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ON THE ORLICZ COHOMOLOGY OF STAR-BOUNDED SIMPLICIAL COMPLEXES

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ABSTRACT. We prove that the Whitney transformation induces an isomorphism between the Orlicz cohomology of a star-bounded simplicial complex and the cohomology of the corresponding Orlicz–Sullivan complex. Our main result is an extension of the isomorphism in the L_p case proved by Gol′dshtein, Kuz′minov, and Shvedov in 1988.

Keywords: star-bounded complex, Orlicz–Sullivan complex, differential form, Orlicz cohomology.

INTRODUCTION

In 1976 in [1], Atiyah defined simplicial L_2-cohomology for free simplicial cocompact group actions. Later in [3], Cheeger and Gromov used simplicial l_2-cohomology to define singular l_2-cohomology for countable groups. For arbitrary p, L_p-methods in the study of simplicial complexes were initiated by Gol′dshtein, Kuz′minov, and Shvedov in [6]. They proved the de Rham isomorphism for L_p-cohomology of a triangulated Riemannian manifold under certain conditions on the triangulation. In [7], Gromov also considered the ℓ_p-cohomology of a simplicial complex with bounded geometry and proved its quasi-isometry invariance in the uniformly contractible case. The ℓ_p-cohomology of simplicial complexes and it quasi-invariance properties was also considered by Pansu [10].

In his thesis [4], Ducret introduced the L_π-cohomology of a simplicial complex of bounded geometry for a sequence of real numbers $\pi = (p_k)$, $1 \leq p_k < \infty$, and, under certain conditions, established the L_π-de Rham isomorphism.

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In 2017 in [2], Carrasco Piaggio introduced simplicial ℓ^Φ-cohomology for a Young function Φ and proved its quasi-isometry invariance in the uniformly contractible case. He also proved the de Rham theorem in degree 1 for Orlicz cohomology. Then, in [11], Sequeira introduced a relative version of simplicial ℓ_p-cohomology and Orlicz cohomology and established its quasi-isometry for Gromov-hyperbolic uniformly contractible simplicial complexes of bounded geometry. He also proved the de Rham isomorphism for Orlicz cohomology of Lie groups.

In the present paper, by analogy with [6], we establish an isomorphism between the Orlicz cohomology of a star-bounded simplicial complex and the cohomology of the corresponding Orlicz Sullivan complex.

The structure of the article is as follows: In Section 1, we recall the main notions and necessary properties of Young functions and Orlicz function spaces. Section 2 deals with the Orlicz cohomology of star-bounded simplicial complexes. First, we introduce an Orlicz Sullivan complex on a Riemannian manifold. Then we recall the definition of the Whitney transformation and prove that it induces an isomorphism between the Orlicz cohomology of a star-bounded simplicial complex and the cohomology of the corresponding Orlicz Sullivan complex (Theorem 2.5).

1. Young Functions and Orlicz Function Spaces

A function $\Phi : \mathbb{R} \to \mathbb{R}_+ \cup \{\infty\}$ is called a *Young function* if

(i) Φ is even and convex;

(ii) $\Phi(0) = 0, \ \lim\limits_{t\to\infty} \Phi(t) = \infty$.

Let Φ be a Young function and let (Ω, Σ, μ) be a measure space.

Given a measurable function $f : \Omega \to \mathbb{R}$, we put

$$\rho_\Phi(f) := \int_\Omega \Phi(f)d\mu.$$

The linear space

$$L^\Phi = L^\Phi(\Omega) = L^\Phi(\Omega, \Sigma, \mu) = \{f : \Omega \to \mathbb{R} \text{ measurable } : \rho_\Phi(af) < \infty \text{ for } \textit{some } a > 0\}$$

is called an *Orlicz space* on (Ω, Σ, μ).

Below we as usual identify two functions equal outside a set of measure zero.

The *gauge* (or *Luxemburg*) *norm* of a function $f \in L^\Phi$ is defined by the formula

$$\|f\|_{(\Phi)} = \|f\|_{L^{(\Phi)}(\Omega)} = \inf\left\{\lambda > 0 \, : \, \rho_\Phi\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

We will need the following familiar assertion, which we prove here for completeness:

Lemma 1.1. *If Φ is a Young function and α is a positive number then the norms $\|\cdot\|_{(\Phi)}$ and $\|\cdot\|_{(\alpha\Phi)}$ are equivalent:*

$$\|\cdot\|_{(\Phi)} \leq \|\cdot\|_{(\alpha\Phi)} \leq \alpha\|\cdot\|_{(\Phi)} \quad \textit{if } \alpha \geq 1;$$
$$\alpha\|\cdot\|_{(\Phi)} \leq \|\cdot\|_{(\alpha\Phi)} \leq \|\cdot\|_{(\Phi)} \quad \textit{if } \alpha < 1.$$

Proof. Assume first that $\alpha \geq 1$.

Put $\widetilde{\Phi} = \alpha\Phi$. We have for any $\lambda > 0$ and $f \in L^\Phi(\Omega)$ due to convexity:

$$\rho_{\widetilde{\Phi}}\left(\frac{f}{\alpha\lambda}\right) = \alpha\rho_\Phi\left(\frac{f}{\alpha\lambda}\right) \leq \rho_\Phi\left(\frac{f}{\lambda}\right). \tag{1.1}$$

Put

$$M_\Phi = \left\{ \lambda \,:\, \rho_\Phi\left(\frac{f}{\lambda}\right) \le 1 \right\}; \quad M_{\widetilde\Phi} = \left\{ \tilde\lambda \,:\, \rho_{\widetilde\Phi}\left(\frac{f}{\tilde\lambda}\right) \le 1 \right\}.$$

If $\lambda \in M_\Phi$ then it follows from (1.1) that $\alpha\lambda \in M_{\widetilde\Phi}$, and hence $\lambda \in \frac{1}{\alpha} M_{\widetilde\Phi}$. Therefore, $M_\Phi \subset \frac{1}{\alpha} M_{\widetilde\Phi}$. This gives $\inf M_\Phi \ge \frac{1}{\alpha} \inf M_{\widetilde\Phi}$, or

$$\|\cdot\|_{(\alpha\Phi)} \le \alpha \|\cdot\|_{(\Phi)}.$$

On the other hand, if $f \in L^{\widetilde\Phi}(\Omega)$ then

$$\rho_\Phi\left(\frac{f}{\lambda}\right) = \frac{1}{\alpha}\rho_{\widetilde\Phi}\left(\frac{f}{\lambda}\right) \le \rho_{\widetilde\Phi}\left(\frac{f}{\lambda}\right). \tag{1.2}$$

Using (1.2), we conclude that $M_{\widetilde\Phi} \subset M_\Phi$, and hence

$$\|\cdot\|_{(\Phi)} = \inf M_\Phi \le \inf M_{\widetilde\Phi} = \|\cdot\|_{(\alpha\Phi)}.$$

The case of $\alpha < 1$ is obtained by noticing that $\Phi = \frac{1}{\alpha}\widetilde\Phi$. □

Remark 1.2. The above considerations also hold for Orlicz sequence spaces with integrals replaced by infinite sums.

2. Orlicz Cohomology of Star-Bounded Simplicial Complexes

2.1. The Sullivan complex. Let K be a simplicial complex and let $|K|$ its geometric realization. We will endow $|K|$ with the standard metric induced by embedding $|K|$ into a Hilbert space, where each vertex of K is mapped to a point having one coordinate equal to one and the remaining coordinates equal to zero. The embedding is affine on each simplex of K.

Denote by $C^k(K)$ the space of real k-cochains of K.

As was observed in [6], given a simplex T and a face T' of T, the identical embedding $j_{T',T} : T' \to T$ induces a restriction mapping of the spaces of forms $j^*_{T',T} : \Omega^*_\infty(T) \to \Omega^*_\infty(T')$. The norm of each of the operators $j^*_{T',T}$ does not exceed 1.

A *Sullivan form*, or, briefly, an *S-form, of degree* k on K is a collection $\omega = \{\omega(T)\}_{T\in K}$, where $\omega(T) \in \Omega^k_\infty(T)$ for each $T \in K$, satisfying the following condition: if T' is a face of T then $j^*_{T',T}\omega(T) = \omega(T')$.

Denote by $S^k(K)$ the vector space of Sullivan forms of degree k on K. The form $\omega(T)$ is called the T-component of the form ω.

Given a Sullivan k-form ω, define the exterior differential Sullivan form $d\omega$ componentwise. Since $dj^*_{T',T} = j^*_{T',T}d$, we have $dS^k(K) \subset S^{k+1}(K)$. Thus, $(S^*(K), d)$ is a cochain complex of real vector spaces. It is called the *Sullivan complex* of the simplicial complex K. Its cohomology space is denoted by $H^k(S^*(K))$.

Define the *integration morphism* [4, 6] as the linear mapping $I : S^k(K) \to C^k(K)$,

$$(I\omega)(T) = \int_T \omega(T),$$

where $\omega \in S^k(K)$, $T \in K$ (recall that T is regarded as a standard Euclidean simplex). Then the transformation $I : S^*(K) \to C^*(K)$ of cochain complexes induces an isomorphism in cohomology (see [12]).

Let M be a smooth manifold and let $\tau : |K| \to M$ be its smooth triangulation. Consider the spaces

$$L^\infty(M, \Lambda^k) = \{\omega \in L^1_{\mathrm{loc}}(M, \Lambda^k) \,:\, \|\omega\|_\infty = \operatorname{ess\,sup} |\omega(x)| < \infty\};$$

$$\Omega^k_\infty(M) = \{\omega \in L^\infty(M, \Lambda^k) \,:\, d\omega \in L^\infty(M, \Lambda^{k+1})\}.$$

The elements of $\Omega^*_\infty(M)$ are sometimes called *flat forms.*

The following assertion can be found in [13]:

Proposition 2.1. *Let $f : M_1 \to M_2$ be a Lipschitz mapping between Riemannian manifolds. Then for any flat form $\omega \in \Omega^k_\infty(M_2)$ the form $f^*\omega$ is well defined and is a flat form. Moreover, $df^*\omega = f^*d\omega$.*

Given $\omega \in \Omega^k_{\infty,\mathrm{loc}}(M)$ and a simplex $T \in K$, we put

$$\omega(T) = (\tau|_T)^*\,\omega.$$

The following assertion is [6, Lemma 1]:

Lemma 2.2. *The collection of forms $\varphi_\tau\omega = \{\omega(T)\}_{T\in K}$ is an S-form on K. The mapping $\varphi_\tau : \Omega^k_{\infty,\mathrm{loc}}(M) \to S^k(K)$ is an isomorphism of vector spaces.*

2.2. The Whitney transformation. In what follows, we assume that the complex K in question is of finite dimension n.

In addition to the mapping $I : S^*(K) \to C^*(K)$, there is a mapping $w : C^*(K) \to S^*(K)$, called the *Whitney transformation* (see [13]), defined as follows.

Let T be an s-simplex of the complex K and let T^k be a k-face of T, $k \le s$. For each of the vertices e_i, the barycentric coodinate function $\beta_i : |K| \to \mathbb{R}$ is defined (see [13]). For $c \in C^k(K)$ and $T = [e_0, e_1, \dots, e_s]$, we put

$$\theta(T) = k! \sum^{k}_{0\le i_0<\dots<i_k\le s} c([e_{i_0}, \dots, e_{i_k}]) \sum_{r=0}^{k} (-1)^r \beta_{i_r} d\beta_{i_0} \wedge \dots \wedge \widehat{d\beta_{i_r}} \wedge \dots \wedge d\beta_{i_k},$$

where $\widehat{\ }$ means the omission of the corresponding term. The k-form $\theta(T)$ does not depend on the order of the vertices of T. If T' is a face of T then $\theta(T') = j_{T',T}\theta(T)$. Therefore, $w(c) = \{\theta(T)\}_{T\in K}$ is an S-form on K. Moreover, $dw(c) = w(\delta c)$ for any cochain $c \in C^*(K)$.

2.3. $\ell_{\Phi_I,\Phi_{II}}$-Cohomology of a simplicial complex. Let $\delta : C^k(K) \to C^{k+1}(K)$ be the coboundary operator. If Φ is a Young function and $c \in C^k(K)$ then consider the space

$$C^k_\Phi(K) = \{c \in C^k(K) \,:\, \sum_{\dim T = k} \Phi(a|c(T)|) < \infty \text{ for some } a > 0\}.$$

Endow the space $C^k_\Phi(K)$ with the gauge norm

$$\|c\|_{(\Phi)} = \inf\left\{\lambda > 0 : \sum_{\dim T = k} \Phi\left(\frac{|c(T)|}{\lambda}\right) \le 1\right\}.$$

$C^k_\Phi(K)$ is a Banach space continuously isomorphic to $\ell^\Phi(\mathbb{Z})$. Put

$$Z^k_\Phi(K) = \{c \in C^k_\Phi(K) \,:\, \delta c = 0\}.$$

Given two Young functions Φ_I and Φ_{II}, we also put

$$B^k_{\Phi_I,\Phi_{II}}(K) = \delta C^{k-1}_{\Phi_I}(K) \cap C^k_{\Phi_{II}}(K) \subset Z^k_{\Phi_{II}}(K);$$

let $\overline{B}^k_{\Phi_I,\Phi_{II}}(K)$ be the closure of $B^k_{\Phi_I,\Phi_{II}}(K)$ in $C^k_{\Phi_{II}}(K)$.

The *kth $\ell_{\Phi_I,\Phi_{II}}$-cohomology* of K is the seminormed space

$$H^k_{\Phi_I,\Phi_{II}}(K) = Z^k_{\Phi_{II}}(K)/B^k_{\Phi_I,\Phi_{II}}(K);$$

the *reduced kth* $\ell_{\Phi_I,\Phi_{II}}$*-cohomology* of K is the Banach space

$$\overline{H}^k_{\Phi_I,\Phi_{II}}(K) = Z^k_{\Phi_{II}}(K)/\overline{B}^k_{\Phi_I,\Phi_{II}}(K).$$

If $\Phi_I = \Phi_{II} = \Phi$ then we will write only one subscript Φ in all the above notations instead of Φ, Φ.

Like in the case of the complex $(C^*_p(K), \delta)$ (see [6]), the following lemma holds. The technique of dealing with estimates of gauge norms here and in some situations below is borrowed from [2, p. 446], where the boundedness of the operator δ in Orlicz spaces was established under different constraints on K.

We call a simplicial complex *star-bounded* if the number of simplices in the star of each of its vertices is bounded from above.

Lemma 2.3. *If K is a star-bounded simplicial complex then $\delta(C^k_\Phi(K)) \subset C^{k+1}_\Phi(K)$ and $\delta : C^k_\Phi(K) \to C^{k+1}_\Phi(K)$ is a bounded operator.*

Proof. Let the number of $(k+1)$-simplices in the star of a k-simplex be bounded by N. For $\lambda > 0$, we have

$$\begin{aligned}\sum_{\dim T = k+1} \Phi\left(\frac{|\delta c(T)|}{\lambda(k+2)}\right) &= \sum_{\dim T = k+1} \Phi\left(\left|\sum_{\dim T' = k} \frac{[T:T']}{\lambda(k+2)} c(T')\right|\right) \\ &\le \sum_{\dim T = k+2} \sum_{\dim T' = k} \frac{|[T:T']|}{(k+2)} \Phi\left(\left|\frac{c(T')}{\lambda}\right|\right) \\ \le \frac{1}{k+2} \sum_{\dim T' = k} \Phi\left(\left|\frac{c(T')}{\lambda}\right|\right) &\sum_{\dim T = k+1} |[T:T']| \le \frac{N}{k+2} \sum_{\dim T' = k} \Phi\left(\frac{|c(T')|}{\lambda}\right).\end{aligned}$$

Here we have put $[T : T'] = 0$ if T' is not a boundary simplex for T. Since $\|\cdot\|_{(\frac{N}{k+1}\Phi)} \sim \|\cdot\|_{(\Phi)}$, the set of λ for which $\frac{N}{k+2}\sum_{\dim T' = k} \Phi\left(\frac{|c(T')|}{\lambda}\right) \le 1$ is not empty, and is in fact an interval, and so such λ can be taken arbitrarily close to $\|c\|_{(\frac{N}{k+2}\Phi)}$. For such λ,

$$\sum_{\dim T = k+1} \Phi\left(\frac{|\delta c(T)|}{\lambda(k+2)}\right) \le 1,$$

and hence $\|\frac{\delta c}{k+2}\| \le \lambda$, or

$$\|\delta c\| \le \lambda(k+2). \tag{2.1}$$

Taking infimum on the right-hand side of (2.1) and applying Lemma 1.1, we obtain

$$\|\delta c\|_{(\Phi)} \le \|c\|_{\left(\frac{N}{k+2}\Phi\right)} \le L\|c\|_{(\Phi)}$$

for some positive constant L depending only on N and k. Thus, the operator $\delta : C^k_\Phi(K) \to C^{k+1}_\Phi(K)$ is bounded. □

Given a closed subcomplex $L \subset K$, denote by $C^k_\Phi(K, L)$ the subspace in $C^k_\Phi(K)$ consisting of chains equal to 0 on L. The exact sequence of complexes

$$0 \to C^*_\Phi(K, L) \to C^*_\Phi(K) \to C^*_\Phi(L) \to 0$$

induces an exact sequence of cohomology spaces

$$\cdots \to H^i_\Phi(K) \to H^i_\Phi(L) \to H^{i+1}_\Phi(K, L) \to H^{i+1}_\Phi(K) \to \dots$$

For an S-form θ of degree k on K and a Young function Φ, we put

$$\|\theta\|_{S^*_\Phi(K)} = \inf\left\{\lambda : \sum_{T\in K} \Phi\left(\frac{\|\theta(T)\|_{\Omega^k_\infty(T)}}{\lambda}\right) \le 1\right\}.$$

It is easy to see that $S^*_\Phi(K)$ is a Banach space (cf. [4, Proposition 2.3]) and in fact can be treated in many aspects as an Orlicz sequence space.

Since the norm of the differential $d : \Omega^k_\infty(T) \to \Omega^{k+1}_\infty(T)$ is at most 1, $d(S^k_\Phi(K)) \subset S^{k+1}_\Phi(K)$. We thus obtain the L_Φ *Sullivan complex* $\{S^*_\Phi(K), d\}$.

If L is a subcomplex of a simplicial complex K then the mapping $j^*_{L,K} : S^k_\Phi(K) \to S^k_\Phi(L)$ is defined by the formula

$$j^*_{L,K}\{\omega(T)\}_{T\in K} = \{\omega(T)\}_{T\in L}.$$

Lemma 2.4. *If L is a subcomplex in a star-bounded finite-dimensional simplicial complex K then $j^*_{L,K}$ is an epimorphism.*

Proof. As was in fact demonstrated in the proof of [6, Lemma 2], if T is a simplex then there exists a bounded linear operator $\gamma : S^*_\Phi(\partial T) \to \Omega^*_\Phi(T)$ such that $j_{T',T} \circ \gamma(\omega)(T) = \omega(T')$ for any face T' of T.

Let now K be a star-bounded finite-dimensional simplicial complex, L be its subcomplex, and $K^{(j)}$ be the jth skeleton of K. Let us prove by induction on j that $j^*_{L,L\cup K^{(j)}}$ is an epimorphism. Put $K_j = L \cup K^{(j)}$. Suppose that $\omega_{j-1} \in S^k_\Phi(K_{j-1})$, and $j^*_{L,K_{j-1}}\omega_{j-1} = \omega$. Put

$$\omega_j(T) = \begin{cases} \omega_{j-1}(T) & \text{if } T \in K_{j-1}, \\ \gamma \circ j^*_{\partial T, K_{j-1}}\omega_{j-1}(T) & \text{if } T \in K_j \setminus K_{j-1}. \end{cases}$$

Then ω_j is an S-form on K_j. Indeed, take a simplex $T \in K$, $T' \in \partial K$ and calculate $j^*_{T',T}\omega_j(T)$. If $T \in K_{j-1}$ then, clearly $j^*_{T',T}\omega_j(T) = \omega(T')$. Let now $T \in K_j \setminus K_{j-1}$. Then

$$j^*_{T',T}\omega_j(T) = j^*_{T',T} \circ \gamma \circ j^*_{\partial T, K_{j-1}}\omega_{j-1}(T) = j^*_{\partial T, K_{j-1}}\omega_{j-1}(T') = \omega_{j-1}(T') = \omega_j(T')$$

by the definition of γ.

Estimate the norm of ω_j. Let $\|\gamma\|$ be the norm of the operator $\gamma : S_\Phi(\partial T) \to \Omega^*_\infty(T)$. We infer for any $\lambda > 0$ (the sums below can be finite or infinite):

$$\begin{aligned}
&\sum_{T\in K_j} \Phi\left(\frac{\|\omega_j(T)\|_{\Omega^k_\infty(T)}}{\lambda(j+1)(\|\gamma\|+1)}\right) \\
&\le \sum_{T\in K_{j-1}} \Phi\left(\frac{\|\omega_{j-1}(T)\|_{\Omega^k_\infty(T)}}{\lambda(j+1)(\|\gamma\|+1)}\right) + \sum_{T\in K_j\setminus K_{j-1}} \Phi\left(\frac{\|\gamma \circ j^*_{\partial T, K_{j-1}}\omega_{j-1}(T)\|_{\Omega^k_\infty(T)}}{\lambda(j+1)(\|\gamma\|+1)}\right) \\
&\qquad\le \sum_{T\in K_{j-1}} \Phi\left(\frac{\|\omega_{j-1}(T)\|_{\Omega^k_\infty(T)}}{\lambda(j+1)(\|\gamma\|+1)}\right) \\
&\qquad + \sum_{T\in K_j\setminus K_{j-1}} \Phi\left(\frac{\|\gamma\|}{\lambda(j+1)(\|\gamma\|+1)} \sum_{T'\in\partial T} \|\omega_{j-1}(T')\|_{\Omega^k_\infty(T')}\right) \\
&\qquad\le \frac{1}{(j+1)(\|\gamma\|+1)} \sum_{T\in K_{j-1}} \Phi\left(\frac{\|\omega_{j-1}(T)\|_{\Omega^k_\infty(T)}}{\lambda}\right)
\end{aligned}$$

$$+\frac{1}{j+1}\sum_{T\in K_j\setminus K_{j-1}}\sum_{T'\in\partial T}\Phi\left(\frac{\|\gamma\|}{\lambda(\|\gamma\|+1)}\|\omega_{j-1}(T')\|_{\Omega^k_\infty(T')}\right)=\mathbb{A}.$$

Let N be the supremum of the number of j-dimensional simplices in the stars of $(j-1)$-dimensional simplices. We proceed as follows:

$$\mathbb{A}\le\frac{1}{j+1}\sum_{T\in K_{j-1}}\Phi\left(\frac{\|\omega_{j-1}(T)\|_{\Omega^k_\infty(T)}}{\lambda}\right)$$
$$+\frac{N}{j+1}\frac{\|\gamma\|}{\|\gamma\|+1}\sum_{T'\in K_{j-1}}\Phi\left(\frac{\|\omega_{j-1}(T')\|_{\Omega^k_\infty(T')}}{\lambda}\right)$$
$$\le\frac{N+1}{j+1}\sum_{T\in K_{j-1}}\Phi\left(\frac{\|\omega_{j-1}(T)\|_{\Omega^k_\infty(T)}}{\lambda}\right).$$

In accordance with Lemma 1.1, $\|\cdot\|_{S^*_{\frac{N+1}{j+1}\Phi}(K)}\sim\|\cdot\|_{S^*_\Phi(K)}$. Therefore, for all λ sufficiently close to $\|\omega_{j-1}\|_{S^*_{\frac{N+1}{j+1}\Phi}}$, we have

$$\frac{N+1}{j+1}\sum_{T\in K_{j-1}}\Phi\left(\frac{\|\omega_{j-1}(T)\|_{\Omega^k_\infty(T)}}{\lambda}\right)\le 1.$$

For such λ,

$$\sum_{T\in K_j}\Phi\left(\frac{\|\omega_j(T)\|_{\Omega^k_\infty(T)}}{\lambda(j+1)(\|\gamma\|+1)}\right)\le 1.$$

Consequently, as in the proof of Lemma 2.3, we obtain

$$\left\|\frac{\omega_j}{\lambda(j+1)(\|\gamma\|+1)}\right\|_{S^*_\Phi(K)}\le\|\omega_{j-1}\|_{S^*_{\frac{N+1}{j+1}\Phi}(K)}.$$

Thus, $\omega_j\in S^*_\Phi(K_j)$ and also $j_{L,K_j}\omega_j=\omega$. Since K is finite-dimensional, the lemma is proved. □

Under the conditions of Lemma 2.4, denote by $S^*_\Phi(K,L)$ the kernel of the mapping $j^*_{L,K}:S^*_\Phi(K)\to S^*_\Phi(L)$. Then we have an exact cohomology sequence

$$\dots\to H^k(S^*_\Phi(K,L))\to H^k(S^*_\Phi(K))\to H^k(S^*_\Phi(L))\to H^{k+1}(S^*_\Phi(K,L))\to\dots$$

2.1. **2.4. The isomorphism between the simplicial and Orlicz–Sullivan cohomology.**

Theorem 2.5. *Let K be an arbitrary star-bounded finite-dimensional simplicial complex and L be its subcomplex. The Whitney transformation takes the cochain complex $\{C^*_\Phi(K,L),d\}$ to $S^*_\Phi(K,L)$ and induces a topological isomorphism of the cohomology spaces.*

Proof. Let $\dim K=n$, $c\in C^k_\Phi(K,L)$, $T\in K$, $\dim T=s$, $\theta=w(c)$. Then the definition of the Whitney transformation implies that

$$\|\theta(T)\|_{\Omega^k_\infty(T)}\le\binom{s+1}{k+1}\sum_{0\le i_0<\dots<i_k\le\dim T}|c([e_{i_0},\dots,e_{i_k}])|.$$

We have for $\lambda>0$:

$$\sum_{T\in K}\Phi\left(\frac{\|\theta(T)\|_{\Omega^k_\infty(T)}}{\lambda\binom{n+1}{k+1}^2}\right)\le\sum_{T\in K}\Phi\left(\frac{\binom{\dim T+1}{k+1}\sum\limits_{0\le i_0<\dots<i_k\le\dim T}|c([e_{i_0},\dots,e_{i_k}])|}{\binom{n+1}{k+1}^2\lambda}\right)$$
$$\le\sum_{T\in K}\Phi\left(\frac{\sum\limits_{0\le i_0<\dots<i_k\le\dim T}|c([e_{i_0},\dots,e_{i_k}])|}{\binom{\dim T+1}{k+1}\lambda}\right)$$
$$\le\sum_{T\in K}\sum_{0\le i_0<\dots<i_k\le\dim T}\frac{1}{\binom{\dim T+1}{k+1}}\Phi\left(\frac{|c([e_{i_0},\dots,e_{i_k}])|}{\lambda}\right)$$
$$\le\sum_{T\in K}\sum_{0\le i_0<\dots<i_k\le\dim T}\Phi\left(\frac{|c([e_{i_0},\dots,e_{i_k}])|}{\lambda}\right)\le\overline{N}\sum_{\dim T=k}\Phi\left(\frac{|c(T)|}{\lambda}\right)$$

for some constant $\overline{N}>0$ due to the star-boundedness of K. Thus,

$$\left\|\frac{w(c)}{\binom{n+1}{k+1}^2}\right\|_{S^*_\Phi(K,L)}\le\|c\|_{C_{\overline{N}\Phi}(K,L)^*},$$

whence (cf. Lemma 1.1)

$$\|w(c)\|_{S_\Phi(K,L)}\le\text{const}\,\|c\|_{C^*_{\overline{N}\Phi}(K,L)},$$

which implies that the operator $c\mapsto w(c)$ is a bounded operator $C^k_\Phi(K,L)\to S^k_\Phi(K,L)$.

Now, let $\omega\in S^k_\Phi(K,L)$. Denote by v_k the volume of the standard k-dimensional simplex. Given a k-dimensional simplex T of K,

$$\left|\int_T\omega(T)\right|\le\|\omega(T)\|_{\Omega^k_\infty(T)}v_k.$$

Hence, for any $\lambda>0$,

$$\sum_{\dim T=k}\Phi\left(\left|\frac{I\omega(T)}{\lambda v_k}\right|\right)\le\sum_{\dim T=k}\Phi\left(\frac{\|\omega(T)\|_{\Omega^k_\infty(T)}}{\lambda v_k}v_k\right)=\sum_{\dim T=k}\Phi\left(\frac{\|\omega(T)\|_{\Omega^k_\infty(T)}}{\lambda}\right).$$

Therefore,

$$\|I\omega(T)\|_{C^k_\Phi(K,L)}\le v_k\|\omega\|_{S^k_\Phi(K,L)}.$$

This means that the operator $\omega\mapsto I\omega$ is a continuous operator $S^k_\Phi(K,L)\to C^k_\Phi(K,L)$.

Since we have two transformations $w:C^*_\Phi(K,L)\to S^*_\Phi(K,L)$ and $I:S^*_\Phi(K,L)\to C_\Phi(K,L)$ with $I\circ w=\mathrm{id}_{C^*_\Phi(K,L)}$, the induced mapping of the cohomology spaces $\varkappa:H^*(S^*_\Phi(K,L))\to H^*(C^*_\Phi(K,L))$ is an epimorphism. We need to prove that $\varkappa$ is a monomorphism. Since I is an epimorphism, for this it suffices to find for every cocycle $\theta\in S_\Phi(K,L)$ with $I\theta=0$ an S-form $\omega\in S^{k-1}_\Phi(K,L)$ with the property $d\omega=\theta$.

Let $K^{(j)}$ be the jth skeleton of K and let $K_j=L\cup K^{(j)}$. Using induction on j, construct S-forms $\omega_j\in S^{k-1}_\Phi(K,L)$, $j\ge k$, such that $d\omega_j=j^*_{K_j,K}\theta$.

Let T be the standard j-simplex. Then $S^*_\Phi(T,\partial T) = S^*(T,\partial T)$, and hence, by Sullivan's theorem,

$$H^k(S^*_\Phi(T,\partial T)) = \begin{cases} \mathbb{R}, & k=j, \\ 0, & k<j. \end{cases}$$

The isomorphism $H^j(S^*_\Phi(T,\partial T)) \cong \mathbb{R}$ is given by the mapping $\theta \mapsto \int_T \theta(T)$.

Denote by B^k the space of k-dimendional boundaries of the complex $S^*_\Phi(T,\partial T)$. The space B^k is closed in $S^k_\Phi(T,\partial T)$, and hence the mapping $d : S^{k-1}_\Phi(T,\partial T) \to B^k$ is an epimorphism of Banach spaces. By Banach's Theorem, there is a constant C satisfying the following condition: for every $\alpha \in B^k$, there exists $\beta \in S^{k-1}_\Phi(T,\partial T)$ such that

$$d\beta = \alpha \quad \text{and} \quad \|\beta\|_{S^{k-1}_\Phi(T,\partial T)} \le C\|\alpha\|_{S^k_\Phi(T,\partial T)}.$$

Turn to constructing the forms ω_j. If $T \in K_{k-1}$ then we put $\omega_k(T) = 0$. Let $T \in K_k \setminus K_{k-1}$. Since $\int_T \theta(T) = 0$ and $d\theta = 0$, we have $j^*_{T,K}\theta \in B^k$. There exists a form $\beta_T \in \Omega^{k-1}_\infty(T,\partial T) = S^{k-1}_\Phi(T,\partial T)$ such that

$$d\beta_T = j^*_{T,K}\theta \quad \text{and} \quad \|\beta_T\|_{\Omega^{k-1}_\infty(T)} \le C'\|\theta(T)\|_{\Omega^k_\infty(T)}.$$

Here the constant C' does not depend on T since all k-dimensional simplices in K are isometric to the standard simplex. Put $\omega_k(t) = \beta_T$. We have

$$\begin{aligned} \sum_{T\in K_k} \Phi\left(\frac{\|\omega_k(T)\|_{\Omega^{k-1}_\infty(T)}}{\lambda C'}\right) &= \sum_{T\in K_k\setminus K_{k-1}} \Phi\left(\frac{\|\beta_T\|_{\Omega^{k-1}_\infty(T)}}{\lambda C'}\right) \\ &\le \sum_{T\in K_k\setminus K_{k-1}} \Phi\left(\frac{C'\|\theta(T)\|_{\Omega^k_\infty(T)}}{\lambda C'}\right) \le \sum_{T\in K_k} \Phi\left(\frac{\|\theta(T)\|_{\Omega^k_\infty(T)}}{\lambda}\right). \end{aligned}$$

This implies that $\omega_k \in S^{k-1}_\Phi(K,L)$ and

$$\|\omega_k\|_{S^{k-1}_\Phi(K,L)} \le C'\|\theta\|_{S^k_\Phi(K,L)}.$$

Suppose that we have constructed a form $\omega_{j-1} \in S^{k-1}(K_{j-1},L)$ for which $d\omega_{j-1} = j^*_{K_j,K}\theta$. Using Lemma 2.4, find a form $\omega' \in S^{k-1}_\Phi(K_j,L)$ such that $j^*_{K_{j-1},K_j}\omega' = \omega_{j-1}$. If $T \in K_{j-1}$ then put $\omega''(T) = 0$. Let $T \in K_j \setminus K_{j-1}$. Then $j^*_{\partial T,K_j}(\theta - d\omega') = 0$. Therefore, $\theta - d\omega'$ is a cocycle of the complex $S^*_\Phi(T,\partial T)$. Since $H^k(S^*_\Phi(T,\partial T)) = 0$ for $k<j$, we have $\theta - d\omega' \in B^k$. There exists a form $\tilde\beta_T$ such that

$$d\tilde\beta_T = \theta - d\omega' \quad \text{and} \quad \|\tilde\beta_T\|_{\Omega^{k-1}_\infty(T,\partial T)} \le C'\|\theta - d\omega'\|_{\Omega^k_\infty(T,\partial T)}.$$

Put $\omega''(T) = \tilde\beta_T$. This gives an S-form $\omega'' \in S^{k-1}_\Phi(K_j,L)$. Let $\omega_j = \omega' + \omega''$. Then $\omega_j \in S^{k-1}_\Phi(K_j,L)$ and $d\omega_j = j^*_{K_j,K}\theta$. This finishes the induction step. Since $\dim K < \infty$, for $j \ge \dim K$ we obtain the S-form $\omega = \omega_j \in S^{k-1}_\Phi(K,L)$ for which $d\omega = 0$. Consequently, I induces an isomorphism of the cohomology groups. Both mappings I and w induce continuous mappings of the cohomology spaces. Hence, I and w induce mutually inverse isomorphisms in cohomology.

Theorem 2.5 is proved. □

Remark 2.6. In [6], the isomorphism between the ℓ_p-cohomology of a star-bounded simplicial complex and the cohomology of the corresponding L_p Sullivan complex was one of key tools in proving the L_p de Rham isomorphism. It reduced the problem to proving that the L_p-cohomology of a Riemannian manifold is isomorphic to the cohomology of the L_p Sullivan complex corresponding to the triangulation under consideration. This involves homotopy operators constructed with the use of regularization. However, in contrast to the L^p spaces, where there is practically no difference between the norm and the corresponding modular, regularization in Orlicz spaces was proved to be bounded only in norm (see [8]), and not in the strong modular sense (cf. [9]), which makes it impossible to glue up a global estimate of the Orlicz norm of a homotopy operator from existing "local" estimates. Thus, the general Orlicz–de Rham isomorphism remains an open problem. The Orlicz–de Rham isomorphism was established by Sequeira in the particular case of Lie groups (see [11, Theorem 1.5]).

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