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VERTEX-VERTEX COLOR ENERGY OF A GRAPH

SAYINATH UDUPA, BHAT R.S.

Abstract. In this paper we introduce new kind of graph energy, vv-color energy of a graph denoted as $E_{cvv}(G)$. It depends both on underlying graph G and its coloring. Upper and lower bounds for $E_{cvv}(G)$ are established.

Keywords: energy of a graph, labeled graph, color energy of a graph, vv-coloring of a graph.

1. Introduction

The terminologies and notations used here are as in [8, 13, 14]. By a graph $G(V, E)$ we mean a connected finite simple graph of order p and size q. A vertex $v \in V$ is a cutvertex if $G - \{v\}$ is disconnected. A graph which has no cutvertex is called non-separable. A maximal non-separable subgraph is a block of G. A coloring of graph G is a coloring of its vertices such that no two adjacent vertices receive the same color. The minimum number of colors needed for coloring of a graph G is called chromatic number and denoted by $\chi(G)$. The eigenvalues of G are the eigenvalues of its adjacency matrix $A(G)$. These eigenvalues arranged in an non-increasing order, will be denoted as $\lambda_1(G), \lambda_2(G) \dots, \lambda_p(G)$.

Then the energy of the graph G is defined as $E(G) = \sum\limits_{i=1}^{p} |\lambda_i(G)|$.

Various properties of energy of the graph may be found in [6]. In connection with graph energy, energy-like quantities were considered for other matrices such as Laplacian [7], distance [9], incidence [10] and vb-dominating [12].

We say that a vertex v is incident with the block b if v is one of the vertex of block b. Two vertices $u, w \in V$ are vv-adjacent if they lie on the same block. *The point graph of a graph G denoted $P_G(G)$ is a graph whose vertex set is same as that of G*

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and any two vertices in $P_G(G)$ are adjacent if and only if they are vv-adjacent in G. For any block graph G, $P_G(G) = G$. Since any tree T is a block graph, $P_G(T) = T$ and $P_G(P_G(G)) = P_G(G)$ for any graph G. For various properties of point graph, one may refer [4].

2. VV-COLOR ENERGY

In [1] it was introduced a matrix of a vertex labeled graph $G = (V, E)$ whose elements are defined as follows. If $l(v_i)$ is the label of the vertex v_i, Then

$$a_{ij} = \begin{cases} 2 & \text{if } v_i \text{ and } v_j \text{ are adjacent with } l(v_i) = l(v_j) \\ 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent with } l(v_i) \neq l(v_j) \\ -1 & \text{if } v_i \text{ and } v_j \text{ are non-adjacent with } l(v_i) = l(v_j) \\ 0 & \text{otherwise} \end{cases}$$

Motivated by the above definition, we define the following.

Definition 1. *vv-color of graph G is the coloring of vertices such that no two vertices in the same block receives same color. Then vv-chromatic number of a graph G denoted by $\chi_{cvv}(G)$ is the minimum number of colors required to vv-color all the vertices of the graph G.*

In this paper, we consider the vv-colored graph. The vv-colored matrix of G is the $p \times p$ matrix $A_{cvv}(G) = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are in the same block and } i \neq j \\ -1 & \text{if } v_i \text{ and } v_j \text{ are in different blocks with } c(v_i) = c(v_j) \\ 0 & \text{otherwise} \end{cases}$$

where $c(v_i)$ is the color of v_i.

The characteristic polynomial of $A_{cvv}(G)$ is defined by the following equality

$$f_p(G, \lambda) = det(\lambda I - A_{cvv}(G))$$

The vv-colored eigenvalues of the graph G are the eigenvalues of $A_{cvv}(G)$. Since $A_{cvv}(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$. The vv-color energy of G is then defined as

$$E_{cvv}(G) = \sum_{i=1}^{p} |\lambda_i(G)|.$$

In this paper we discuss some basic properties of vv-color energy of a graph $E_{cvv}(G)$ and derive an upper and lower bound for $E_{cvv}(G)$ and we compute vv-color energy of some family of graphs. First we compute vv-color energy of the graph shown in FIG. 1.

Let G be a graph with 10 vertices which are vv-colored using colors a, b, c and d (see Figure 1). Then

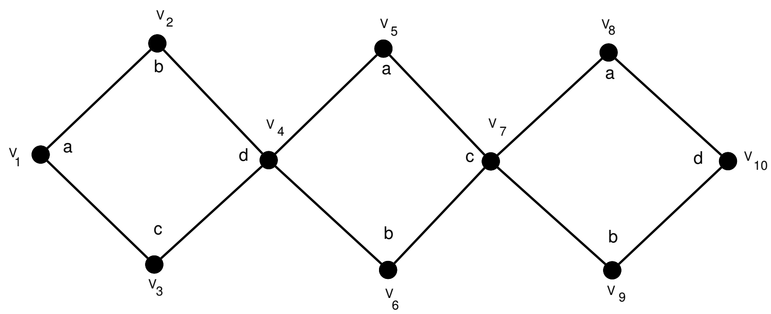

FIG. 1. vv-colored graph G

$$A_{cvv}(G) = \begin{bmatrix} 0 & 1 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

The characteristic polynomial of $A_{cvv}(G)$ is $\lambda^{10} - 26\lambda^8 - 8\lambda^7 + 213\lambda^6 + 104\lambda^5 - 536\lambda^4 - 288\lambda^3 - 36\lambda^2$. Then the vv-colored eigenvalues are $-3.0000, -2.8558, -2.5780, -0.3216, -0.2051, 0.0000, 0.0000, 2.1774, 3.0000$ and 3.7831. Therefore the vv-color energy $E_{cvv}(G) = 17.9210$. Note that the vv-color energy of the graph G depends on its vv-coloring of G.

2.1. **Basic properties of vv-colored eigenvalues of a graph.**

Theorem 1. *If $\lambda_1(G), \lambda_2(G) \ldots, \lambda_p(G)$ are the vv-colored eigenvalues of $A_{cvv}(G)$, then*

$$\sum_{i=1}^{p} \lambda_i = 0 \tag{1}$$

$$\sum_{i=1}^{p} \lambda_i^2 = 2(q_p + V_s). \tag{2}$$

where q_p is the number of edges in the point graph $P_G(G)$ and V_s is the number of unordered pairs of vertices that receive the same color in vv-colored graph G.

Proof. (1) We know that the sum of the eigenvalues of $A_{cvv}(G)$ is equal to trace of $A_{cvv}(G)$. Therefore $\sum_{i=1}^{p} \lambda_i = \sum_{i=1}^{p} a_{ii} = 0$.

(2) The sum of the squares of the eigenvalues of $A_{cvv}(G)$ is the trace of $(A_{cvv}(G))^2$.

Therefore

$$\sum_{i=1}^{p} \lambda_i^2 = \sum_{i=1}^{p} \sum_{j=1}^{p} (a_{ij} a_{ji})$$
$$= 2 \sum_{i<j} (a_{ij})^2 + \sum_{i=1}^{p} (a_{ii})^2$$
$$= 2(q_p + V_s).$$

□

Let P_s denote the number of paths of length 2 with end vertices having same colors in $P_G(G)$, T_s be the number of pairwise non vv-adjacent triplet of vertices having same colors in $P_G(G)$, $\triangle$ be the number of triangles in the point graph $P_G(G)$ and q_p denote the number of edges in the point graph $P_G(G)$.

Theorem 2. *Let G be a graph with p vertices. Let $f_p(G, \lambda) = c_0 \lambda^p + c_1 \lambda^{p-1} + c_2 \lambda^{p-2} + \cdots + c_p$ be the characteristic polynomial of G, Then*

$$c_0 = 1 \tag{3}$$
$$c_1 = 0 \tag{4}$$
$$c_2 = -(q_p + V_s) \tag{5}$$
$$c_3 = -2(\triangle - P_s - T_s) \tag{6}$$

Proof. (3) directly follows from the definition of $f_p(G, \lambda)$.
(4) The sum of determinants of all 1×1 principal sub-matrices of $A_{cvv}(G)$ is the trace of $A_{cvv}(G)$, which is equal to 0. Thus $(-1)^1 c_1 = 0$.
(5) $(-1)^2 c_2$ is equal to the sum of determinants of all 2×2 principal sub-matrices of $A_{bb}(G)$, that is

$$c_2 = \sum_{1 \le i < j \le p} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}$$
$$= \sum_{1 \le i < j \le p} (a_{ii} a_{jj} - a_{ij} a_{ji})$$
$$= \sum_{1 \le i < j \le p} (a_{ii} a_{jj}) - \sum_{i \le i < j \le p} (a_{ij})^2$$
$$= 0 - (q_p + V_s).$$

(6) We have

$$
\begin{aligned}
c_3 &= (-1)^3 \sum_{1\le i<j<k\le p} \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix} \\
&= -\sum_{1\le i<j<k\le p} \left[a_{ii}\left(a_{jj}a_{kk} - a_{jk}a_{kj}\right) - a_{ij}\left(a_{ji}a_{kk} - a_{ki}a_{jk}\right) + a_{ik}\left(a_{ji}a_{kj} - a_{ki}a_{jj}\right)\right] \\
&= -\sum_{1\le i<j<k\le p} (a_{ii}a_{jj}a_{kk}) + \sum_{1\le i<j<k\le p} (a_{ii}a_{jk} + a_{jj}a_{ik} + a_{kk}a_{ij}) \\
&\quad - \sum_{1\le i<j<k\le p} (a_{ij}a_{jk}a_{ki}) - \sum_{1\le i<j<k\le p} (a_{ik}a_{kj}a_{ji}) \\
&= -2\Big(\sum_{1\le i<j<k\le p} (a_{ij}a_{jk}a_{ki})\Big) \\
&= -2\left(\triangle - P_s - T_s\right).
\end{aligned}
$$

□

Theorem 3. *If $\lambda_1(G)$ is the largest eigenvalue of the vv-colored matrix $A_{cvv}(G)$, then*

$$\lambda_1(G) \ge \frac{2(q_p - V_s)}{p}. \tag{7}$$

Proof. Let X be any non zero vector, then we have

$\lambda_1\left(A_{cvv}(G)\right) = \begin{matrix} max \\ X \neq 0 \end{matrix} \left[\frac{X'AX}{X'X}\right]$ (see [2])

$\lambda_1(A_{cvv}(G)) \ge \left[\frac{J'AJ}{J'J}\right] = \frac{2(q_p - V_s)}{p}$, where J is the all one's vector. □

In [3] it is proved that if energy of the graph is a rational number, then it is an even integer. Similar result for vv-color energy is established in the following proposition.

Theorem 4. *If the vv-color energy of a graph $E_{cvv}(G)$ is a rational number, then $E_{cvv}(G)$ must be an even positive integer.*

Proof. Let $\lambda_1, \lambda_2 \ldots, \lambda_p$ be the eigenvalues of the vv-colored matrix $A_{cvv}(G)$ of a graph G of which $\lambda_1, \lambda_2 \ldots, \lambda_r$ are positive and the rest of the vv-colored eigenvalues are non-positive, then

$$
\begin{aligned}
\sum_{i=1}^{p} |\lambda_i| &= (\lambda_1 + \lambda_2 + \cdots + \lambda_r) - (\lambda_{r+1} + \lambda_{r+2} \cdots + \lambda_p) \\
&= 2(\lambda_1 + \lambda_2 + \cdots + \lambda_r) - (\lambda_1 + \lambda_2 + \cdots + \lambda_p) \\
E_{cvv}(G) &= 2(\lambda_1 + \lambda_2 + \cdots + \lambda_r) - \sum_{i=1}^{p} \lambda_i \\
E_{cvv}(G) &= 2(\lambda_1 + \lambda_2 + \cdots + \lambda_r) - 0 \quad \text{[From the Theorem 1]}
\end{aligned}
$$

Therefore $E_{cvv}(G) \equiv 0 (\text{mod } 2)$

Hence $E_{cvv}G$ must be an even positive integer. □

Theorem 5. *Let G_1 and G_2 be two vv-colored graphs with p vertices, q_p and $q_{p'}$ be the number of edges in $P_G(G_1)$ and $P_G(G_2)$ respectively and V_s and $V_{s'}$ are the number of unordered pairs of vertices that receive the same color in vv-colored graph G_1 and G_2 respectively. If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$ and $\lambda_1^{'} \geq \lambda_2^{'} \geq \cdots \geq \lambda_p^{'}$ are the vv-colored eigenvalues of G_1 and G_2 respectively, then*

$$\sum_{i=1}^{p} \lambda_i \lambda_i^{'} \leq 2\sqrt{(q_p + V_s)(q_{p'} + V_{s'})}. \tag{8}$$

Proof. Using Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^{p} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{p} a_i^2\right)\left(\sum_{i=1}^{p} b_i^2\right) \tag{9}$$

and taking $a_i = \lambda_i, b_i = \lambda_i^{'}$, by Theorem 1, we get

$$\begin{aligned}\left(\sum_{i=1}^{p} \lambda_i \lambda_i^{'}\right)^2 &\leq \left(\sum_{i=1}^{p} \lambda_i^2\right)\left(\sum_{i=1}^{p} (\lambda_i^{'})^2\right) \\ &= 4(q_p + V_s)(q_{p'} + V_{s'}).\end{aligned}$$

then the result follows. □

3. Upper and lower bounds on vv-color energy of a graph

Similar to [11], we have given bounds for vv-color energy of the graph in the following proposition.

Theorem 6. *Let G be the graph with p vertices, q_p edges in $P_G(G)$ and V_s unordered pairs of vertices receive the same color in G, Then*

$$E_{cvv}(G) \leq \sqrt{2p\,(q_p + V_s)}. \tag{10}$$

Proof. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p$ be the vv-colored eigenvalues of $A_{cvv}(G)$.
Using (9) and taking $a_i = 1, b_i = |\lambda_i|$, we get

$$\begin{aligned}(E_{cvv}(G))^2 = \left(\sum_{i=1}^{p} |\lambda_i|\right)^2 &\leq \left(\sum_{i=1}^{p} 1\right)\left(\sum_{i=1}^{p} |\lambda_i|^2\right) \\ &= p\sum_{i=1}^{p} \lambda_i^2 \\ &= 2p(q_p + V_s) \text{ [Using Theorem 1]}\end{aligned}$$

Then the result follows. □

Following proposition gives lower bound for $E_{cvv}(G)$ in terms of number of vertices in G and number of edges q_p in the point graph $P_G(G)$.

Theorem 7. *Let G be the graph with p vertices and q_p edges in $P_G(G)$. If $K =$ det $A_{cvv}(G)$, then*

$$E_{cvv}(G) \geq \sqrt{2(q_p + V_s) + p(p-1)K^{\frac{2}{p}}}. \tag{11}$$

Proof. Consider

$$
\begin{aligned}
(E_{cvv}(G))^2 &= \left(\sum_{i=1}^{p} |\lambda_i|\right)^2 \\
&= \left(\sum_{i=1}^{p} |\lambda_i|\right)\left(\sum_{j=1}^{p} |\lambda_j|\right) \\
&= \sum_{i=1}^{p} |\lambda_i|^2 + \sum_{i \neq j} |\lambda_i||\lambda_j|.
\end{aligned}
$$

Now employing the inequality between the arithmetic and geometric means, we obtain

$$
\begin{aligned}
\frac{1}{p(p-1)} \sum_{i \neq j} |\lambda_i||\lambda_j| &\geq \left(\prod_{i \neq j} |\lambda_i||\lambda_j|\right)^{\frac{1}{p(p-1)}} \\
\sum_{i \neq j} |\lambda_i||\lambda_j| &\geq p(p-1)\left(\prod_{i \neq j} |\lambda_i||\lambda_j|\right)^{\frac{1}{p(p-1)}} \\
\text{Thus, } (E_{cvv}(G))^2 &\geq \sum_{i=1}^{p} |\lambda_i|^2 + p(p-1)\left(\prod_{i \neq j} |\lambda_i||\lambda_j|\right)^{\frac{1}{p(p-1)}} \\
&\geq \sum_{i=1}^{p} |\lambda_i|^2 + p(p-1)\left(\prod_{i=1}^{p} |\lambda_i|^{2(p-1)}\right)^{\frac{1}{p(p-1)}} \\
&\geq \sum_{i=1}^{p} |\lambda_i|^2 + p(p-1)\left(\prod_{i=1}^{p} |\lambda_i|\right)^{\frac{2}{p}} \\
&\geq 2(q_p + V_s) + p(p-1)K^{\frac{2}{p}} \text{ [from the Theorem 1]}
\end{aligned}
$$

Then the result follows. □

4. Conclusion

The notion of color energy of the graph made the way for us to define vv-color energy of the graph. Few basic properties of vv-colored eigenvalues is discussed. Several upper and lower bounds of vv-color energy are obtained.

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Department of Mathematics,
Manipal Institute of Technology,
Manipal Academy of Higher Education,
Manipal, India-576104
Email address: sayinath.udupa@gmail.com

Department of Mathematics,
Manipal Institute of Technology,
Manipal Academy of Higher Education,
Manipal, India-576104
Email address: rs.bhat@manipal.edu