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RECOVERY OF A VECTOR FIELD IN THE CYLINDER BY ITS JOINTLY KNOWN NMR IMAGES AND RAY TRANSFORMS

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Abstract. In the paper we consider a problem of recovering a 3D vector field given in cylinder by means of jointly known nuclear magnetic resonance (NMR) images and ray transforms. The NRM images and 2D longitudinal and transverse ray transforms are known in every plane orthogonal to the cylinder axis. The 3D ray transforms of new type connected with a family of the parallel planes are defined. Simulation confirms the legitimacy and further perspective of the proposed approach.

Keywords: vector field, cylindrical domain, NMR image, ray transform, inversion formula, boundary value problem, numerical simulation.

1. Introduction

The approach of reducing a problem of recovery of a function of three variables to a series of recovery problems of the functions depending on two variables is well known in tomography. The essence of this approach contains in the term “tomography” (tomos — Greek “slice”) by itself. This approach has also developed and for solving the problems of vector and tensor tomography. Thus, the author of paper [1] offered a method of slice-by-slice reconstruction of a solenoidal vector and symmetric 2-tensor fields defined in a ball by longitudinal ray transforms. Further, the method was modified and numerically realized, [2]–[4]. The methods of recovery of a function in a cylindrical area by its geodesic ray transforms and Riemannian

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metrics of a special kind were suggested and numerically realized in [5], [6]. A number of papers have proposed to use instead of ray transforms of vector fields their nuclear magnetic resonance (NMR) images (see, for example, [7]).

We consider the problem of reconstruction of an arbitrary 3D vector field by its jointly known NMR images, longitudinal and transverse 2D ray transforms given for every section orthogonal to the cylinder axis. Both solenoidal and arbitrary vector fields containing a potential part are taken as sought-for ones.

1.1. **Magnetic resonance imaging.** The magnetic resonance imaging (MRI) is a nondestructive method of the investigation of the inner structure of objects widely used in medical diagnostics. MRI is based on a phenomenon of NMR that involves reorientation of magnetic moments of atomic nuclei with a nonzero spin in an exterior magnetic field. The most representative element in a chemical composition of a live organism is hydrogen, whose nucleus has a nonzero spin. Hence this fact provides a necessary condition for application of MRI in studies on different kinds of biotissues and bioliquids. A specialized method of MRI was developed for the investigation of the circulatory system of live organisms and is called magnetic resonance angiography (angio — Greek "vessel") (MRA).

With usage of the general approach to recovering the vascular networks by MRA data [8] the object under investigation is scanned by a set of parallel planes orthogonal to the direction in which the majority of the main vessels lie (Fig. 1a). The obtained images of sections make up a data packet (Fig. 1a). Fig. 1b and Fig. 1c demonstrate real tomographic slices of parts of the head of a laboratory mouse. As a result, a set of sections of the object is obtained, and the blood flow is registered by the normal to the section of the scanning plane.

MRA-images create a 3D set of data with information about presence or absence of the blood flow orthogonal to the slice plane. Due to magnetization saturation the stationary tissues are visualized in a hypo-intense way, while moving liquids (including the blood in particular) possessing absolute magnetization are visualized hyper-intensively. From mathematical point of view, the tomographic data $A(x, y, z)$ (brightness at a point) is the dependence

$$A(x, y, z) = F(\mathbf{v}(x, y, z), \nu)$$

on the vector $\mathbf{v}(x, y, z)$ of the velocity of the blood flow at a point (x, y, z) and on the vector of the normal ν to the scanning plane. The function $F(\mathbf{v}(x, y, z), \nu)$ is the value of brightness at the point (x, y, z) of the slice and takes small values as $\langle \mathbf{v}(x, y, z), \nu\rangle \approx 0$. There are two cases that correspond to a low intensity in the slice images:

1) the flow is absent or too low in a given section: $\mathbf{v} \approx 0$;
2) the vectors of velocity $\mathbf{v}$ and normal ν are almost orthogonal.

In the papers [9], [10], a method providing a possibility to overcome the absence of information which arises due to non-collinearity of the vectors $\mathbf{v}$ and ν is described. Thus, it is possible to construct a vessel chain that is more informative compared to the chain obtained using a single set of parallel planes.

1.2. **Vector tomography.** The Doppler effect is the main physical phenomenon allowing to detect the vector characteristics of the medium. Thus, in [11] a possibility of reconstruction of a 3D function of distribution $F(v_1, v_2, v_3) = F(v)$ of molecules with respect to their velocities using the method of Doppler spectroscopy

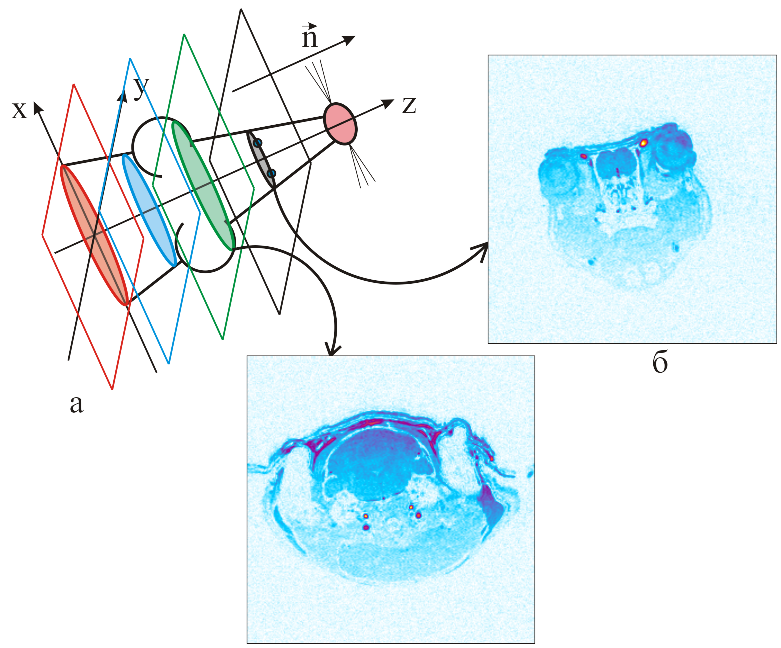

FIG. 1. The scanned object is intersected by a set of planes

is described. The profile of the absorption band $D(w, \theta, \phi)$ registered in the experiment is defined by the integral

$$D(w, \theta, \phi) = \int d^3 v F(v) \delta(w - \langle n\,,\, v \rangle), \tag{1}$$

where the polar and the azimuth angles (θ, ϕ) determine the direction of observation n. The integral (1) is the Radon transform in the space of velocities of a distribution function of particles velocities $F(v)$ in the given point (x, y, z). Given the fixed angles θ, ϕ, the measured profile $D(w)$ is a set of integrals over the planes with a fixed direction of normal n. Therefore, in the general case there arises a 6-dimensional inverse problem on defining a function of three variables $F_r(v)$ at every point $r \in \mathbb{R}^3$ of the space.

It is important to note that the simplified formulation is considered in the vast majority of problems of vector tomography. In particular, the function of particle velocity distribution is replaced by the average velocity at every point of the space, and, as a result, there arises a problem of defining a vector field of average velocities by its longitudinal ray transform.

Paper [12] describes a method of reconstruction of a velocity field of a liquid flow in the cylinder using acoustic Doppler measurements. The goal of such studies is to model the measurements of velocity field of blood in vessels. The assumptions about incompressibility of a liquid and absence of sources in it lead to a conclusion about the solenoidality of the sought-for field which have to be reconstructed by the longitudinal ray transform known at every point of the boundary of the cylinder and any vector of direction.

In papers [9], [10], data replenishment is realized using variation of a slope angle of the scanning plane. In our work, the replenishment of the missing data is realized by the integral ray transforms. The values of ray transforms aim to compensate for the lack of the given information, moreover, it suffices to use 2D ray transforms given at every scanning planes.

The operators of longitudinal and transverse ray transforms acting on vector fields possess nonzero kernels [13]. Hence, it is only possible to recover the solenoidal part of the field by its longitudinal ray transform, and its potential part by its transverse ray transform. The recovery of the entire field may be possible only in the case of jointly known transforms. Guided by [13], [14], we will provide the information about 2D vector fields and their ray transforms which will be further needed.

2. Preliminary information

Let $B=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2<1\}$ be a unit disk, $\partial B=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\}$ be a unit circle. Along with the introduced denotations (x,y), for the coordinates of points on $\mathbb{R}^2$ we sometimes use the index denotation (x^1,x^2), (y^1,y^2), ...

Let a function $\varphi(x,y)$ belongs to C^k, k is an integer, $k\geq -1$ and is defined in B, moreover, it equals 0 on the set $D=\bar{B}\setminus \operatorname{supp}\varphi$. The function $\varphi(x,y)$ is infinitely differentiable at the points $(x,y)\in D$, and it is continuously differentiable up to k-th order inclusively at the boundary points of the support $\operatorname{supp}\varphi$. Moreover, if $(x,y)\in\partial D$, then the function φ and all its derivatives up to k-th order inclusively equals 0, and the derivatives of order $k+1$ have a discontinuity of the first kind. In particular, if the function f has the discontinuity of the first kind at the points $(x,y)\in\partial D$, then we put $k=-1$ and write $f\in C^{-1}(B)$. Sometimes we say that $\varphi\in C^k(B)$ is the *C^k-potential* on $\mathbb{R}^2$.

Along with the C^k-potentials, the same way will be chosen for definition of the fields $u=(u_j)$, $v=(v_j)$, ..., $j=1,2$ which are C^k-vector on $\mathbb{R}^2$. Therefore, the *C^k-vector field* is a field (u_1,u_2) such that $u_j\in C^k(B)$, $j=1,2$. We denote the set of such fields by $C^k(S^1(B))$.

Operators of gradient and orthogonal gradient ∇, $\nabla^\perp : C^k(B)\to C^{k-1}(S^1(B))$ are defined in coordinates by the formulas

$$(\nabla\,\varphi)_k=\frac{\partial\varphi}{\partial x^k},\quad \nabla\,\varphi=\Big(\frac{\partial\varphi}{\partial x^1}\,,\,\frac{\partial\varphi}{\partial x^2}\Big),$$

$$(\nabla^\perp\psi)_k=(-1)^k\frac{\partial\psi}{\partial x^{(3-k)}},\quad \nabla^\perp\psi=\Big(-\frac{\partial\psi}{\partial x^2}\,,\,\frac{\partial\psi}{\partial x^1}\Big).$$

Operators of divergence and orthogonal divergence δ, $\delta^\perp : C^k(S^1(B))\to C^{k-1}(B)$ are defined in coordinates by the formulas

$$\delta u=\frac{\partial u_1}{\partial x^1}+\frac{\partial u_2}{\partial x^2},\quad \delta^\perp u=-\frac{\partial u_1}{\partial x^2}+\frac{\partial u_2}{\partial x^1}.$$

The vector field $u\in C^k(S^1(B))$ is potential if there exists a potential $\chi\in C^{k+1}(B)$ such that $u=\nabla\chi$. The field $v\in C^k(S^1(B))$ is solenoidal if its divergence $\delta v\in C^{k-1}(B)$ equals to zero.

We denote by $H^1(S^1(B))$ the Sobolev space of vector fields whose components are integrable with a square together with their first derivatives. It is known [15], [16], that every vector field $w\in H^1(S^1(B))$ can be decomposed into a sum of its potential $\nabla\psi$, solenoidal v, and harmonic ∇h parts,

$$w = v + \nabla h + \nabla\varphi, \quad \delta v = 0, \quad \varphi\,|_{\partial B} = 0, \quad \langle v, \nu\rangle\,|_{\partial B} = 0, \tag{2}$$

where h is a harmonic function in B, ν is an exterior normal to the boundary ∂B, $v \in H^1(S^1(B))$, $\varphi \in H^2(B)$, where $H^2(B)$ is the Sobolev space of functions integrable with a square together with their second derivatives. The decomposition (2) is unique. The harmonic part ∇h is both solenoidal and potential vector field. Based on physical considerations, it is natural to assume that vector fields have discontinuities (if they exist) of the first kind only. Hence, the potentials of vector fields equal zero on the set ∂D, and, therefore, harmonic vector fields are absent in the decomposition (2), because the function that is harmonic in the disk B cannot be equal zero on the boundary ∂B. Moreover, it is known [14] that a solenoidal 2D field v is representable in the form $v = \nabla^\perp\psi$ for some potential $\psi \in C^k(B)$, $k = 0, 1, \ldots$. Therefore, the decomposition of a vector field can be represented in the following way:

$$w = \nabla^\perp\psi + \nabla\varphi, \quad \varphi,\ \psi\,|_{\partial B} = 0. \tag{3}$$

2.1. **Ray transforms and inversion formulas.** A straight line $L(\xi, s)$ is defined by the unit normal vector $\xi \in \partial B$, $\xi = (\xi^1, \xi^2) = (\cos\alpha, \sin\alpha)$ and the parameter s, $-1 \le s \le 1$, where $|s|$ is a distance from the line to the origin. The direction vector of the line $L(\xi, s)$ is $\eta = \xi^\perp = \big((\xi^\perp)^1, (\xi^\perp)^2\big) = (-\sin\alpha, \cos\alpha)$. Thus, every line of a parallel bundle of lines is uniquely determined by the vector ξ and the real number $s \in \mathbb{R}$. Parametrical definition of the line $L(\xi, s)$ is as follows: $x = s\cos\alpha - t\sin\alpha$, $y = s\sin\alpha + t\cos\alpha$. Multiplying the first equation by $\cos\alpha$, the second one by $\sin\alpha$, and summarizing, we obtain the formula $s = s(x, y, \alpha) = x\cos\alpha + y\sin\alpha$ with dependence of s on x, y, α.

Let $f(x, y)$ be a potential of C^k, $k \ge 1$, $\operatorname{supp} f \subset B$. The Radon transform of the potential f is defined by the relation

$$\big(\mathcal{R}f\big)(\xi, s) = \int\limits_{L(\xi,s)} f(x, y)dL = \int\limits_{-\infty}^{\infty} f(s\xi + t\eta)dt,$$

where $\mathcal{R} : f \to g \equiv \mathcal{R}f$, $g(\xi, s) \in C^k(Z)$, $Z = \{(\xi, s) \mid \xi \in \partial B\,, |s| < 1\}$ is a cylinder, $C^k(Z)$ is a set of functions g, differentiable with their derivatives up to k-th order inclusively, up to the boundary $\overline{Z}\setminus Z$. Below, we use a rule of summation which requires to sum up from 1 to 2 with respect to the repeating upper and lower indices of the same monomial.

The longitudinal ray transform $\mathcal{P} : C^k(S^1(B)) \to C^k(Z)$ maps the C^k-vector field v into the function $g \in C^k(Z)$,

$$\big(\mathcal{P}v\big)(\eta, s) = \int\limits_{-\infty}^{\infty} \langle v(s\xi + t\eta), \eta\rangle dt = \int\limits_{-\infty}^{\infty} \big(v_1\eta^1 + v_2\eta^2\big)dt. \tag{4}$$

The transverse ray transform $\mathcal{P}^\perp : C^k(S^1(B)) \to C^k(Z)$, $u \in C^k(S^1(B))$ is defined by the formula

$$\big(\mathcal{P}^\perp u\big)(\xi, s) = \int_{-\infty}^{\infty} \langle u(s\xi + t\eta), \xi\rangle dt = \int_{-\infty}^{\infty} \big(u_1\xi^1 + u_2\xi^2\big)dt. \tag{5}$$

The main properties and relations between vector fields, differential operators and ray transforms [13] are listed below.

Proposition 1. *Let $v \in C^k(S^1(B))$ be a solenoidal vector field, $u \in C^k(S^1(B))$ be a potential one, k is an integer, $k \geq -1$.*

1. There exist functions $\varphi, \psi \in C^{k+1}(B)$ such that $v = \nabla^\perp \psi$, $u = \nabla\varphi$. Then for $k \geq 1$,

$$\delta(\nabla\psi) = \Delta\psi, \quad \delta(\nabla^\perp\psi) = 0, \quad \delta^\perp(\nabla^\perp\varphi) = \Delta\varphi, \quad \delta^\perp(\nabla\varphi) = 0.$$

2. The ray transforms $\mathcal{P}$, $\mathcal{P}^\perp$ of the fields u, $v \in C^k(S^1(B))$ possess nonzero kernels and are connected with the Radon transforms of their potentials φ, $\psi \in C^{k+1}(B)$,

$$(\mathcal{P}u)(\eta, s) \equiv 0, \quad \big(\mathcal{P}v\big)(\eta, s) = \frac{\partial}{\partial s}\mathcal{R}\psi(\xi, s),$$

$$(\mathcal{P}^\perp v)(\xi, s) \equiv 0, \quad \big(\mathcal{P}^\perp u\big)(\xi, s) = \frac{\partial}{\partial s}\mathcal{R}\varphi(\xi, s).$$

Below we restrict ourselves by the class $C^k(B)$ of potentials (at suitable k) and use the decomposition (3) for vector fields. Let the longitudinal (4) and transverse (5) transforms of a vector field w be given. These transforms can be written in the form of the following equations,

$$\begin{aligned} \mathcal{P}w &= \eta^1 \cdot \mathcal{R}w_1 + \eta^2 \cdot \mathcal{R}w_2 = -\sin\alpha \cdot \mathcal{R}w_1 + \cos\alpha \cdot \mathcal{R}w_2, \\ \mathcal{P}^\perp w &= \xi^1 \cdot \mathcal{R}w_1 + \xi^2 \cdot \mathcal{R}w_2 = \cos\alpha \cdot \mathcal{R}w_1 + \sin\alpha \cdot \mathcal{R}w_2, \end{aligned} \tag{6}$$

where $\mathcal{R}w_1$, $\mathcal{R}w_2$ are unknown variables. The solution of this system is given by the expressions

$$\begin{aligned} \mathcal{R}w_1 &= \eta^1 \cdot \mathcal{P}w + \eta^2 \cdot \mathcal{P}^\perp w = -\sin\alpha \cdot \mathcal{P}w + \cos\alpha \cdot \mathcal{P}^\perp w, \\ \mathcal{R}w_2 &= \xi^1 \cdot \mathcal{P}w + \xi^2 \cdot \mathcal{P}^\perp w = \cos\alpha \cdot \mathcal{P}w + \sin\alpha \cdot \mathcal{P}^\perp w \end{aligned} \tag{7}$$

for the Radon transforms $\mathcal{R}w_1$, $\mathcal{R}w_2$ of components of the sought-for vector field w depending on the known ray transforms $\mathcal{P}w$ and $\mathcal{P}^\perp w$. Applying any of the numerous inversion formulas to both sides of the obtained expressions we get the components w_1, w_2 of the sought-for field.

If we know only one of the transforms, for example the longitudinal ray transform (4) of a solenoidal vector field v, then using Proposition 1 we obtain the expressions

$$\mathcal{R}v_1 = \eta^1\mathcal{P}w \equiv -\sin\alpha \cdot \mathcal{P}w, \quad \mathcal{R}v_2 = \eta^2\mathcal{P}w \equiv \cos\alpha \cdot \mathcal{P}w$$

for the Radon transforms of the components of the solenoidal field v. Again, we can use any of the known inversion formulas for the Radon transform (see, for example, [17]).

3. Recovery of a field in a cylinder

We denote by Z a cylinder, $Z = \{x \in \mathbb{R}^3 \,|\, (x^1)^2 + (x^2)^2 \leq 1,\ -1 \leq x^3 \leq 1\}$. Along with the divergence δ acting on the vector field defined on $\mathbb{R}^3$ we define its constrictions δ_{jk} acting on the vector fields $p = (p_1, p_2, p_3)$ given in the planes parallel to the coordinate ones,

$$\delta_{jk}p = \frac{\partial p_j}{\partial x^j} + \frac{\partial p_k}{\partial x^k}, \quad j, k, = 1, 2, 3, \quad j < k. \tag{8}$$

The operator $\delta^\perp$ is not defined in $\mathbb{R}^3$, but we can refer to the operators $\delta^\perp_{jk}$ acting on the vector fields defined in the planes parallel to the coordinate ones,

$$\delta^\perp_{jk} p = -\frac{\partial p_j}{\partial x^k} + \frac{\partial p_k}{\partial x^j}, \quad j,k,=1,2,3, \quad j<k. \tag{9}$$

Consider the definitions (8), (9). The coordinate of the vector field absent in right-hand parts of the equalities is not taken into consideration. The variable which does not take part in those is assumed as a constant with absolute value $|x^l| = \mathrm{const}, l \neq j,k$ being a distance from the plane to the origin. The equivalency of denotations of the coordinates $x^1 \leftrightarrow x$, $x^2 \leftrightarrow y$, $x^3 \leftrightarrow z$ is assumed, and below we use those which are convenient in a particular situation.

3.1. **Solenoidal field.** Let a solenoidal field $v=(v_1,v_2,v_3)$ be given in the cylinder Z. This means that $\delta v = 0$ and moreover there exists a potential being a vector field $w=(w_1,w_2,w_3)$ such that $v = \operatorname{rot} w$, that is,

$$v = \Big(\frac{\partial w_3}{\partial y} - \frac{\partial w_2}{\partial z}, \frac{\partial w_1}{\partial z} - \frac{\partial w_3}{\partial x}, \frac{\partial w_2}{\partial x} - \frac{\partial w_1}{\partial y}\Big) \tag{10}$$

We assume that the values $v_3(x,y,z)$ are known for (x,y,z) such that $x^2+y^2 \leq 1$, $-1 \leq z \leq 1$. Applying the operator δ to (10) we obtain

$$\delta v = \frac{\partial^2 w_3}{\partial x \partial y} - \frac{\partial^2 w_2}{\partial x \partial z} + \frac{\partial^2 w_1}{\partial y \partial z} - \frac{\partial^2 w_3}{\partial x \partial y} + \frac{\partial^2 w_2}{\partial x \partial z} - \frac{\partial^2 w_1}{\partial y \partial z} = 0.$$

It is clear that

$$\delta_{12} v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = -\frac{\partial}{\partial z} v_3.$$

Note that if v_3 is known at every cross-section of the cylinder then its derivative $\dfrac{\partial v_3}{\partial z}$ with respect to z can also be calculated, therefore below we assume that it is known.

On the other hand, by the theorem about the decomposition of a 2D vector field (3) we have that

$$\widetilde{v} = \nabla^\perp \psi + \nabla \varphi = \Big(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}\Big) + \Big(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}\Big),$$

where $\widetilde{v}$ is a 2D vector field $\widetilde{v} = (v_1, v_2)$, $z = \mathrm{const}$, being a projection of the field v on the plane OXY. Hence,

$$\delta_{12} v = -\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y \partial x} + \frac{\partial^2 \varphi}{\partial y^2} = \Delta_{12}\varphi,$$

where Δ_{jk} is the Laplace operator acting on functions of the variables x^j, x^k. In particular, the operator $\Delta_{12} \equiv \dfrac{\partial^2}{\partial (x^1)^2} + \dfrac{\partial^2}{\partial (x^2)^2} \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ acts on the restrictions of the functions $f(x,y,z)$ in the planes parallel to the coordinate plane OXY. We have obtained a homogeneous Dirichlet boundary value problem for the Poisson equation. The potential φ of the potential part $\nabla\varphi$ of the vector field $\widetilde{v}$ is a solution of this problem,

$$\begin{cases} \Delta_{12}\varphi(x,y) = -\dfrac{\partial}{\partial z} v_3(x,y,z)\,|_{z=\mathrm{const}}, \\ \varphi\,|_{\partial B} = 0. \end{cases}$$

As for the solenoidal part $\nabla^{\perp}\varphi$ of the 2D vector field $\widetilde{v}$, it can be uniquely reconstructed by its known longitudinal ray transform $\mathcal{P}$ (at $z = \text{const}$). Remind (Proposition 1) that potential fields with a potential equals zero on the boundary of the disk $x^2 + y^2 \leq 1$ belong to the kernel of the operator $\mathcal{P}$.

Consider the action of the operator $\delta_{12}^{\perp}$ on the vector field v,

$$\begin{aligned}\delta_{12}^{\perp}v &= -\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x} = -\frac{\partial}{\partial y}\Big(\frac{\partial w_3}{\partial y} - \frac{\partial w_2}{\partial z}\Big) + \frac{\partial}{\partial x}\Big(\frac{\partial w_1}{\partial z} - \frac{\partial w_3}{\partial x}\Big) \\ &= -\Delta_{12}w_3 + \frac{\partial}{\partial z}\Big(\frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y}\Big) = -\Delta_{12}w_3 + \frac{\partial}{\partial z}\delta_{12}w.\end{aligned}$$

On the other hand, $\widetilde{v} = \nabla^{\perp}\psi + \nabla\varphi$, and then $\delta_{12}^{\perp}v = \Delta_{12}\psi$. We obtain the relations

$$\delta_{12}^{\perp}v = \Delta_{12}\psi = -\Delta_{12}w_3 + \frac{\partial}{\partial z}\delta_{12}w.$$

One of them connects the potential ψ of the solenoidal part of the field $\widetilde{v}$ with its orthogonal divergence. The other one connects ψ with the second derivatives of the vector potential $w = (w_1, w_2, w_3)$. Therefore, we have obtained the following proposition. By the boundary ∂Z of the cylinder Z we mean its lateral area. We do not set boundary conditions on the parts of the boundary $Z \cap \{z = \pm 1\}$.

Proposition 2. *Let the vector field v be solenoidal in Z, $\delta v = 0$. The scalar potentials $\varphi, \psi \in C^1(B)$ are such that $\widetilde{v} = (v_1, v_2) = \nabla^{\perp}\psi + \nabla\varphi$, $\varphi\,|_{\partial B} = 0$, $\psi\,|_{\partial B} = 0$, $B = Z \cap \{z = const, |const| < 1\}$. The vector potential $w \in C^1(Z)$ is such that $v = \mathrm{rot}\, w$, $w\,|_{\partial Z} = 0$. Then at $z = const$, we have that*

$$\begin{cases} \Delta_{12}\varphi(x,y) = -\dfrac{\partial}{\partial z}v_3(x,y,z)\,|_{z=const}, \\ \varphi\,|_{\partial B} = 0, \end{cases} \tag{11}$$

$$\delta_{12}v = \Delta_{12}\varphi = -\frac{\partial v_3}{\partial z} = -\frac{\partial}{\partial z}\big(\delta_{12}^{\perp}w\big), \tag{12}$$

$$\delta_{12}^{\perp}v = \Delta_{12}\psi = -\Delta_{12}w_3 + \frac{\partial}{\partial z}\big(\delta_{12}w\big). \tag{13}$$

Corollary 1. *The following statements hold under the conditions of Proposition 2.*

— If the component v_3 of the field v does not depend on the variable z, then the 2D vector field $\widetilde{v}$ is solenoidal, $\delta_{12}v = 0$.

— The 2D vector field $\widetilde{v}$ is potential if and only if one of the following conditions is fulfilled:
1) the potential ψ is identically zero in B;
2) w_3 is a harmonic function in Z, and δw does not depend on z;
3) $-\Delta w_3 + \dfrac{\partial}{\partial z}\big(\delta w\big) = 0$ is valid identically in Z.

— The 2D vector field $\widetilde{v}$ is solenoidal if and only if one of the following conditions is fulfilled:
1) the potential φ is identically zero in B;
2) the component v_3 of the field v does not depend on z;
3) the 2D field $\widetilde{w}$ is potential, or $\delta_{12}^{\perp}w$ does not depend on z.

Consider a solenoidal field q, $q = \mathrm{rot} v$, where $v = \mathrm{rot} w$. We will express the field q in terms of w,

$$\begin{aligned} q \quad &= \Big(-\Delta_{23} w_1 + \frac{\partial}{\partial x}\Big(\frac{\partial w_2}{\partial y} + \frac{\partial w_3}{\partial z}\Big)\,,\, -\Delta_{13} w_2 + \frac{\partial}{\partial y}\Big(\frac{\partial w_1}{\partial x} + \frac{\partial w_3}{\partial z}\Big)\,, \\ &\quad -\Delta_{12} w_3 + \frac{\partial}{\partial z}\Big(\frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y}\Big)\Big) \\ &= \Big(-\Delta_{23} w_1 + \frac{\partial}{\partial x}(\delta_{23} w)\,,\, -\Delta_{13} w_2 + \frac{\partial}{\partial y}(\delta_{13} w)\,,\, -\Delta_{12} w_3 + \frac{\partial}{\partial z}(\delta_{12} w)\Big). \end{aligned}$$

Consider the action of the operator δ_{12} on the vector field q. Since $\delta q = 0$, we have that

$$\delta_{12} q = \frac{\partial}{\partial z}(\Delta_{xy} w_3) - \frac{\partial^2}{\partial z^2}\Big(\frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y}\Big) = \frac{\partial}{\partial z}\Big(\Delta_{12} w_3 - \frac{\partial}{\partial z}(\delta_{12} w)\Big).$$

Using trivial transformations we can obtain the following formulas,

$$\delta_{12} q = \frac{\partial^2}{\partial y \partial z}\Big(\frac{\partial w_3}{\partial y} - \frac{\partial w_2}{\partial z}\Big) - \frac{\partial^2}{\partial x \partial z}\Big(\frac{\partial w_1}{\partial z} - \frac{\partial w_3}{\partial x}\Big) = -\frac{\partial}{\partial z}(\delta_{12}^{\perp} v) = -\frac{\partial}{\partial z} q_3.$$

On the other hand since the 2D vector field $\widetilde{q}$ can be represented in the form $\widetilde{q} = \nabla^{\perp}\psi + \nabla\varphi$, we have that

$$\delta_{12} q = \frac{\partial}{\partial x}\Big(-\frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(\frac{\partial \psi}{\partial x} + \frac{\partial \varphi}{\partial y}\Big) = \Delta_{12}\varphi.$$

Acting by the operator $\delta_{12}^{\perp}$ on the vector field q we get the relations

$$\begin{aligned} \delta_{12}^{\perp} q \quad &= -\frac{\partial}{\partial y}\Big(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\Big) + \frac{\partial}{\partial x}\Big(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\Big) \\ &= -\Delta_{12} v_3 + \frac{\partial}{\partial z}\Big(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}\Big) = -\Delta_{12} v_3 + \frac{\partial}{\partial z}(\delta_{12} v) \\ &= -\Delta(\delta_{12}^{\perp} w) = -\Delta v_3. \end{aligned}$$

By means of the representation $\widetilde{q} = \nabla^{\perp}\psi + \nabla\varphi$ we obtain that $\delta_{12}^{\perp} q = \Delta_{12}\psi$.

Proposition 3. *Let the vector field q be solenoidal in Z, $\delta q = 0$. The potentials $\varphi, \psi \in C^1(B)$ are such that $\widetilde{q} = (q_1, q_2) = \nabla^{\perp}\psi + \nabla\varphi$, $\varphi\,|_{\partial B} = 0$, $\psi\,|_{\partial B} = 0$, $B = Z \cap \{z = const, |const| < 1\}$. The vector potential $w \in C^2(S^1(Z))$ is such that $q = rot\, rot\, w$, $w\,|_{\partial Z} = 0$, $v = rot\, w\,|_{\partial Z} = 0$. Then if $z = const$,*

$$\begin{cases} \Delta_{12}\varphi(x,y) = -\dfrac{\partial}{\partial z} q_3(x,y,z)\,|_{z=const}, \\ \varphi\,|_{\partial B} = 0, \end{cases}$$

$$\begin{aligned} \delta_{12} q = \Delta_{12}\varphi = -\frac{\partial}{\partial z}(\delta_{12}^{\perp} v) = \frac{\partial}{\partial z}\Delta_{12} w_3 - \frac{\partial^2}{\partial z^2}(\delta_{12} w), \\ \delta_{12}^{\perp} q = \Delta_{12}\psi = -\Delta_{12} v_3 + \frac{\partial}{\partial z}(\delta_{12} v). = -\Delta(\delta_{12}^{\perp} w). \end{aligned}$$

Corollary 2. *The following statements hold under the conditions of Proposition 3.*

— If the component q_3 of the field q does not depend on the variable z, then the 2D vector field $\widetilde{q}$ is solenoidal, $\delta_{12} q = 0$.

— the 2D vector field $\widetilde{q}$ is potential if and only if one of the following conditions is fulfilled:
1) the potential $\psi \equiv 0$ in B;
2) the value $-\Delta v_3 + \dfrac{\partial}{\partial z}\big(\delta v\big)$ equals zero in Z;
3) the orthogonal divergence $\delta_{12}^{\perp} w$ of the field w equals zero in B or it is a harmonic function in Z.

— The 2D vector field $\widetilde{q}$ is solenoidal if and only if one of the following conditions is fulfilled:
1) the potential $\varphi \equiv 0$ in B;
2) the orthogonal divergence $\delta_{12}^{\perp} v$ of the field v does not depend on the variable z, or the field $\widetilde{v}$ is potential;
3) the value $\Delta w_3 - \dfrac{\partial}{\partial z}\big(\delta w\big)$ does not depend on z or equals zero in Z.

3.2. **Arbitrary field.** We now provide another formulation of the problem under consideration abstracting from the way of obtaining one of the components of the arbitrary vector field p containing both solenoidal and potential parts.

We consider a rectangular Cartesian coordinate system and a family of planes $z = d$, $d \in \mathbb{R}$, parallel to the coordinate plane OXY and denoted by P_d in $\mathbb{R}^3$. Every plane of the family possesses a normal $\zeta = (0, 0, 1)$ and is separated from the origin at the distance $|d|$. A coordinate system defined at every plane is induced by the coordinate system of the coordinate plane OXY. Therefore, the points M of the plane P_d have the coordinates (x, y, d). Find the relation between the three types of ray transforms of vector fields and the family of planes P_d. We define the three orthonormal vectors $\xi = (\cos\alpha, \sin\alpha, 0)$, $\eta = (-\sin\alpha, \cos\alpha, 0)$, $\zeta = (0, 0, 1)$. The vectors ξ, η are coplanar to every plane of the family P_d, the vector ζ is orthogonal to it. Consider the vector field $w \in C^k(S^1)$. We define three types of ray transforms of the field w connected with the family of planes P_d. Those types include the well-known 2D longitudinal one, with a modified definition

$$\big(\mathcal{P}_3 w\big)(d, \eta, s) = \int\limits_{-\infty}^{\infty} \langle w(d\zeta + s\xi + t\eta), \eta\rangle dt = \int\limits_{-\infty}^{\infty} w_j \eta^j\, dt, \tag{14}$$

and the transverse one, defined by the formula

$$\big(\mathcal{P}_3^{\perp} w\big)(d, \xi, s) = \int\limits_{-\infty}^{\infty} \langle w(d\zeta + s\xi + t\eta), \xi\rangle dt = \int\limits_{-\infty}^{\infty} w_k \xi^k dt. \tag{15}$$

The third type of ray transform which we refer to as *normal* is defined by the formula

$$\big(\mathcal{P}_3^{\ddagger} w\big)(d, \zeta, s) = \int\limits_{-\infty}^{\infty} \langle w(d\zeta + s\xi + t\eta), \zeta\rangle dt = \int\limits_{-\infty}^{\infty} w_j \zeta^j\, dt. \tag{16}$$

Obviously, the normal ray transform "cuts out" the third component of the field $w = (w_1, w_2, w_3)$. The kernels of thus defined ray transforms can be easily described.

The kernel of the longitudinal ray transform (14) consists of 3D vector fields $U = (u_1, u_2, u_3)$ with components $u_j(x, y, d)$, $j = 1, 2, 3$ depending on d as the parameter; $u_k(x,y,d) = \dfrac{\partial\varphi}{\partial x^k}$, $k = 1,2$, $\varphi \in C^m(B)$, $m \geq 0$ is an integer. The third component $u_3(x,y,d)$ of the field U is arbitrary, $u_3 \in C^m(B)$. Similarly, the kernel of the transverse ray transform (15) consists of 3D vector fields $V = (v_1, v_2, v_3)$ with components $v_j(x,y,d)$, $j = 1,2,3$ depending on d as the parameter; $v_k(x,y,d) = (-1)^k \dfrac{\partial\psi}{\partial x^(3-k)}$, $k = 1,2$, $\psi \in C^m(B)$, $m \geq 0$ is an integer. The third component $v_3(x,y,d)$ of the field V is arbitrary, $v_3 \in C^m(B)$. The kernel of the normal ray transform (16) consists of all 3D vector fields W of the form $(w_1, w_2, 0)$, $w_k(x,y,d)$, $k = 1,2$, orthogonal to the vector ζ and depending on d as the parameter.

Note that the family P_d of parallel planes can be chosen in an arbitrary way. Using transformation of the origin rotation the family may be easily reduced to the canonical one described above.

Consider a 3D field $p = v + u$ being a sum of the solenoidal $v = \operatorname{rot} w$ and the potential $u = \nabla\chi$ fields. The action of divergence on the field p leads to the relation $\delta\, p = \Delta\chi$. We assume that the third component of the field p is known, that is,

$$v_3 + u_3 = \delta_{12}^{\perp} w + \frac{\partial\chi}{\partial z} = \frac{\partial w_2}{\partial x} - \frac{\partial w_1}{\partial y} + \frac{\partial\chi}{\partial z}.$$

We act on p by the operator δ_{12},

$$\delta_{12} p = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \Delta_{12}\chi = -\frac{\partial}{\partial z} v_3 + \delta_{12} u = -\frac{\partial}{\partial z}\big(\delta_{12}^{\perp} w\big) + \Delta_{12}\chi.$$

On the other hand, $\widetilde{v} = \nabla^{\perp}\psi + \nabla\varphi$, and $\widetilde{p} = \nabla^{\perp}\psi + \nabla\varphi + \nabla_{12}\chi$. Then

$$\delta_{12} p = \delta_{12}(\nabla^{\perp}\psi) + \delta_{12}(\nabla\varphi) + \delta_{12}(\nabla_{12}\chi) = \Delta_{12}\big(\varphi + \chi\big).$$

Therefore, we obtain

$$\Delta_{12}(\varphi + \chi) = -\frac{\partial}{\partial z}\big(\delta_{12}^{\perp} w\big) + \Delta_{12}\chi,$$

or

$$\Delta_{12}\varphi = -\frac{\partial}{\partial z}\big(\delta_{12}^{\perp} w\big) = -\frac{\partial}{\partial z} v_3.$$

We can see that this relation does not contain the potential χ. Using slightly different way of considerations and saving the known component $v_3 + u_3$ of the sought-for field we obtain the relation

$$\Delta_{12}\varphi + \frac{\partial^2}{\partial z^2}\big(-\chi\big) = -\frac{\partial}{\partial z}\big(v_3 + u_3\big), \tag{17}$$

with known right-hand side. It is incomprehensible how to interpret the operators in its left-hand side, which are actually acting on different potentials. If we transfer the second derivative with respect to z to the right-hand side on the left we obtain the Laplace operator but at the same time the right-hand side becomes partly unknown.

We now check the action of the operator $\delta_{12}^{\perp}$ on the field p,

$$\begin{aligned}
\delta_{12}^{\perp} p \quad &= -\frac{\partial p_1}{\partial y} + \frac{\partial p_2}{\partial x} = -\frac{\partial}{\partial y}\Big(v_1 + \frac{\partial\chi}{\partial x}\Big) + \frac{\partial}{\partial x}\Big(v_2 + \frac{\partial\chi}{\partial y}\Big) \\
&= -\frac{\partial}{\partial y}\Big(\frac{\partial w_3}{\partial y} - \frac{\partial w_2}{\partial x}\Big) + \frac{\partial}{\partial x}\Big(\frac{\partial w_1}{\partial z} - \frac{\partial w_3}{\partial x}\Big) \\
&= -\frac{\partial^2 w_3}{\partial y^2} + \frac{\partial^2 w_2}{\partial x \partial y} + \frac{\partial^2 w_1}{\partial x \partial z} - \frac{\partial^2 w_3}{\partial x^2} = -\Delta_{12} w_3 + \frac{\partial}{\partial z}\big(\delta_{12} w\big).
\end{aligned}$$

On the other hand,

$$\begin{aligned}\delta_{12}^{\perp}p &= \delta_{12}^{\perp}\big(\nabla^{\perp}\psi + \nabla(\varphi+\chi)\big)\\ &= -\frac{\partial}{\partial y}\Big(-\frac{\partial\psi}{\partial y} + \frac{\partial(\varphi+\chi)}{\partial x}\Big) + \frac{\partial}{\partial x}\Big(\frac{\partial\psi}{\partial x} + \frac{\partial(\varphi+\chi)}{\partial y}\Big)\\ &= \frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2(\varphi+\chi)}{\partial x\partial y} + \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2(\varphi+\chi)}{\partial x\partial y} = \Delta_{12}\psi,\end{aligned}$$

and therefore,

$$\Delta_{12}\psi = -\Delta_{12}w_3 + \frac{\partial}{\partial z}\big(\delta_{12}w\big).$$

The obtained relations including (17) do not lead to the homogeneous Dirichlet boundary value problem for the Poisson equation with a known right-hand side, formulated in Proposition 2. Two of the relations coincide with (12), (13) and do not contain neither the potential χ nor the components of the potential field u generated by them. Although the relation (17) contains the known component of the required vector field but it is impossible to be treated as a boundary value problem with a known right-hand side of the type similar to the ones which have arisen earlier in the case of the sought-for field being a solenoidal one. That means that for recovering an arbitrary field we need to use the values of both longitudinal and transverse ray transforms.

4. Algorithms and numerical experiments

The potentials (scalar and vector ones) listed below and the generated by them vector fields are considered in the cylinder Z.

Example 1. A vector potential w,

$$w = (1-r^2)\big(x,y,z\big),$$

generates the solenoidal vector field

$$v = \mathrm{rot} w = 2(-yz, xz, 0),$$

$r^2 = x^2 + y^2$. The field v is solenoidal both in the cylinder Z and in every disk B, i.e. $\delta v = 0$, $\delta_{12}v = 0$. The component v_3 equals zero, and the potential φ of the field $\nabla\varphi$ also equals zero. The potential ψ satisfies the equation $\Delta_{12}\psi = 4z$.

Example 2. A vector potential w is given by the formula

$$w = (1-r^2)\big(x^2yz, xy^2z, xyz^2\big),$$

then its rotor $v = \mathrm{rot} w$ is

$$v = \big(-2xy^2z^2 + x(1-r^2)(z^2-y^2), 2x^2yz^2 + y(1-r^2)(x^2-z^2), z(1-r^2)(y^2-x^2)\big).$$

The field v is 3D solenoidal, $\delta\, v = 0$, but not 2D solenoidal, that is, $\delta_{12}v \neq 0$. The right-hand side of the Poisson equation is $\dfrac{\partial v_3}{\partial z} = (1-r^2)(y^2-x^2)$ and does not depend on z.

Example 3. A potential $\chi = \sin\tau$, $\tau = \dfrac{20(x^2+y^2+z^2)}{\pi}$, and the potential vector field generated by it,

$$u = \nabla\chi = \left(\frac{40x}{\pi}\cos\tau, \frac{40y}{\pi}\cos\tau, \frac{40z}{\pi}\cos\tau\right),$$

are considered.

Example 4. We choose as a vector field $p = v + u$ a field which solenoidal part v is taken from Example 1 and potential part u is taken from Example 3.

Example 5. A scalar potential $\chi = (1 - r^2)^2 \ln(1 + \rho^2)$, $\rho^2 = x^2 + y^2 + z^2$, $|z| \le 1$, and the continuously differentiable potential field $u = \nabla\chi$,

$$u = (1 - r^2)\left(\frac{2x(1 - r^2)}{1 + \rho^2} - 4x\ln(1 + \rho^2), \frac{2y(1 - r^2)}{1 + \rho^2} - 4y\ln(1 + \rho^2), \frac{2z(1 - r^2)}{1 + \rho^2}\right),$$

generated by it, are considered.

Example 6. We choose as a vector field $p = v + u$ a field which solenoidal part v is taken from Example 2, and potential part u is taken from Example 5.

Algorithm I

Applicable, if we known that the sought-for field v is solenoidal.

The steps of Algorithm I of recovering the 3D field v are as follows.

– 1. Differentiate the component v_3 (known from MRI measurements) of the field v with respect to z. Solve the homogeneous Dirichlet boundary value problem for the Poisson equation (11) — the one with the right-hand side $-\partial v_3/\partial z$, — and find the potential φ.

– 2. Apply the operator ∇ to the found potential φ and obtain the potential vector field $\nabla\varphi = \big((\nabla\varphi)_1, (\nabla\varphi)_2\big)$.

– 3. Using the values of the longitudinal ray transform $\mathcal{P}$ being applied to the field $\widetilde{v}$ we find the solenoidal part $\nabla^\perp\psi = \big((\nabla^\perp\psi)_1, (\nabla^\perp\psi)_2\big)$.

– 4. Construct the required field $v = (v_1, v_2, v_3)$, where $v_1 = (\nabla^\perp\psi)_1 + (\nabla\varphi)_1$, $v_2 = (\nabla^\perp\psi)_2 + (\nabla\varphi)_2$, the component v_3 of the field v is known.

Algorithm II

Applicable for the arbitrary vector field w.

The steps of Algorithm II of the recovering the 3D field v are as follows.

– 1. Using the values of the longitudinal ray transform $\mathcal{P}$ being applied to the field $\widetilde{w}$ find the potential ψ of the solenoidal part of the 2D field $\widetilde{w}$. Apply the operator $\nabla^\perp$ to the potential ψ and obtain the field $\big((\nabla^\perp\psi)_1, (\nabla^\perp\psi)_2\big)$.

– 2. Using the values of the transverse ray transform $\mathcal{P}^\perp$ being applied to the field $\widetilde{w}$ find the potential $\varphi + \chi$ of the potential part of the field $\widetilde{w}$. Apply the operator ∇ to the potential $\varphi + \chi$ and obtain the field $\big((\nabla\varphi + \nabla\chi)_1, (\nabla\varphi + \nabla\chi)_2\big)$.

– 3. Construct the required field $w = (w_1, w_2, w_3)$, where $w_1 = (\nabla^\perp\psi)_1 + (\nabla(\varphi + \chi))_1$, $w_2 = (\nabla^\perp\psi)_2 + (\nabla(\varphi + \chi))_2$, w_3 is known.

Algorithm III

Applicable for the arbitrary vector field p.

It is advisable to express $\mathcal{R}p_1$, $\mathcal{R}p_2$ through the known values $\mathcal{P}p$, $\mathcal{P}^\perp p$, using equations (6), (7). Then, use any of the numerous inversion algorithms. These actions led to the components $p_1(x, y, c)$, $p_2(x, y, c)$ of the sought-for field $p(x, y, z)$.

The steps of Algorithm III of the reconstruction of the 3D field p are as follows.

– 1. Fix the values of the longitudinal ray transform $\mathcal{P}$ being applied to the 2D field $\widetilde{p}$ defined in the unit disk (an axial section of the cylinder with $z = c$, $|c| < 1$).

– 2. Fix the values of the transverse ray transform $\mathcal{P}^\perp$ being applied to the 2D field $\widetilde{p}$ defined in the unit disk (an axial section of the cylinder with $z = c$, $|c| < 1$).

– 3. Construct the right-hand sides of the system of equations (7) depending on α, s.

– 4. Apply the inversion formula to the known $\mathcal{R}p_1$, $\mathcal{R}p_2$, and find $p_1(x, y, c)$, $p_2(x, y, c)$, c is fixed.

– 5. In every slice construct the 3D vector field $p(x, y, c)$ adding the third known component to the two obtained ones p_1, p_2.
– 6. Fix another slice $z = c$ and repeat the entire procedure from the beginning. Test all the values $c = z_k$ and obtain the values of the 3D vector field $p(x_i, y_j, z_k)$ in the nodes of a rectangular mesh defined in the cylinder.
– 7. If desired, construct an approximation of the field p, for example by orthogonal polynomials which constitute a product of polynomials orthogonal in the disk B and the ones orthogonal on the segment $[-1, 1]$; or an approximation by B-splines.

Remark 1. *Algorithms II and III are universal and applicable for the arbitrary vector field. Algorithm I should be used if the required field is known to be solenoidal. The advantage of the algorithm is that it suffices to apply only the longitudinal ray transform without using the transverse one. Instead, on every section it is necessary to solve numerically the homogeneous Dirichlet boundary value problem for the Poisson equation.*

4.1. **Numerical modeling.** In this section, the results of the numerical tests on recovering the vector fields in the cylinder are provided. The carried out tests do not pursue as their goal the entire investigation of the possibilities of all three algorithms, but only a demonstration of legitimacy of the proposed approach.

We use 3D vector fields from Examples 1–6 as the test fields. The recovery of the fields is realized slice-by-slice using Algorithm III.

The recovery of the components of a 2D field by the longitudinal and transverse ray transforms is realized using the formulas (7). Moreover, the inversion of the Radon transform of the components of the 2D field under recovery is realized with the help of back projection and Riesz potential. The discretization of the ray transforms by s and α is 512×512. The components of the field are reconstructed in a square $[-1, 1]^2$ with a uniform grid with the step $1/32$ on each of the axes.

The back projection is calculated in the square $[-8, 8]^2$ with a uniform grid with the step $1/32$ on each of the axes. The number of rays for calculation of the back projection is 512. To evaluate the quality of the approximation of the 2D vector field $v = (v_1, v_2)$ relatively to the exact value $u = (u_1, u_2)$ we use a relative error of recovery (as a percentage), calculated by the formula

$$\varepsilon = \left(\sqrt{\sum_{i,j=0}^{64} \Big((v_1^{ij} - u_1^{ij})^2 + (v_2^{ij} - u_2^{ij})^2 \Big)} \Big/ \sqrt{\sum_{i,j=0}^{64} \Big((u_1^{ij})^2 + (u_2^{ij})^2 \Big)} \right) \cdot 100\%,$$

where the upper index ij denotes the node of the recovery mesh.

In Table 1, the values of the relative recovery error of the test fields are provided. The lines of the table correspond to the different slices (by z axis) of the cylinder. The rows correspond to different test fields. The Fig. 3, 4, 5 (a, b) show the components of the 2D vector fields obtained in Examples $3, 5, 6$ in the plane $z = -0.8$, and their reconstructions (c, d).

As can be seen of the table, the obtained relative errors of recovery are quite significant. This is due to the fact that the test fields are discontinuous. To compare, Fig. 4 (a, b) shows the components of the 2D potential vector field constructed with the help of the potential $\chi = (1 - r^2)^2 \ln(1 + \rho^2)$, $\rho^2 = x^2 + y^2 + z^2$. The field u possesses continuous components and we obtain an error 1.15% in the plane $z = -0.8$. The reason for the coincidence of the recovery error of field from Example 1 on different layers by z is that it does not depend on the variable z.

TABLE 1. Recovery error of the test fields (as a percentage)

z \ field number	1	2	3	4	5	6
-0.8	5.30	7.74	3.80	3.85	0.99	3.36
-0.4	5.30	5.80	3.33	3.35	1.18	1.86
0.2	5.30	2.53	6.76	6.76	1.30	1.36
0.6	5.30	7.24	6.02	6.00	1.05	2.72

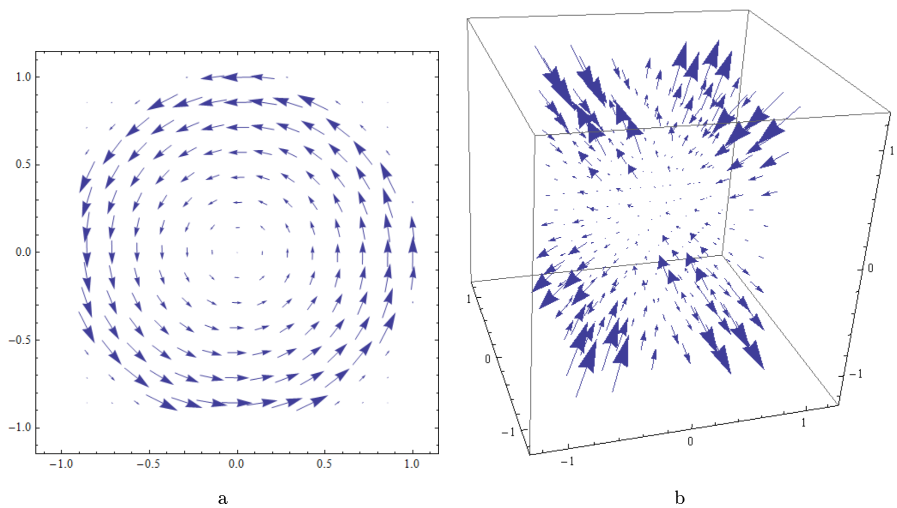

FIG. 2. Vector fields from Example 1 (a), relative recovery error $\varepsilon = 5.3\%$, and from Example 2 (b), relative error $\varepsilon = 5.8\%$ with $z = -0.4$

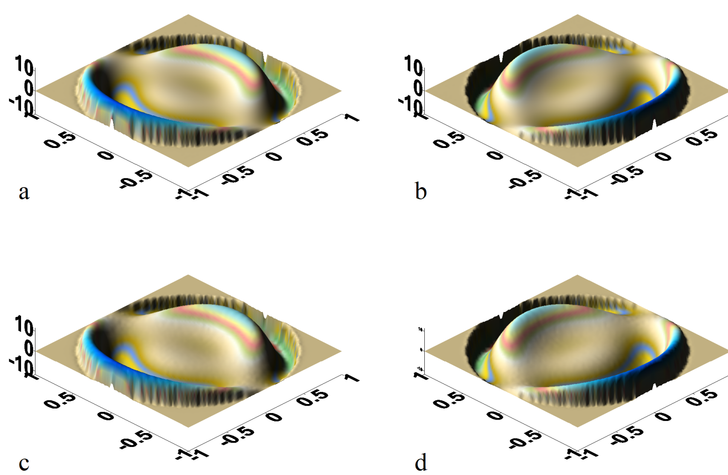

FIG. 3. The components of the vector field from Example 3 (a, b) and its reconstructions (c, d) given $z = -0.8$, $\varepsilon = 3.8\%$

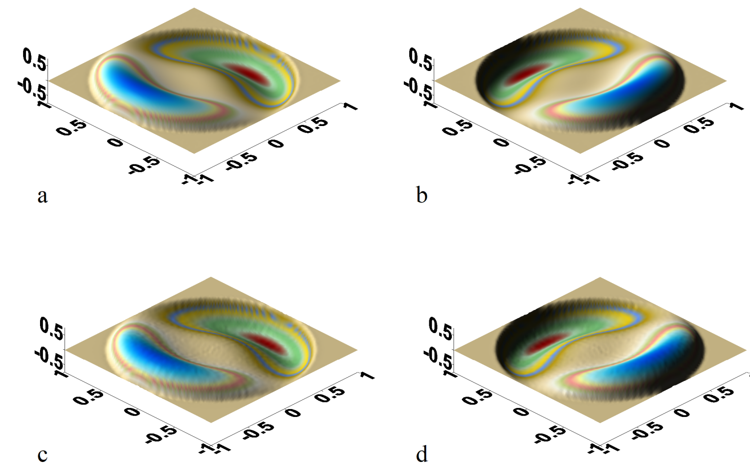

FIG. 4. The components of the vector field from Example 5 (a, b) and its reconstructions (c, d) given $z = -0.8$, $\varepsilon = 1.0\%$

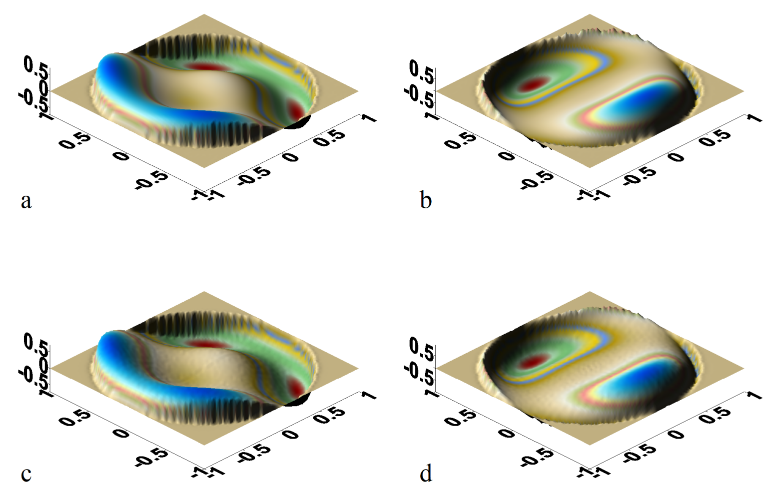

FIG. 5. The components of the vector field from Example 6 (a, b) and its reconstructions (c, d) given $z = -0.8$, $\varepsilon = 3.4\%$

In our paper, the approaches to solving the problem of recovery of a 3D vector field in the cylinder by its known ray transforms and NMR images are described. We have proposed three algorithms for solving the problem. These algorithms are applicable both for a solenoidal field and for the one of an arbitrary type. The results of numerical tests aimed on confirmation of the applicability of the proposed approaches for solving the problem of vector tomography given in the cylinder are provided. We have proposed an original interpretation of the data of the problem. In particular, it can be represented in the form of the values of three types of ray transforms of a 3D vector field, closely connected with some family of parallel planes which can be defined arbitrary.

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