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FIXED POINTS OF CYCLIC GROUPS ACTING PURELY HARMONICALLY ON A GRAPH

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Abstract. Let X be a finite connected graph, possibly with loops and multiple edges. An automorphism group of X acts purely harmonically if it acts freely on the set of directed edges of X and has no invertible edges. Define a genus g of the graph X to be the rank of the first homology group. A discrete version of the Wiman theorem states that the order of a cyclic group $\mathbb{Z}_n$ acting purely harmonically on a graph X of genus $g > 1$ is bounded from above by $2g + 2$. In the present paper, we investigate how many fixed points has an automorphism generating a «large» cyclic group $\mathbb{Z}_n$ of order $n \geq 2g - 1$. We show that in the most cases, the automorphism acts fixed point free, while for groups of order $2g$ and $2g - 1$ it can have one or two fixed points.

Keywords: graph, homological genus, harmonic automorphism, fixed point, Wiman theorem

1. Introduction

Let X be a finite connected graph. Loops and multiple edges are admitted. We provide each edge of X (including loops) by two possible orientations. Define the genus g of the graph X to be the rank of its first homology group. An automorphism group of a graph is said to act *harmonically* if it acts freely on the set of its directed edges and *purely harmonically* if it also has no invertible edges.

By [1] and [2], a finite group acting harmonically on a graph of genus g is a discrete analogue of a finite group of automorphisms of a closed Riemann surface

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of genus g. In papers [2, 3], a discrete version of the classical $84(g-1)$ Hurwitz theorem is established. Discrete versions of the Oikawa and the Arakawa theorems that refine the Hurwitz theorem for various classes of groups were obtained in [5].

An automorphism of a graph X is said to be *harmonic* if it generates a cyclic group acting harmonically on X. In paper [6] a discrete analogue of the Wiman theorem has been established. More precisely, it was shown that the order of a harmonic automorphism of a graph X of genus $g \geq 2$ does not exceed $2g+2$ and this bound is achieved for any even g. The size of cyclic group acting harmonically on X with given number of fixed points was estimated from the above in [4].

In [7] the following problem for cyclic group $\mathbb{Z}_n$ acting purely harmonically on a graph X of genus g with fixed points is solved. Given subgroup $\mathbb{Z}_d < \mathbb{Z}_n$, the signature of orbifold $X/\mathbb{Z}_d$ through the signature of $X/\mathbb{Z}_n$ is expressed. As a result, the formulas are given for the number of fixed points for generators of group $\mathbb{Z}_d$ and for genus of orbifold $X/\mathbb{Z}_d$. For Riemann surfaces, similar results were obtained earlier by M. J. Moore [8].

In the present paper, we deal with cyclic group $\mathbb{Z}_n$ acting purely harmonically on a graph X of genus $g \geq 2$. We investigate on how many fixed points are there for a generator of group $\mathbb{Z}_n$ in the case of the three largest possible orders $n = 2g+2, 2g, 2g-1$. The respective results are given by Theorems $1, 2$ and 3. In the last section, we illustrate the obtained results by series of examples.

2. Basic definitions and preliminary facts

In this paper, a graph X is a finite connected multigraph, possibly with loops. We provide each edge of X including loops, by the two possible orientations. Denote by $V(X)$ the set of vertices and by $E(X)$ the set of directed edges of X. Given $e \in E(X)$, by $\bar{e}$ we denote edge e taking with the opposite orientation. Let $G \leq \mathrm{Aut}(X)$ be a group of automorphisms of a graph X. An edge $e \in E(X)$ is called *invertible* if there is $g \in G$ such that $g(e) = \bar{e}$. Let G act without invertible edges. Define the quotient graph X/G so that its vertices and edges are G-orbits of the vertices and edges of X. Denote by $\varphi : G \to X/G$ the respective canonical map. Note that if the endpoints of an edge $e \in E(X)$ lie in the same G-orbit then the G-orbit of e is a loop in the quotient graph X/G. We say that the group G acts harmonically on a graph X if it acts freely on the set of directed edges $E(G)$ which simply means that, each element of G that fixes an edge $e \in E(G)$ is the identity. If G acts harmonically and without invertible edges, we say that G acts purely harmonically on X.

Let G be a finite group acting purely harmonically on a graph X. For every $\tilde{v} \in V(X)$ denote by $G_{\tilde{v}}$ the stabilizer of $\tilde{v}$ in the group G and by $|G_{\tilde{v}}|$ its order. For each vertex $v \in V(X/G)$ we prescribe the number $m_v = |G_{\tilde{v}}|$, where $\tilde{v} \in \varphi^{-1}(v)$. Since G acts transitively on each fibre of φ, these numbers are well-defined. The point v, for which $m_v \geq 2$, will be called *branch point* of order m_v.

Define the *genus* $\gamma = \gamma(X)$ of a graph X as its cyclomatic number or Betti number (equivalently, rank of the first homology group). More precisely,

$$\gamma(X) = 1 - |V(X)| + |E(X)|,$$

where $|V(X)|$ and $|E(X)|$ is the number of vertices and edges of X respectively.

We prefer to view the quotient graph X/G as a one-dimensional orbifold. In this case, the notion of signature is very important. If the group G acts purely harmonically on X, the *signature* of X/G is defined as the sequence $(\gamma; m_1, \ldots, m_r)$, where γ is genus of X/G and $m_1, m_2, \ldots, m_r$ are branch orders of the covering $\varphi : X \to X/G$.

The following theorem has been proved in [7].

Theorem A (Moore formula for graphs). *Let $\mathbb{Z}_n$ be a cyclic group acting harmonically on a graph X and h be an element of order d, $d > 1$ in the group $\mathbb{Z}_n$. Denote by $(\gamma; m_1, \ldots, m_r)$ the signature of the orbifold $X/\mathbb{Z}_n$. Then the number of fixed points of h is given by the formula*

$$\sum_{d \mid m_i} \frac{n}{m_i}.$$

As an important consequence of Theorem A we have the following proposition.

Proposition 1. *Let $\mathbb{Z}_n$ be a cyclic group acting harmonically on a graph X and $(\gamma; m_1, \ldots, m_r)$ be the signature of the orbifold $X/\mathbb{Z}_n$. Then the number of fixed points of a generator of $\mathbb{Z}_n$ coincides with the number of entities m_i in the signature which are equal to n.*

Let $\mathbb{Z}_n$ be a cyclic group acting harmonically a graph X of genus $g \geq 2$. Recall the following results proved in [6]. First of all, we get $n \leq 2g+2$. The upper bound $n = 2g + 2$ is attained for any even g. In this case, the signature of the orbifold $X/\mathbb{Z}_n$ is $(0; 2, g+1)$, that is $X/\mathbb{Z}_n$ is a tree with two branch points of order 2 and $g+1$ respectively. Moreover, if $n < 2g+2$, then $n \leq 2g$. The upper bound $n = 2g$ is attained when $X/\mathbb{Z}_n$ is an orbifold of the signature $(0; 2, 2g)$, and also for $n = 12$ when $X/\mathbb{Z}_n$ is an orbifold of the signature $(0; 3, 4)$. The third largest cyclic group $\mathbb{Z}_n$ of order $n = 2g - 1$ appeared only in two cases: for $n = 3$ and $X/\mathbb{Z}_n$ is an orbifold of the signature $(0; 3, 3)$, and for $n = 15$ and $X/\mathbb{Z}_n$ is an orbifold of the signature $(0; 3, 5)$.

In the next section, we describe the number of fixed points of an automorphism generating the above group $\mathbb{Z}_n$ for the three largest possible values $n = 2g+2, 2g$ and $2g - 1$.

3. Main results

The main results of the paper are given by the following three theorems.

Theorem 1. *Let X be a graph of genus $g \geq 2$ and $\mathbb{Z}_n = \langle T : T^n = 1 \rangle$ be a cyclic group acting purely harmonically on X. Suppose that $n = 2g + 2$. Then T acts on the graph X without fixed points.*

Proof. By Theorem 3 from [6], we know that if T is an automorphism of order $n = 2g + 2$ then the signature of the orbifold $X/\mathbb{Z}_n$ is $(0; 2, g + 1)$. Note that by Proposition 1 if the signature of orbifold $X/\mathbb{Z}_n$ is equal to $(\gamma; m_1, \ldots, m_r)$, then the number of fixed points of a generator of $\mathbb{Z}_n$ coincides with the number of entities m_i in the signature which are equal to n.

In our case, $\gamma = 0, r = 2, m_1 = 2, m_2 = g + 1$. Since $n = 2g + 2 > m_1 = 2$ and $n = 2g+2 > m_2 = g+1$, there are no numbers m_i equal to n. Hence, automorphism T of the largest possible order $2g + 2$ acts on a graph X of genus g without fixed points.

Theorem 2. *Let X be a graph of genus $g \geq 2$ and $\mathbb{Z}_n = \langle T : T^n = 1 \rangle$ be a cyclic group acting purely harmonically on X. Suppose that $n = 2g$. Then for $n \neq 12$ automorphism T acts on the graph X with one fixed point. If $n = 12$, then either T has one fixed point or acts fixed point free on the graph X.*

Proof. Now, by virtue of Theorem 4 from [6], we get either (i) $(\gamma; m_1, \ldots, m_r) = (0; 2, 2g)$, where $g \geq 2$, or (ii) $(\gamma; m_1, \ldots, m_r) = (0; 3, 4)$, where $g = 6$. In the first case, exactly one of the numbers m_i is equal to $n = 2g$. By Proposition 1, automorphism T has one fixed point. In the second case, all the numbers m_i are differ from n, that is T has no fixed points.

Theorem 3. *Let X be a graph of genus $g \geq 2$ and $\mathbb{Z}_n = \langle T : T^n = 1 \rangle$ be a cyclic group acting purely harmonically on X. Suppose that $n = 2g - 1$. Then either T has two fixed points or acts fixed point free on the graph X. In the first case, $g = 2$; in the second $g = 8$.*

Proof. By ([6], Theorem 4), the cyclic group $\mathbb{Z}_n$ of order $n = 2g - 1$ appears only in the following two cases:

(iii) $n = 3$ and $X/\mathbb{Z}_n$ is an orbifold of the signature $(0; 3, 3)$, $g = 2$;

(iv) $n = 15$ and $X/\mathbb{Z}_n$ is an orbifold of the signature $(0; 3, 5)$, $g = 8$.

In case (a), $\gamma = 0$, $r = 2$, $m_1 = 3$, $m_2 = 3$. Hence $m_1 = m_2 = n$ and we get exactly two fixed points for automorphism T.

In case (b), $\gamma = 0$, $r = 2$, $m_1 = 3$, $m_2 = 5$. Now $n > m_1, m_2$. By Proposition 1 automorphism T has no fixed points.

In the next section, we illustrate all the obtained results by examples.

4. Examples

Example 1. To illustrate Theorem 1 consider the complete bipartite graph $X = K_{2,g+1}$ of genus g with vertices $u_1, u_2, v_1, v_2, \ldots, v_{g+1}$ and edges $u_i v_j$, $i = 1, 2$, $j = 1, 2, \ldots, g+1$. Let g be even and cyclic group $\mathbb{Z}_{2g+2} = \mathbb{Z}_2 \oplus \mathbb{Z}_{g+1}$ acts on $K_{2,g+1}$ by substitution $(u_1, u_2)(v_1, v_2, \ldots, v_{g+1})$. Then the orbifold $X/\mathbb{Z}_{2g+2}$ is a path graph with vertices that are branched points of order 2 and $g+1$ respectively.

Example 2. (i) To describe an automorphism of order $n = 2g$ with one fixed point acting purely harmonically on a graph of genus g consider a graph X on $g+1$ vertices u, v_j, $j = 1, 2, \ldots, g$ such that for each j it has two edges e_j, e_{j+g} between the vertices u, and v_j (see Fig. 1). Let T be an automorphism of X of order n fixing u and sending v_i to v_{i+1} and e_j to e_{j+g}, where the indices i and j are taken modulo g and $2g$ respectively. To see the action of $\mathbb{Z}_{2g}$ on X one can imagine X as a star graph with $2g$ edges whose opposite vertices are identified. Then the factor-graph $X/\mathbb{Z}_{2g}$ is an orbifold of signature $(0; 2, 2g)$ consisting of two vertices and one edge between them.

(ii) To construct an automorphism T of order $n = 12$ acting fixed point free on a graph X of genus 6 we set $X = K_{3,4}$. Here $K_{3,4}$ is the complete bipartite graph with vertices $u_1, u_2, u_3, v_1, v_2, v_3, v_4$ and edges $u_i v_j$, $i = 1, 2, 3$, $j = 1, 2, 3, 4$. The action of automorphism T on vertices of X is given by the order 12 substitution $T = (u_1, u_2, u_3)(v_1, v_2, v_3, v_4)$. If $\mathbb{Z}_{12} = \langle T \rangle$, then the respective orbifold $X/\mathbb{Z}_{12}$ has signature $(0; 3, 4)$.

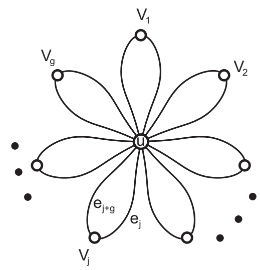

FIG. 1. Genus g graph X with $\mathbb{Z}_{2g}$ harmonic action.

Example 3. (iii) Let X be the theta graph consisting of two vertices u and v joined by three edges e_1, e_2, e_3. Let T be an order three automorphism of X circularly permutated edges e_1, e_2, e_3 and leaving vertices u and v fixed. Then the group $\mathbb{Z}_3 = \langle T \rangle$ acts purely harmonically on X and the factor graph $X/\mathbb{Z}_3$ is an orbifold of signature $(0; 3, 3)$.

(iv) Let $K_{3,5}$ be the complete bipartite graph with vertices $u_1, u_2, u_3, v_1, v_2, v_3, v_4, v_5$ and edges $u_i v_j$, $i = 1, 2, 3$, $j = 1, 2, 3, 4, 5$. We note that $K_{3,5}$ is a graph of genus eight and define an automorphism T of X by the order 15 substitution $T = (u_1, u_2, u_3)(v_1, v_2, v_3, v_4, v_5)$. Then T acts fixed point free on X and the orbifold $X/\langle T \rangle$ has signature $(0; 3, 5)$.

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