differential equations with a discontinuous sign–variable coefficient at the highest derivative with respect to the time variable. Since the equation has a discontinuous leading coefficient, in addition to setting the boundary conditions it is also necessary to set some conjugation conditions. For the problems under study, existence and uniqueness theorems are proved for the class of regular solutions, i.e., for the solutions that have all Sobolev weak derivatives up to the third order in time variable and up to the second order in spatial variables.

Keywords: third order quasi-parabolic equations, discontinuous sign-variable coefficient, boundary value problems, conjugation conditions, regular solutions, existence, uniqueness.

1. Introduction

In this work, we study the solvability of boundary value problems with conjugation conditions for the differential equation

$$\varphi(t)D_t^3 u + \Delta u = f(x,t) \qquad (*)$$

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($D_t^k = \frac{\partial^k}{\partial t^k}$, Δ is the Laplace operator by spacial variables) with a discontinuous sign–variable coefficient $\varphi(t)$.

Differential equations $(*)$, in the case when a continuous coefficient $\varphi(t)$ does not turn into zero, can be called quasi-parabolic equations by analogy to quasi-elliptic ones, since they represent a special case of equations

$$\varphi(t)D_t^{2p+1}u + \Delta u + \Phi u = f(x,t) \qquad (**)$$

with a differential operator Φ of first order by spatial variables and order $2p$ by the chosen variable t, which are parabolic given $p = 0$.

Boundary value problems for quasi-parabolic equations $(**)$ for the case of a continuous sign–variable coefficient are well studied (see works [1]–[5]).

If in the quasi-parabolic equation $(**)$ the coefficient $\varphi(t)$ is continuous but can turn into zero and vary in sign, then the statements of the boundary value problems and the conditions of their solvability for such equations can be found in works [6]–[8].

Boundary value problems for quasi-parabolic equations $(*)$ with a discontinuous coefficient and with conjugation conditions on the line of discontinuity were studied in articles [9]–[11].

This work can be regarded a sequel of the article [9]. In particular, in this paper, the solvability is studied of boundary value problems for the equations more general compared to the ones in [9], and the conjugation conditions on the discontinuity line of the function $\varphi(t)$ along with boundary conditions will differ from the conjugation conditions and boundary conditions from [9].

Everywhere in the work, we will refer as a regular solution of one equation or another to a function that has all square summable by a corresponding area S. L. Sobolev generalized derivatives entering the equation under study [12]–[14].

We should specify that the goal of this work is to prove the existence and uniqueness of solutions of one problem or another for the equation $(*)$, regular in areas that do not contain the discontinuity line.

2. Statement of the problems

Let Ω be a bounded domain in the space $\mathbb{R}^n$ with a smooth (for simplicity, infinitely differentiable) boundary Γ, T is a given positive number, Q_1 and Q_2 are cylinders $\Omega \times (-T, 0)$ and $\Omega \times (0, T)$ respectively. Further, let $\varphi(t)$, $\psi(x,t)$ and $f(x,t)$ be given functions, defined when $t \in [-T, T]$, $x \in \overline{\Omega}$, $\alpha = (\alpha_i)$, $\beta = (\beta_i)$, $i = \overline{1,6}$ be given vectors with real coordinates, L be a differential operator whose action on the given function $v(x,t)$ is defined by the equality

$$Lv = \varphi(t)D_t^3 v + \Delta v.$$

The goal of the work is to define the correctness conditions for the problem of finding the function $u(x,t)$, representing in the cylinders Q_1 and Q_2 the solution of the equation

$$Lu = f(x,t) \qquad (1)$$

and such that for this function, the conjugation conditions

$$\alpha_1 u(x,-0) + \alpha_2 u(x,+0) + \alpha_3 u_t(x,-0) + \alpha_4 u_t(x,+0) +$$
$$+\alpha_5 u_{tt}(x,-0) + \alpha_6 u_{tt}(x,+0) = 0, \quad x \in \Omega, \qquad (2)$$
$$\beta_1 u(x,-0) + \beta_2 u(x,+0) + \beta_3 u_t(x,-0) + \beta_4 u_t(x,+0) +$$

$$+\beta_5 u_{tt}(x,-0)+\beta_6 u_{tt}(x,+0)=0, \quad x\in\Omega, \tag{3}$$

the homogenous condition of the first boundary value problem on the lateral boundaries of the cylinders Q_1 and Q_2

$$u(x,t)|_{\Gamma\times(-T,0)}=0, \quad u(x,t)|_{\Gamma\times(0,T)}=0, \tag{4}$$

and also necessary boundary conditions on the bases $t=-T$ and $t=T$ are met.

In the paper [9] that has been mentioned above, the case $\varphi(t)=\text{sign}(-t)$, $\alpha_i=0$, $i=\overline{3,6}$, $\beta_i=0$, $i=\overline{1,4}$ was studied. In this work, we will consider the case of an arbitrary alternating function $\varphi(t)$ which has a discontinuity of the first kind given $t=0$, and also the cases of conjugation coefficients α_i and β_i distinct from the ones that have been mentioned above.

In what follows, we will assume that the function $\varphi(t)$ has a representation

$$\varphi(t)=\begin{cases}\varphi_1(t) & t\in[-T,0],\\ \varphi_2(t) & t\in(0,T].\end{cases}$$

and moreover, the function $\varphi_2(t)$ has a finite limit $\lim\limits_{t\to+0}\varphi_2(t)$. We denote by $\widetilde{\varphi}_2(t)$ the function

$$\widetilde{\varphi}_2(t)=\begin{cases}\varphi_2(t), & t\in(0,T],\\ \lim\limits_{t\to+0}\varphi_2(t), & t=0.\end{cases}$$

Suppose that the following condition is met:

$$\varphi_1(t)\in C^1([-T,0]), \quad \widetilde{\varphi}_2(t)\in C^1([0,T]), \quad \varphi_1(t)>0 \quad \textit{when} \quad t\in[-T,0],$$
$$\widetilde{\varphi}_2(t)<0 \quad \textit{when} \quad t\in[0,T]. \tag{5}$$

We will show the way to define the exact form of conjugation conditions (2) and (3), necessary for the correctness of the problem. With the help of the given functions $a(t)$, $b(t)$, $c(t)$, $A(t)$, $B(t)$, and $C(t)$, we will define the operators M_1 and M_2:

$$M_1 v=a(t)v_{tt}+b(t)v_t+c(t)v,$$
$$M_2 v=A(t)v_{tt}+B(t)v_t+C(t)v.$$

Assuming that for the function $u(x,t)$ the inclusions $u(x,t)\in W_2^{2,3}(Q_1)$, $u(x,t)\in W_2^{2,3}(Q_2)$ are true, and that it is the solution of equation (1) in the cylinders Q_1 and Q_2, and also that it satisfies the condition (4), consider the equality

$$\int\limits_{Q_1} LuM_1u\,dx\,dt+\int\limits_{Q_2} LuM_2u\,dx\,dt=$$
$$=\int\limits_{Q_1} fM_1u\,dx\,dt+\int\limits_{Q_2} fM_2u\,dx\,dt. \tag{6}$$

Integrating over parts, it is easy to transform this equality into quadratic form. The condition that this form should be positive semidefinite (with some additional conditions) will allow, first, to define the admissible boundary conditions for $t=-T$ and $t=T$, and second, to establish in a standard way the uniqueness of regular solutions of some or another conjugation problems for equations (1). We will provide two examples.

1. The first example corresponds to the operators M_1 and M_2 of the simplest form:

$$M_1u=\frac{c}{\varphi_1(t)}u,\ M_2v=\frac{C}{\varphi_2(t)}u,\ c=const,\ C=const.$$

In this case, equality (6) takes the form

$$-c\int\limits_{Q_1}\frac{1}{\varphi_1(t)}\left(\sum_{i=1}^{n}u_{x_i}^2\right)dx\,dt-C\int\limits_{Q_2}\frac{1}{\varphi_2(t)}\left(\sum_{i=1}^{n}u_{x_i}^2\right)dx\,dt-\frac{c}{2}\int\limits_{\Omega}u_t^2(x,-0)\,dx+$$
$$+c\int\limits_{\Omega}u_{tt}(x,-0)u(x,-0)\,dx+\frac{C}{2}\int\limits_{\Omega}u_t^2(x,+0)\,dx-C\int\limits_{\Omega}u_{tt}(x,+0)u(x,+0)\,dx-$$
$$-c\int\limits_{\Omega}u_{tt}(x,-T)u(x,-T)\,dx+C\int\limits_{\Omega}u_{tt}(x,T)u(x,T)\,dx+\frac{c}{2}\int\limits_{\Omega}u_t^2(x,-T)\,dx-$$
$$-\frac{C}{2}\int\limits_{\Omega}u_t^2(x,T)\,dx=c\int\limits_{Q_1}\frac{fu}{\varphi_1(t)}\,dx\,dt+C\int\limits_{Q_2}\frac{fu}{\varphi_2(t)}\,dx\,dt. \tag{7}$$

Now, if the function $f(x,t)$ identically equals zero, then the uniqueness of the solutions will hold if the following conditions are met:
1_1) $c<0$, $C>0$;
2_1) $u(x,-T)=u(x,T)=u_t(x,-T)=u_t(x,T)=0$ given $x\in\Omega$;
3_1) $\int\limits_{\Omega}\Big[cu_{tt}(x,-0)u(x,-0)-Cu_{tt}(x,+0)u(x,+0)-\frac{c}{2}u_t^2(x,-0)+$
$+\frac{C}{2}u_t^2(x,+0)\Big]\,dx\geq 0.$
Here, condition 2_1) defines the boundary values for the function $u(x,t)$ on the lower and upper bases of the cylinders Q_1 and Q_2 (generally speaking, these conditions can be flexible, for example, they can define values of the second derivative by t given $t=-T$ or (and) $t=T$). Condition 3_1) defines admissible conjugation conditions (exact statements will be provided below).

2. The second example corresponds to the operators M_1 and M_2 of the following form:

$$M_1v=\frac{b}{\varphi_1(t)}v_t,\ M_2v=\frac{B}{\varphi_2(t)}v_t,\ b=const,\ B=const.$$

In this case, equality (6) takes the form

$$-b\int\limits_{Q_1}u_{tt}^2\,dx\,dt+\frac{b}{2}\int\limits_{Q_1}\left(\frac{1}{\varphi_1(t)}\right)_t\left(\sum_{i=1}^{n}u_{x_i}^2\right)dx\,dt-B\int\limits_{Q_2}u_{tt}^2\,dx\,dt+$$
$$+\frac{B}{2}\int\limits_{Q_2}\left(\frac{1}{\varphi_2(t)}\right)_t\left(\sum_{i=1}^{n}u_{x_i}^2\right)dx\,dt-\frac{b}{2\varphi_1(-0)}\sum_{i=1}^{n}\int\limits_{\Omega}u_{x_i}^2(x,-0)\,dx+$$
$$+\frac{B}{2\varphi_2(+0)}\sum_{i=1}^{n}\int\limits_{\Omega}u_{x_i}^2(x,+0)\,dx+b\int\limits_{\Omega}u_{tt}(x,-0)u_t(x,-0)\,dx-$$
$$-B\int\limits_{\Omega}u_{tt}(x,+0)u_t(x,+0)\,dx-b\int\limits_{\Omega}u_{tt}(x,-T)u_t(x,-T)\,dx+$$
$$+B\int\limits_{\Omega}u_{tt}(x,T)u_t(x,T)\,dx+\frac{b}{2\varphi_1(-T)}\sum_{i=1}^{n}\int\limits_{\Omega}u_{x_i}^2(x,-T)\,dx-$$

$$-\frac{B}{2\varphi_2(T)}\sum_{i=1}^{n}\int_{\Omega} u_{x_i}^2(x,T)\,dx = b\int_{Q_1}\frac{fu_t}{\varphi_1(t)}\,dx\,dt + B\int_{Q_2}\frac{fu_t}{\varphi_2(t)}\,dx\,dt. \tag{8}$$

Again, in the case when $f(x,t)\equiv 0$, we can define the conditions that provide the uniqueness of solutions:

1_2) $b<0$, $B<0$, $\varphi_1'(t)\ge 0$ with $t\le 0$, $\widetilde{\varphi}_2'(t)\ge 0$ with $t\ge 0$;

2_2) $u(x,-T)=u(x,T)=u_t(x,-T)=u_t(x,T)=0$ with $x\in\Omega$;

3_2) $\int_{\Omega}\Big[bu_{tt}(x,-0)u_t(x,-0) - Bu_{tt}(x,+0)u_t(x,+0) - \frac{b}{2\varphi_1(-0)}\sum_{i=1}^{n} u_{x_i}^2(x,-0) +$
$+\frac{B}{2\varphi_2(+0)}\sum_{i=1}^{n} u_{x_i}^2(x,+0)\Big]\,dx \ge 0.$

Condition 2_2) defines again the boundary data on the lower and upper bases of the cylinders Q_1 and Q_2 (which is also flexible), while condition 3_2) defines the necessary conjugation conditions again.

The choice of the operators M_1 and M_2 distinct from the one in the examples 1 and 2 can lead to different conditions compared to the ones mentioned above given $t=-T$ and $t=T$, and also to different conjugation conditions.

In this paper, we will study some model conjugation problems for equation (1), and also some strengthenings and generalisations of the obtained results will be described.

We will now provide exact statements of the studied problems.

Conjugation problem I: *find a function $u(x,t)$ that represents in the cylinders Q_1 and Q_2 a solution of equation (1), such that for this function the boundary condition (4), the condition*

$$u(x,-T)=u(x,T)=u_t(x,-T)=u_t(x,T)=0, \quad x\in\Omega, \tag{9}$$

and also the following conjugation conditions are satisfied

$$u(x,-0)=a_1u(x,+0)+b_1u_{tt}(x,-0),$$
$$u_{tt}(x,+0)=a_2u(x,+0)+b_2u_{tt}(x,-0), \quad x\in\Omega. \tag{10}$$

Conjugation problem II: *find a function $u(x,t)$ that represents in the cylinders Q_1 and Q_2 a solution of equation (1), such that for this function the conditions (4) and (9), and also the following conjugation conditions are satisfied*

$$u_t(x,-0)=a_1u_t(x,+0)+b_1u_{tt}(x,-0),$$
$$u_{tt}(x,+0)=a_2u_t(x,+0)+b_2u_{tt}(x,-0), \quad x\in\Omega. \tag{11}$$

Conjugation problem III: *find a function $u(x,t)$ that represents in the cylinders Q_1 and Q_2 the solution of equation (1), such that for this functions the boundary conditions (4), the conditions*

$$u(x,-T)=u(x,T)=u_{tt}(x,-T)=u_{tt}(x,T)=0, \quad x\in\Omega, \tag{12}$$

and also the following conjugation conditions are satisfied

$$u(x,-0)=a_1u(x,+0)+b_1u_t(x,-0),$$
$$u_t(x,+0)=a_2u(x,+0)+b_2u_t(x,-0), \quad x\in\Omega. \tag{13}$$

Conjugation problem IV: *find a function $u(x,t)$ that represents in the cylinders Q_1 and Q_2 a solution of equation (1), such that for that function the boundary conditions (4) and (9), as well as conjugation conditions (13) are satisfied.*

3. Uniqueness of solutions

We will now define the exact conditions, which, if satisfied, will provide the uniqueness of solution of the conjugation problems I–IV.

We should specify that throughout what follows we will refer as a regular solution of one problem or another for equation (1) to a function $u(x,t)$, belonging to the spaces $W_2^{2,3}(Q_1)$ and $W_2^{2,3}(Q_2)$.

Theorem 1. *Suppose that condition (5) is satisfied, as well as the condition*

$$a_2 \le 0, \quad b_1 \le 0, \quad a_2 b_1 - a_1 b_2 \ge 0. \tag{14}$$

Then conjugation problem I cannot have more than one regular solution.

Proof. Let $f(x,t) \equiv 0$ hold in equation (1). First, consider the case $a_2 b_1 \neq 0$. In equality (7), we put $c = -1$. We will show that if condition (14) is satisfied, there exists a positive number C such that the form in the left-hand side of equality (7) is positive semidefinite.

Conjugation conditions (10) yield the equalities

$$\int_\Omega \{-u_{tt}(x,-0)u(x,-0) - Cu_{tt}(x,+0)u(x,+0)\}\,dx =$$

$$= \int_\Omega \{-u_{tt}(x,-0)[a_1 u(x,+0) + b_1 u_{tt}(x,-0)] -$$

$$-Cu(x,+0)[a_2 u(x,+0) + b_2 u_{tt}(x,-0)]\}\,dx =$$

$$= \int_\Omega \{-b_1 u_{tt}^2(x,-0) - (Cb_2 + a_1)u_{tt}(x,-0)u(x,-0) - Ca_2 u^2(x,+0)\}\,dx. \tag{15}$$

If $b_2 = 0$, then for the required number C we can take an arbitrary number from the interval $\left(\frac{a_1^2}{4a_2b_1}, +\infty\right)$. Suppose now that $b_2 \neq 0$. We put $c_1 = Ca_2$, $c_2 = \frac{c_1 b_2}{a_2} + a_1$. We will define a quadratic form $F_1(\xi,\eta)$:

$$F_1(\xi,\eta) = -b_1\xi^2 - c_1\eta^2 - c_2\xi\eta.$$

This form is positive semidefinite in the case when $b_1 < 0$ (which is the case) and when the following equality holds:

$$4b_1c_1 - c_2^2 \ge 0.$$

In other words, there exists a number c_1, such that the following quadratic inequality holds:

$$b_2^2c_1^2 + (2a_1a_2b_2 - 4b_1a_2^2)c_1 + a_1^2a_2^2 \le 0. \tag{16}$$

Due to the final inequality of condition (14), a corresponding quadratic equation has real roots z_1 and z_2, moreover, $z_1 < z_2$ in the case when $a_2b_1 - a_1b_2 > 0$, $z_1 = z_2$ in the case when $a_2b_1 - a_1b_2 = 0$, and in this case, the root z_1 is negative. Assume that $c_1 = z_1$, $C = \frac{c_1}{a_2}$. The number C is the required positive number for which the integral (15) has a non-negative value.

Given the mentioned choice for the number C, it follows from equality (7) that all special derivatives $u_{x_i}(x,t)$ are functions identically zero in the cylinders Q_1 and Q_2. Due to conditions (9), the function $u(x,t)$ also identically equals zero in the cylinders Q_1 and Q_2. That means that for conjugation problem I the condition of uniqueness of regular solution is met.

Now suppose that $a_2b_1 = 0$ is true. If $b_2 \neq 0$, $a_1 \neq 0$, then we put $C = -\frac{a_1}{b_2}$. Condition (14) means that this number is positive. But then again for conjugation problem I the condition of uniqueness of regular solution is met.

Finally, if the equalities $a_2b_1 = 0$, $a_1b_2 = 0$ hold, then conjugation problem I splits into two independent problems, whose uniqueness of solutions is obvious due to condition (14).

The theorem is completely proved. □

Theorem 2. *Let condition (5) hold along with the conditions*

$$\varphi_1'(t) \geq 0 \quad \textit{when} \quad t \in [-T, 0], \quad \widetilde{\varphi}_2'(t) \geq 0 \quad \textit{when} \quad t \in [0, T]; \tag{17}$$

$$a_2 \geq 0, \quad b_1 \leq 0, \quad a_2b_1 - a_1b_2 \leq 0. \tag{18}$$

Then conjugation problem II cannot have more than one regular solution.

Proof. Let $f(x,t)$ be a function identically zero in the cylinders Q_1 and Q_2, and suppose that the condition $a_2b_1 \neq 0$ is satisfied. In equality (8), we put $b = -1$. We will show that if conjugation conditions (11) are satisfied, there exists a negative number B such that the following inequality holds:

$$\int\limits_{\Omega} \{-u_{tt}(x,-0)u_t(x,-0) - Bu_{tt}(x,+0)u_t(x,+0)\}\, dx \geq 0. \tag{19}$$

In the case when $b_2 = 0$, we can take for the required number B an arbitrary number from the interval $\left(-\infty, \frac{a_1^2}{4a_2b_1}\right)$. In the case $b_2 \neq 0$, we put $c_1 = Ba_2$, $c_2 = \frac{c_1b_2}{a_2} + a_1$. The required number B exists, if there is a negative number c_1 for which inequality (16) holds. A quadratic equation that corresponds to inequality (17) (due to condition (17)) also has real roots, moreover, one of the roots (the least one) is negative. This root z_1 is the one to define the numbers c_1 and B: $c_1 = z_1$, $B = \frac{c_1}{a_2}$.

For the obtained number B, inequality (19) holds. Due to this inequality, equality (8), and conditions (4) and (9) it follows that in the case when $f(x,t) \equiv 0$, the function $u(x,t)$ is identically zero in the cylinders Q_1 and Q_2. This implies the uniqueness of regular solutions of conjugation problem II.

In the case $a_2b_1 = 0$, the uniqueness of solutions is established in an obvious way (see the final lines of the proof of Theorem 1).

The theorem is completely proved. □

Theorem 3. *Suppose that condition* (5) *is fulfilled and*

$$\varphi_1'(t) \leq 0 \quad \textit{when} \quad t \in [-T, 0], \quad \widetilde{\varphi}_2'(t) \leq 0 \quad \textit{when} \quad t \in [0, T]; \tag{20}$$

$$a_2 \geq 0, \quad b_1 \leq 0, \quad a_1b_2 \geq 0. \tag{21}$$

Then conjugation problem III cannot have more than one regular solution.

Proof. Suppose that in equation (1) we have $f(x,t) \equiv 0$. Assume that $M_1v = v_{tt}$, $M_2v = Av_{tt}$, $A = const > 0$. In this case, taking into account boundary conditions (12) and conjugation conditions (13), equality (6) takes the form:

$$-\frac{1}{2}\int\limits_{Q_1} \varphi_1'(t)u_{tt}^2\, dx\, dt - \frac{A}{2}\int\limits_{Q_1} \varphi_2'(t)u_{tt}^2\, dx\, dt +$$

$$+\sum_{i=1}^{n}\int\limits_{Q_1} u_{x_it}^2\, dx\, dt + A\sum_{i=1}^{n}\int\limits_{Q_2} u_{x_it}^2\, dx\, dt +$$

$$+\frac{1}{2}\int\limits_{\Omega}\varphi_1(-0)u_{tt}^2(x,-0)\,dx-\frac{A}{2}\int\limits_{\Omega}\varphi_2(+0)u_{tt}^2(x,+0)\,dx+$$

$$+\sum_{i=1}^{n}\int\limits_{\Omega}(Aa_2u_{x_i}^2(x,+0)+(Ab_2-a_1)u_{x_i}(x,+0)u_{tx_i}(x,-0)-b_1u_{tx_i}^2(x,-0))\,dx=0. \tag{22}$$

We will show that if condition (21) is fulfilled, there exists a positive number A such that the last term of the left-hand side of equality (22) has a non-negative value.

Consider the quadratic form $F(\xi,\eta)$

$$F(\xi,\eta)=Aa_2\xi^2+(Ab_2-a_1)\xi\eta-b_1\eta^2.$$

If $b_2=0$, $a_2b_1<0$ is true, then we can take as the required number A any number from the interval $(0,-\frac{a_1^2}{4a_2b_1}]$. If $b_2\neq 0$, $a_2b_1<0$ holds, then the form $F(\xi,\eta)$ is positive semidefinite, if there exists a positive number A for which the following inequality holds:

$$b_2^2A^2+(4a_2b_1-2a_1b_2)A+a_1^2\leq 0.$$

We can take as the required number A any number from the interval $(0,z_2)$, where z_2 is a positive root of the equation

$$b_2^2z^2+(4a_2b_1-2a_1b_2)z+a_1^2=0$$

(such root exists due to the conditions of the theorem).

From the mentioned reasonings, it follows that in the case $a_2b_1<0$, if the conditions of the theorem are satisfied and if the number A is chosen as mentioned above, the functions $u_{x_it}(x,t)$, $i=1,2...n$, are identically zero in Q_1 and Q_2. But then the function $u_t(x,t)$ also identically equals zero in the cylinders Q_1 and Q_2, and further, the function $u(x,t)$ is identically zero as well. That means that for conjugation problem III the condition of uniqueness of regular solution is met.

Now, if $a_2b_1=0$, then conjugation problem III splits into two independent problems in the cylinders Q_1 and Q_2, whose uniqueness of solutions is obvious.

These arguments yield that Theorem 3 is true.

The theorem is completely proved. □

Conjugation problem IV will be studied only for a piecewise constant function $\varphi(t)$:

$$\varphi(t)=\begin{cases}\varphi_1, & t\in[-T,0],\\ \varphi_2, & t\in(0,T].\end{cases}$$

Theorem 4. *Let $\varphi(t)$ be a piecewise constant function, and suppose that the following conditions are satisfied:*

$$\varphi_1>0,\quad \varphi_2<0; \tag{23}$$

$$a_1b_2\geq 0,\quad b_1\leq 0,\quad a_2\geq 0. \tag{24}$$

Then conjugation problem IV cannot have more than one regular solution.

Proof. First, consider the case $a_1b_2\neq 0$. Let M_1 and M_2 be operators such that the following holds:

$$a(t)=a(T+t),\quad a=const>0,\quad b(t)=-a,\quad c(t)\equiv 0,$$

$$A(t) = \frac{aa_1(T-t)}{b_2}, \quad B(t) = \frac{aa_1}{b_2}, \quad C(t) \equiv 0.$$

In equality (6), we put $f(x,t) \equiv 0$. After simple transformations, using conditions (4), (9), and (12), we obtain the equality

$$\frac{a\varphi_1}{2} \int\limits_{Q_1} u_{tt}^2\, dx\, dt + a \sum_{i=1}^{n} \int\limits_{Q_1} (T+t) u_{x_i t}^2\, dx\, dt - \frac{aa_1\varphi_2}{2b_2} \int\limits_{Q_2} u_{tt}^2\, dx\, dt +$$

$$+ \frac{aa_1}{b_2} \sum_{i=1}^{n} \int\limits_{Q_2} (T-t) u_{x_i t}^2\, dx\, dt + a \sum_{i=1}^{n} \int\limits_{\Omega} u_{x_i}^2(x,-0)\, dx +$$

$$+ \frac{a\varphi_1 T}{2} \int\limits_{\Omega} u_{tt}^2(x,-0)\, dx - ab_1 T \sum_{i=1}^{n} \int\limits_{\Omega} u_{x_i t}^2(x,-0)\, dx -$$

$$-a\varphi_1 \int\limits_{\Omega} u_{tt}(x,-0) u_t(x,-0)\, dx + \frac{aa_1(1+a_2T)}{b_2} \sum_{i=1}^{n} \int\limits_{\Omega} u_{x_i}^2(x,+0)\, dx -$$

$$- \frac{aa_1\varphi_2 T}{2b_2} \int\limits_{\Omega} u_{tt}^2(x,+0)\, dx - \frac{aa_1\varphi_2}{b_2} \int\limits_{\Omega} u_{tt}(x,+0) u_t(x,+0)\, dx = 0. \tag{25}$$

The following inequalities hold:

$$\left| \int\limits_{\Omega} u_{tt}(x,-0) u_t(x,-0)\, dx \right| \le$$

$$\le \left(\int\limits_{\Omega} u_{tt}^2(x,-0)\, dx \right)^{1/2} \left(\int\limits_{\Omega} u_t^2(x,-0)\, dx \right)^{1/2} \le$$

$$\le \left(T \int\limits_{\Omega} u_{tt}^2(x,-0)\, dx \right)^{1/2} \left(\int\limits_{Q_1} u_{tt}^2\, dx\, dt \right)^{1/2},$$

$$\left| \int\limits_{\Omega} u_{tt}(x,+0) u_t(x,+0)\, dx \right| \le$$

$$\le \left(\int\limits_{\Omega} u_{tt}^2(x,+0)\, dx \right)^{1/2} \left(\int\limits_{\Omega} u_t^2(x,+0)\, dx \right)^{1/2} \le$$

$$\le \left(T \int\limits_{\Omega} u_{tt}^2(x,+0)\, dx \right)^{1/2} \left(\int\limits_{Q_2} u_{tt}^2\, dx\, dt \right)^{1/2},$$

Using these inequalities, it is easy to show that (22) yields the inequality

$$\frac{a\varphi_1}{2} \sum_{i=1}^{n} \int\limits_{Q_1} (T+t) u_{x_i t}^2\, dx\, dt + \frac{aa_1}{b_2} \sum_{i=1}^{n} \int\limits_{Q_2} (T-t) u_{x_i t}^2\, dx\, dt + a \sum_{i=1}^{n} \int\limits_{\Omega} u_{x_i}^2(x,-0)\, dx +$$

$$+\frac{aa_1(1+a_2T)}{b_2}\sum_{i=1}^{n}\int\limits_{\Omega} u_{x_i}^2(x,+0)\,dx - ab_1T\sum_{i=1}^{n}\int\limits_{\Omega} u_{x_it}^2(x,-0)\,dx \le 0.$$

From this inequality, it follows that the functions $u(x,-0)$, $b_1u_t(x,-0)$ and $u(x,+0)$ are identically zero in Ω. But then conjugation problem IV in the cylinder Q_1 represents a problem whose uniqueness of solutions is obvious. Further, we obtain a problem in the cylinder Q_2 with zero boundary data, for which the uniqueness of solutions is true as well. From here, it follows that $u(x,t)\equiv 0$ everywhere.

In the case $a_1b_2 = 0$, conjugation problem IV splits right away, hence, the uniqueness of regular solutions is obvious for it.

The theorem is completely proved. □

Now, we turn to the proof of the theorems on solvability of conjugation problems I–IV.

Let λ be a number from the segment $[0,1]$. We put

$$d_1(\lambda) = \left[1-\frac{2\lambda b_1}{T^2}\right]\cdot\left[1-\frac{\lambda T^2 a_2}{2}\right] - \lambda^2 a_1 b_2,$$

$$p_{11}(t,\lambda) = -\left[\frac{t^2}{T^2}+\frac{2t}{T}+1\right]\cdot\frac{\lambda}{d_1(\lambda)}\left[b_1+\frac{\lambda T^2(a_1b_2-a_2b_1)}{2}\right],$$

$$p_{21}(t,\lambda) = -\left[\frac{t^2}{T^2}+\frac{2t}{T}+1\right]\cdot\frac{\lambda a_1}{d_1(\lambda)},$$

$$q_{11}(t,\lambda) = -\left[\frac{t^2}{2}-tT+\frac{T^2}{2}\right]\cdot\frac{\lambda b_2}{d_1(\lambda)},$$

$$q_{21}(t,\lambda) = -\left[\frac{t^2}{2}-tT+\frac{T^2}{2}\right]\cdot\frac{\lambda}{d_1(\lambda)}\left[a_2+\frac{2\lambda(a_1b_2-a_2b_1)}{T^2}\right].$$

Theorem 5. *Suppose that conditions* (5) *and* (14) *are fulfilled, along with the condition $a_1b_2\neq 0$. Then for every function $f(x,t)$ from the space $L_2(Q)$, conjugation problem I has a solution $u(x,t)$ such that $u(x,t)\in W_2^{2,3}(Q_1)$, $u(x,t)\in W_2^{2,3}(Q_2)$.*

Proof. We denote by $f_1(x,t)$ the restriction of the function $f(x,t)$ to the cylinder Q_1, and by $f_2(x,t)$ the restriction of the function $f(x,t)$ to the cylinder Q_2. Let ε be a positive number, λ be a number from the interval $[0,1]$. Consider the boundary value problem: *find functions $w(x,t)$ and $z(x,t)$, defined in the cylinders Q_1 and Q_2 respectively, which are the solutions of the equations*

$$-\varepsilon\varphi_1(t)\Delta w_{ttt}+\varphi_1(t)w_{ttt}+\Delta w =$$
$$= f_1(x,t)+p_{11}(t,\lambda)\Delta w_{tt}(x,-0)+p_{21}(t,\lambda)\Delta z(x,+0),\quad (x,t)\in Q_1 \qquad (26)$$

$$-\varepsilon\varphi_2(t)\Delta z_{ttt}+\varphi_2(t)z_{ttt}+\Delta z =$$
$$f_2(x,t)+q_{11}(t,\lambda)\Delta w_{tt}(x,-0)+q_{21}(t,\lambda)\Delta z(x,+0),\quad (x,t)\in Q_2, \qquad (27)$$

and such that the following conditions are satisfied:

$$w(x,t)|_{\Gamma\times(-T,0)} = 0, \qquad (28)$$

$$w(x,-T) = w_t(x,-T) = w(x,-0) = 0,\quad x\in\Omega, \qquad (29)$$

$$z(x,t)|_{\Gamma\times(0,T)} = 0, \qquad (30)$$

$$z(x,T) = z_t(x,T) = z_{tt}(x,+0) = 0,\quad x\in\Omega. \qquad (31)$$

We denote by $V(Q_i)$, $i=1,2$, the linear spaces $V(Q_i)=\{v(x,t): v(x,t)\in W_2^{2,3}(Q_i), \Delta v_{ttt}(x,t)\in L_2(Q_i)\}$. We normalize the given spaces with the help of the norms

$$\|v\|_{V(Q_i)}=\left(\|v\|^2_{W_2^{2,3}(Q_i)}+\int\limits_{Q_i}(\Delta V_{ttt})^2dxdt\right)^{\frac{1}{2}}.$$

It is obvious that the spaces $V(Q_i)$ with the above norms are Banach spaces.

We will show that for a fixed ε and when the functions $f_1(x,t)$ and $f_2(x,t)$ belong to the spaces $L_2(Q_1)$ and $L_2(Q_2)$, boundary value problems (26)–(31) for every λ from the segment $[0,1]$ have a solution $\{w(x,t),z(x,t)\}$, such that $w(x,t)\in V(Q_1)$, $z(x,t)\in V(Q_2)$.

First of all, note that given $\lambda=0$, problem (26)-(31) splits into two independent problems in the cylinders Q_1 and Q_2 respectively, and their solvability in the spaces $V(Q_1)$ and $V(Q_2)$ is obvious.

The functions $p_{11}(t,\lambda), p_{21}(t,\lambda), q_{11}(t,\lambda), q_{21}(t,\lambda)$ are continuous by λ. According to the theorem on of continuation in a parameter [, Ch. III, §14], boundary value problem (26)-(31) has a solution $\{w(x,t),z(x,t)\}$, such that $w(x,t)\in V(Q_1)$, $z(x,t)\in V(Q_2)$, if for all possible solutions of this problem, the a priori estimate

$$\|w\|_{V(Q_1)}+\|z\|_{V(Q_2)}\le R_0, \tag{32}$$

holds with a constant R_0 that does not depend on λ.
We will show that the required estimate really exists.

We define the function $u(x,t)$:

$$u(x,t)=\begin{cases} w(x,t)-p_{11}(t,\lambda)w_{tt}(x,-0)-p_{21}(t,\lambda)z(x,+0), & (x,t)\in Q_1,\\ z(x,t)-q_{11}(t,\lambda)w_{tt}(x,-0)-q_{21}(t,\lambda)z(x,+0), & (x,t)\in Q_2.\end{cases}$$

For this function in the cylinders Q_1 and Q_2, the equation

$$-\varepsilon\varphi_i(t)\Delta u_{ttt}(x,t)+\varphi_i(t)u_{ttt}(x,t)+\Delta u(x,t)=f_i(x,t),\quad i=1,2, \tag{33}$$

is true, and condition (2) is satisfied, and the following conditions are fulfilled:

$$u(x,-0)=\lambda[a_1u(x,+0)+b_1u_{tt}(x,-0)],$$
$$u_{tt}(x,+0)=\lambda[a_2u(x,+0)+b_2u_{tt}(x,-0)],\quad x\in\Omega. \tag{34}$$

Consider sequentially the equalities

$$\int\limits_{Q_1}[-\varepsilon\varphi_1(t)\Delta u_{ttt}+\varphi_1(t)u_{ttt}+\Delta u]\frac{u}{\varphi_1(t)}dxdt+$$
$$+C\int\limits_{Q_2}[-\varepsilon\varphi_2(t)\Delta u_{ttt}+\varphi_2(t)u_{ttt}+\Delta u]\frac{u}{\varphi_2(t)}dxdt=$$
$$=-\int\limits_{Q_1}\frac{f_1u}{\varphi_1(t)}dxdt+C\int\limits_{Q_2}\frac{f_2u}{\varphi_2(t)}dxdt,$$

$$\int\limits_{Q_1}[-\varepsilon\varphi_1(t)\Delta u_{ttt}+\varphi_1(t)u_{ttt}+\Delta u]\frac{\Delta u}{\varphi_1(t)}dxdt+$$
$$+C\int\limits_{Q_2}[-\varepsilon\varphi_2(t)\Delta u_{ttt}+\varphi_2(t)u_{ttt}+\Delta u]\frac{\Delta u}{\varphi_2(t)}dxdt=$$

$$= \int\limits_{Q_1} \frac{f_1 \Delta u}{\varphi_1(t)} dx dt + C \int\limits_{Q_2} \frac{f_2 \Delta u}{\varphi_2(t)} dx dt,$$

in which the number C is chosen in the same way as in the proof of Theorem I. Integrating over parts and using conditions (5) and (13), and also applying the Young's inequality, it is easy to obtain that for the function $u(x,t)$ the a priori estimate

$$\sum_{i=1}^{n} \int\limits_{Q_1} u_{x_i}^2 \, dx \, dt + \sum_{i=1}^{n} \int\limits_{Q_2} u_{x_i}^2 \, dx \, dt + \int\limits_{Q_1} (\Delta u)^2 \, dx \, dt + \int\limits_{Q_2} (\Delta u)^2 \, dx \, dt \leq$$

$$\leq K_1 \left(\int\limits_{Q_1} f_1^2 \, dx \, dt + \int\limits_{Q_2} f_2^2 \, dx \, dt \right), \tag{35}$$

holds, and its constant K_1 is only defined by the area Ω, and also by the functions $\varphi_1(t)$ and $\varphi_2(t)$.

With the help of inequality (35), it is easy to obtain the second a priori estimate for the function $u(x,t)$:

$$\varepsilon^2 \int\limits_{Q_1} (\Delta u_{ttt})^2 \, dx \, dt + \varepsilon^2 \int\limits_{Q_2} (\Delta u_{ttt})^2 \, dx \, dt + \int\limits_{Q_1} u_{ttt}^2 \, dx \, dt + \int\limits_{Q_2} u_{ttt}^2 \, dx \, dt$$

$$\leq K_2 \left(\int\limits_{Q_1} f_1^2 \, dx \, dt + \int\limits_{Q_2} f_2^2 \, dx \, dt \right), \tag{36}$$

whose constant K_2 is only defined by the area Ω, and by the functions $\varphi_1(t)$ and $\varphi_2(t)$.

The following inequalities hold:

$$w(x,t) = u(x,t) - \frac{(t+T)^2}{T^2} [b_1 u_{tt}(x,-0) + a_1 u(x,+0)]$$

$$z(x,t) = u(x,t) - \frac{(t+T)^2}{T^2} [b_2 u_{tt}(x,-0) + a_2 u(x,+0)].$$

From these equalities, estimates (35) and (36), and also from the elementary estimates of the traces of the function on the plane $t = 0$ in terms of derivatives by the variable t, it follows that for the functions $w(x,t)$ and $z(x,t)$ there exist a priori estimates similar to the estimates (35) and (36), but with constants K_1' and K_2', defined by the area Ω, numbers T, a_1, a_2, b_1, b_2. But then for the functions $w(x,t)$ and $z(x,t)$ for a fixed ε the required estimate is true.

As it was mentioned above, the existence of the estimate (32) means that the boundary value problem $(26) - (31)$ has a solution $\{w(x,t), z(x,t)\}$, such that $w(x,t) \in V(Q_1), z(x,t) \in V(Q_2)$, with every λ from the segment $[0,1]$, including $\lambda = 1$.

Let $\{\varepsilon_m\}_{m=1}^{\infty}$ be a sequence of positive numbers monotonously tending to zero, $w_m(x,t)$ and $z_m(x,t)$ be solutions of the problem $(26)-(31)$ given $\lambda = 1$ and $\varepsilon = \varepsilon_m$. For the function $u_m(x,t)$ generated by these functions, estimate (35) holds, along with estimate (36), in which $\varepsilon = \varepsilon_m$. Using the reflexive property of a Hilbert space, it is easy to obtain that there is a function $u(x,t)$, such that for some sequence $\{m_k\}$ of natural numbers, given $k \to \infty$, there exist convergences

$u_{m_k}(x,t) \to u(x,t)$ *weakly in the space* $V(Q_1)$,
$u_{m_k}(x,t) \to u(x,t)$ *weakly in the space* $V(Q_2)$,
$u_{m_k}(x,+0) \to u(x,+0)$ *weakly in the space* $L_2(\Omega)$,
$u_{m_k tt}(x,-0) \to u_{tt}(x,-0)$ *weakly in the space* $L_2(\Omega)$,
$\varepsilon_{m_x}\Delta u_{m_k ttt}(x,t) \to 0$ *weakly in the space* $L_2(Q_1)$,
$\varepsilon_{m_x}\Delta u_{m_k ttt}(x,t) \to 0$ *weakly in the space* $L_2(Q_2)$.

From these convergences, it follows that for a limit function $u(x,t)$ in the cylinders Q_1 and Q_2 equation (1) holds, along with boundary conditions (4) and (9), and also conjugation conditions (10). That fact that the function $u(x,t)$ belongs to the spaces $V(Q_1)$ and $V(Q_2)$ is obvious. Therefore, the obtained function $u(x,t)$ represents the required solution of conjugation problem I.

The theorem is proved. □

Now, we turn to studying the solvability of conjugation problem II.

We put

$$d_2(\lambda) = [1+\lambda T a_2]\left(1-\frac{\lambda b_1}{T}\right)+\lambda^2 a_1 b_2,$$

$$p_{12}(t,\lambda) = -\left[\frac{t^2}{2T}+t+\frac{T}{2}\right]\cdot\frac{\lambda b_1+\lambda^2 T(a_2b_1-a_1b_2)}{d_2(\lambda)},$$

$$p_{22}(t,\lambda) = -\left[\frac{t^2}{2T}+t+\frac{T}{2}\right]\cdot\frac{\lambda a_1}{d_2(\lambda)},$$

$$q_{12}(t,\lambda) = -\left[\frac{t^2}{2}-Tt+\frac{T^2}{2}\right]\cdot\frac{\lambda b_2}{d_2(\lambda)},$$

$$q_{22}(t,\lambda) = -\left[\frac{t^2}{2}-Tt+\frac{T^2}{2}\right]\cdot\frac{\lambda a_2+\frac{\lambda^2}{T}(a_1b_2-a_2b_1)}{d_2(\lambda)}.$$

Theorem 6. *Let conditions* (5), (17), *and* (18) *hold, along with condition* $a_1b_2 \neq 0$. *Then for every function* $f(x,t)$ *such that* $f_1(x,t) \in L_2(-T,0;W_2^2(\Omega)\cap \overset{\circ}{W}{}_2^1(\Omega))$, $f_2(x,t) \in L_2(0,T;W_2^2(\Omega)\cap \overset{\circ}{W}{}_2^1(\Omega))$, *conjugation problem II has a solution* $u(x,t)$, *such that* $u(x,t) \in L_2(0,T;W_2^2(\Omega)\cap \overset{\circ}{W}{}_2^1(\Omega))$, $u_{ttt}(x,t) \in L_2(0,T;W_2^2(\Omega)\cap \overset{\circ}{W}{}_2^1(\Omega))$.

Proof. Let ε be a positive number, λ be a number from the segment $[0,1]$. Consider a boundary value problem: *find functions* $w(x,t)$ *and* $z(x,t)$, *defined in the cylinders* Q_1 *and* Q_2 *respectively, which are a solution of the equations*

$$\begin{gathered}\varepsilon\varphi_1(t)\Delta^2 w_{ttt}+\varphi_1(t)w_{ttt}+\Delta w = \\ = f_1(x,t)+p_{12}(t,\lambda)\Delta^2 w_{tt}(x,-0)+ \\ +p_{22}(t,\lambda)\Delta^2 z_t(x,+0), \quad (x,t)\in Q_1\end{gathered} \tag{37}$$

$$\begin{gathered}\varepsilon\varphi_2(t)\Delta^2 z_{ttt}+\varphi_2(t)z_{ttt}+\Delta z = \\ f_2(x,t)+q_{12}(t,\lambda)\Delta^2 w_{tt}(x,-0)+q_{22}(t,\lambda)\Delta^2 z_t(x,+0), \quad (x,t)\in Q_2,\end{gathered} \tag{38}$$

and such that the following conditions hold:

$$w(x,t)|_{\Gamma\times(-T,0)} = \Delta w(x,t)|_{\Gamma\times(-T,0)} = 0, \tag{39}$$

$$w(x,-T) = w_t(x,-T) = w_t(x,-0) = 0, \quad x\in\Omega, \tag{40}$$

$$z(x,t)|_{\Gamma\times(0,T)} = \Delta z(x,t)|_{\Gamma\times(0,T)} = 0, \tag{41}$$

$$z(x,T) = z_t(x,T) = z_{tt}(x,+0) = 0, \quad x\in\Omega. \tag{42}$$

With the help of the functions $w(x,t)$ and $z(x,t)$ we define the function $u(x,t)$:

$$u(x,t) = \begin{cases} w(x,t) - p_{12}(t,\lambda)w_{tt}(x,-0) - p_{22}(t,\lambda)z_t(x,+0), & (x,t) \in Q_1, \\ z(x,t) - q_{12}(t,\lambda)w_{tt}(x,-0) - q_{22}(t,\lambda)z_t(x,+0), & (x,t) \in Q_2. \end{cases}$$

For the function $u(x,t)$ in the cylinders Q_1 and Q_2, the equations

$$\varepsilon\varphi_1(t)\Delta^2 u_{ttt}(x,t) + \varphi_1(t)u_{ttt}(x,t) + \Delta u(x,t) = f_1(x,t), \quad (x,t) \in Q_1, \tag{43}$$

$$\varepsilon\varphi_2(t)\Delta^2 u_{ttt}(x,t) + \varphi_2(t)u_{ttt}(x,t) + \Delta u(x,t) = f_2(x,t), \quad (x,t) \in Q_2, \tag{44}$$

and also the following conditions hold:

$$u(x,t)|_{\Gamma\times(-T,0)} = \Delta u(x,t)|_{\Gamma\times(-T,0)} = 0, \tag{45}$$

$$u(x,t)|_{\Gamma\times(0,T)} = \Delta u(x,t)|_{\Gamma\times(0,T)} = 0, \tag{46}$$

$$u_t(x,-0) = \lambda[a_1 u_t(x,+0) + b_1 u_{tt}(x,-0)], \quad x \in \Omega, \tag{47}$$

$$u_{tt}(x,+0) = \lambda[a_2 u_t(x,+0) + b_2 u_{tt}(x,-0)], \quad x \in \Omega. \tag{48}$$

Given $\lambda = 0$, the boundary value problem (37)–(42) splits into two independent problems for the functions $w(x,t)$ and $z(x,t)$ in the cylinders Q_1 and Q_2 respectively, and solvability of each of them in the classes of regular solutions is obvious. Further, it is easy to obtain the estimates in the space L_2 of all derivatives belonging to equations (37) and (38). The required estimates are first established for the function $u(x,t)$ with the help of sequential analysis of the equalities

$$-\int\limits_{Q_1} [\varepsilon\varphi_1(t)\Delta^2 u_{ttt} + \varphi_1(t)u_{ttt} + \Delta u - f_1]\frac{\Delta^k u}{\varphi_1(t)}\,dx\,dt -$$

$$-B\int\limits_{Q_2} [\varepsilon\varphi_2(t)\Delta^2 u_{ttt} + \varphi_2(t)u_{ttt} + \Delta u - f_2]\frac{\Delta^k u}{\varphi_2(t)}\,dx\,dt = 0, \tag{49}$$

in which $k = 0, 1, 2$, the number B is a number defined in the proof of Theorem 2, and further, for the functions $w(x,t)$ and $z(x,t)$. We should specify that during the first stage, the a priori estimates are established for the function $u(x,t)$ and thereby for the functions $w(x,t)$ and $z(x,t)$ for a fixed ε, and the constants on the right-hand sides of these estimates are defined by the norms of the functions $f_1(x,t)$ and $f_2(x,t)$ in the spaces $L_2(Q_1)$ and $L_2(Q_2)$, and also by the number ε and the functions $\varphi_1(t)$ and $\varphi_2(t)$. These estimates allow to apply the theorem on continuation in a parameter, and to obtain thereby the solvability of the boundary value problem (37)–(42) in the class of regular solutions for every λ from the segment $[0,1]$. On the second stage, the estimates uniform by the parameter ε are obtained. Here again we use the equalities (49), but at the same time, we integrate over parts in the terms with the functions $f_1(x,t)$ and $f_2(x,t)$.

The obtained estimates, uniform by ε, along with the reflexivity property of a Hilbert space allow to organize the process of passage to the limit, choosing first the sequence $\{\varepsilon_m\}_{m=1}^{\infty}$, that monotonously tends to zero, and further, choosing a functional subsequence $\{u_{m_k}\}_{k=1}^{\infty}$, weakly converging in the spaces $L_2(Q_1)$ and $L_2(Q_2)$ to the solution $u(x,t)$ of conjugation problem II.

The theorem is proved. □

Note that, actually, to prove the solvability of conjugation problems I and II, it suffices to obtain suitable a priori estimates of regular solutions of one problem or another for equation (33) (the transition to the functions $w(x,t)$ and $z(x,t)$ was necessary only to substantiate the application of continuation in a parameter for a fixed ε).

This fact will be used for the proof of solvability of conjugation problems III and IV.

Theorem 7. *Suppose that conditions* (5), (20), *and* (21) *are satisfied, along with the condition* $a_1 b_2 \neq 0$. *Then for every function* $f(x,t)$, *such that* $f_1(x,t) \in L_2(Q_1)$, $f_{1t}(x,t) \in L_2(Q_1)$, $f_2(x,t) \in L_2(Q_2)$, $f_{2t}(x,t) \in L_2(Q_2)$, $f(x,-T) = f(x,0) = f(x,T) = 0$ *given* $x \in \Omega$, *conjugation problem III has a solution* $u(x,t)$, *such that*

$$u(x,t) \in L_\infty(-T,0;W_2^2(\Omega)), \quad u_{ttt}(x,t) \in L_\infty(-T,0;L_2(\Omega)),$$
$$u(x,t) \in L_\infty(0,T;W_2^2(\Omega)), \quad u_{ttt}(x,t) \in L_\infty(0,T;L_2(\Omega)).$$

Proof. For a positive number ε, consider a problem: *find a function* $u(x,t)$ *such that in the cylinders* Q_1 *and* Q_2 *equation* (33) *holds for this function, along with conditions* (4), (12), *and* (13). Let $u(x,t)$ be a regular solution of this problem. We multiply equation (33) by the function $-\Delta u_{tt}(x,t) - \Delta u_{ttt}(x,t)$ and integrate over the cylinder Q_1, further, we multiply by the function $-A\Delta u_{tt}(x,t) - A\Delta u_{ttt}(x,t)$ (the number A was defined in the proof of Theorem 3) and integrate over the cylinder Q_2. Summing the two obtained results, using the statement of the theorem, and applying the Young's inequality, we obtain that there exists an a priori estimate

$$\varepsilon \int\limits_{Q_1} (\Delta u_{ttt})^2 \, dx\, dt + \varepsilon \int\limits_{Q_2} (\Delta u_{ttt})^2 \, dx\, dt + \sum_{i=1}^n \int\limits_{Q_1} u_{x_i ttt}^2 \, dx\, dt +$$
$$+ \sum_{i=1}^n \int\limits_{Q_2} u_{x_i ttt}^2 \, dx\, dt + \int\limits_{Q_1} (\Delta u_t)^2 \, dx\, dt + \int\limits_{Q_2} (\Delta u_t)^2 \, dx\, dt \leq$$
$$\leq K_3 \left(\int\limits_{Q_1} f_1^2 \, dx\, dt + \int\limits_{Q_2} f_2^2 \, dx\, dt \right),$$

in which the constant K_3 is defined by the numbers a_1, a_2, b_1, b_2, T, and ε. With the help of this estimate, transition to the functions $w(x,t)$ and $z(x,t)$, and also the continuation in a parameter, it is easy to obtain that, given a fixed ε, boundary value problem (33), (4), (12), (13) has a regular solution. Further, for the family of solutions of this problem, the second a priori estimate is true

$$\varepsilon^2 \int\limits_{Q_1} (\Delta u_{ttt})^2 \, dx\, dt + \varepsilon^2 \int\limits_{Q_2} (\Delta u_{ttt})^2 \, dx\, dt + \int\limits_{Q_1} u_{ttt}^2 \, dx\, dt + \int\limits_{Q_2} u_{ttt}^2 \, dx\, dt +$$
$$+ \int\limits_{Q_1} (\Delta u_t)^2 \, dx\, dt + \int\limits_{Q_2} (\Delta u_t)^2 \, dx\, dt \leq K_4 \left(\int\limits_{Q_1} (f_1^2 + f_{1t}^2) \, dx\, dt + \int\limits_{Q_2} (f_2^2 + f_{2t}^2) \, dx\, dt \right), \tag{50}$$

in which the constant K_4 is only defined by the numbers a_1, a_2, b_1, and b_2 (to prove this estimate it suffices to multiply equation (33) by the function $-\varepsilon\Delta u_{ttt} + u_{ttt} - \Delta u_{tt}$, integrate over cylinder Q_1, further, to multiply equation (33) by the function $A(-\varepsilon\Delta u_{ttt}+u_{ttt}-\Delta u_{tt})$, integrate over cylinder Q_2; applying additionally the formula for integrating over parts in the terms corresponding to the functions $f_1(x,t)$ and $f_2(x,t)$, and having summed the two obtained equalities, after applying the statements of the theorem, we will obtain the required estimate (50)). From this estimate and the reflexivity property of a Hilbert space, it follows that after choosing the sequence $\{\varepsilon_m\}_{m=1}^{\infty}$, tending to zero, and transition to the weakly converging functional sequence $\{u_{\varepsilon_{m_k}}\}_{k=1}^{\infty}$ in the limit we will obtain the solution of conjugation problem III.

The theorem is proved. □

Theorem 8. *Let $\varphi(t)$ be a piecewise constant function, and suppose that condition* (23) *is fulfilled, along with the conditions $a_1b_2>0$, $b_1<0$, $a_2\geq 0$. Then for every function $f(x,t)$ such that*

$$f_1(x,t)\in L_2(-T,0;\overset{\circ}{W}{}_2^1(\Omega)),\quad f_2(x,t)\in L_2(0,T;\overset{\circ}{W}{}_2^1(\Omega)),$$

conjugation problem IV has a solution $u(x,t)$, such that

$$u(x,t)\in W_2^3(Q_1),\quad u(x,t)\in W_2^3(Q_2).$$

Proof. For a positive number ε, consider a problem: *find a function $u(x,t)$ such that in the cylinders Q_1 and Q_2, equation* (33) *holds for this function, along with conditions* (4), (9), *and* (13). Let $u(x,t)$ be a regular solution of this problem. We multiply sequentially equation (33) first by the functions

$$(T+t)u_{tt}(x,t)-u_t(x,t),\quad (T+t)\Delta u_{tt}(x,t)-\Delta u_t(x,t),$$

and $u_{ttt}(x,t)$, later by the functions

$$\frac{a_1}{b_2}\left[(T-t)u_{tt}(x,t)+u_t(x,t)\right],\quad \frac{a_1}{b_2}\left[(T-t)\Delta u_{tt}(x,t)-\Delta u_t(x,t)\right],$$

and $-u_{ttt}(x,t)$. Integrating the first three equalities over the cylinder Q_1, the next three over the cylinder Q_2, further, pairwise summing, using the integral inequalities from the proof of Theorem 4 and the Young's inequality, it is easy to show that for a fixed ε and when the functions $f_i(x,t)$ belong to the spaces $L_2(Q_i)$, $i=1,2$, there exists a regular solution $u^{\varepsilon}(x,t)$ of boundary value problem (33), (4), (9), (13). Further, additionally integrating over parts by spacial variables in the integrals of the functions $[(T-t)\Delta u_{tt}^{\varepsilon}(x,t)-\Delta u_t^{\varepsilon}(x,t)]\,f_1(x,t)$ and

$$\frac{a_1}{b_2}\left[(T+t)\Delta u_{tt}^{\varepsilon}(x,t)+\Delta u_t^{\varepsilon}(x,t)\right]f_2(x,t)$$

and again applying the Young's inequality, it is easy to obtain the a priori estimate uniform by ε:

$$\varepsilon^2\left[\int\limits_{Q_1}(\Delta u_{ttt}^{\varepsilon})^2\,dx\,dt+\int\limits_{Q_2}(\Delta u_{ttt}^{\varepsilon})^2\,dx\,dt\right]+\int\limits_{Q_1}\left[(u_{ttt}^{\varepsilon})^2+(\Delta u^{\varepsilon})^2\right]\,dx\,dt+$$

$$+\int\limits_{Q_2}\left[(u_{ttt}^{\varepsilon})^2+(\Delta u^{\varepsilon})^2\right]\,dx\,dt\leq K_5\left[\int\limits_{Q_1}\left(f_1^2+\sum_{i=1}^{n}f_{1x_i}^2\right)\,dx\,dt+\right.$$

$$+\int\limits_{Q_2}\left(f_2^2+\sum_{i=1}^{n}f_{2x_i}^2\right)\,dx\,dt\Bigg].$$

This estimate and the procedure of choosing the converging subsequence described in the proof of Theorem 5 provide the possibility to obtain the required solution of conjugation problem IV.

The theorem is proved. □

4. Conclusion

The equations studied in the work have a model form. Obviously, the results similar to the ones provided above can be also obtained for more general equations, for example, the Laplace operator can be substituted by a general elliptic operator of second order with all lowest coefficients.

The condition $a_1b_2 \neq 0$ in the existence theorems means that the cases of splitting problems are not considered. The existence of regular solutions of splitting problems is obvious.

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