S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 18, №2, стр. 1046–1057 (2021) УДК 517.922
DOI 10.33048/semi.2021.18.080 MSC 35E15

ON SOLVABILITY OF THE BOUNDARY VALUE PROBLEM FOR ONE PSEUDOHYPERBOLIC EQUATION

L.N. BONDAR, V.S. NURMAKHMATOV

Abstract. The paper considers the first boundary value problem in a rectangle for the equation describing torsional vibrations of an elastic rod. Existence and uniqueness theorems are proved for a generalized solution of the first boundary value problem in Sobolev space.

Keywords: generalized solution, pseudohyperbolic equation, torsional vibrations, Sobolev space, unique solvability, Galerkin method.

1. Introduction

In the work, we consider the first boundary value problem in a rectangle for the equation

$$(I - D_x^2)D_t^2 u + D_x^4 u - a^2 D_x^2 u = f(t, x). \tag{1.1}$$

Equation (1.1) is not resolved with respect to the highest-order derivative. Such equations are often called Sobolev type equations, since it was S.L. Sobolev whose works were the beginning of a systematic study on such equations. In the works by S.L. Sobolev [1], detailed study on one equation nonresolved with respect to the highest-order derivative was first performed and a number of new mathematical problems were formulated. In particular, Sergei LпїSvovich Sobolev stated a problem of constructing a theory on boundary value problems for differential equations nonresolved with respect to the highest-order time derivative.

Bondar, L.N., Nurmakhmatov, V.S., On solvability of the boundary value problem for one pseudohyperbolic equation.

The work is supported by the Mathematical Center in Akademgorodok under agreement No. 075-15-2019-1613 with the Ministry of Science and Higher Education of the Russian Federation.

Received June, 15, 2021, published October, 19, 2021.

The study [2] was devoted to solving that problem. This book distinguishes between three classes of equations of the form

$$L_0(D_x)D_t^l u + \sum_{k=0}^{l-1} L_{l-k}(D_x)D_t^k u = f(t,x).$$

In particular, the equations of Sobolev type, pseudoparabolic, and pseudohyperbolic equations. For these equations, the Cauchy problem and general mixed boundary value problems in the quarter of the space have been studied. The works [3–7] are dedicated to studying of the Cauchy problem for pseudohyperbolic equations. In [2], also the boundary value problems for systems of Sobolev type and pseudoparabolic ones were investigated.

The class of pseudohyperbolic equations, introduced by G.V. Demidenko (see [2]), includes the equation that describes torsional vibrations of an elastic rod (see, for example, [8])

$$\rho I_p \theta_{tt} - \mu I_k \theta_{xx} + E I_d \theta_{xxxx} - \rho I_d \theta_{xxtt} = f(t,x), \tag{1.2}$$

where $\theta(t,x)$ is the angle of cross-sectional rotation or twist angle, I_p is the polar moment of inertia, μ is the Lame constant, I_k is the torsional moment of inertia, E is the Young's modulus, I_d is the moment of deplanation, $f(t,x)$ is the external force. In the literature, equation (1.2) is referred to as Vlasov's equation [8, 9].

It is easy to see that by performing the following substitution of variables

$$\tilde{x} = x\sqrt{I_p/I_d}, \quad \tilde{t} = t\sqrt{E/\rho}\sqrt{I_p/I_d}, \quad \theta(t,x) \equiv \frac{I_d}{EI_p^2}u(\tilde{t},\tilde{x}), \quad a^2 = \frac{\mu I_k}{EI_p},$$

equation (1.2) is reduced to the considered equation of the form (1.1). Note that the equation describing longitudinal vibrations of the bar, the Rayleigh-Bishop equation, can also be reduced to the equation of the form (1.1) [10, 11].

Our goal is to prove existence and uniqueness of a generalized solution of the first boundary value problem in a rectangle.

2. The statement of the problem

Consider the first boundary value problem for equation (1.1) in a rectangle: $\Pi = \{(t,x) : t \in (0,T),\, x \in (0,1)\}$:

$$\begin{cases} (I - D_x^2)D_t^2 u + D_x^4 u - a^2 D_x^2 u = f(t,x), & (t,x) \in \Pi, \\ u|_{x=0} = 0, \quad D_x u|_{x=0} = 0, \\ u|_{x=1} = 0, \quad D_x u|_{x=1} = 0, \\ u|_{t=0} = \varphi_1(x), \\ D_t u|_{t=0} = \varphi_2(x). \end{cases} \tag{2.1}$$

Definition. The function $u(t,x)$ belongs to the anisotropic Sobolev space $W_2^{1,2}(\Pi)$, if $u(t,x) \in L_2(\Pi)$, there exist generalized derivatives

$$D_t^{\alpha_1} D_x^{\alpha_2} u(t,x) \in L_2(\Pi), \quad (\alpha_1,\alpha_2) \in \Lambda = \left\{ \frac{\alpha_1}{1} + \frac{\alpha_2}{2} \le 1 \right\},$$

the norm is defined in the following way:

$$\|u(t,x), W_2^{1,2}(\Pi)\| = \sum_{(\alpha_1,\alpha_2)\in\Lambda} \|D_t^{\alpha_1} D_x^{\alpha_2} u(t,x), L_2(\Pi)\|.$$

We denote

$$W^{1,2}_{2,add}(\Pi) = \{u(t,x) \in W^{1,2}_2(\Pi) : \exists\, D^2_{tx}u(t,x) \in L_2(\Pi)\}.$$

We will formulate the notion of a generalized solution to the boundary value problem (2.1).

Suppose that $f(t,x) \in L_2(\Pi)$, $\varphi_1(x) \in \overset{\circ}{W}{}^2_2(0,1)$, $\varphi_2(x) \in \overset{\circ}{W}{}^1_2(0,1)$.

Definition. The function $u(t,x) \in W^{1,2}_{2,add}(\Pi)$ such that

$$u|_{t=0} = \varphi_1(x),\ u|_{x=0} = 0,\ D_x u|_{x=0} = 0,\ u|_{x=1} = 0,\ D_x u|_{x=1} = 0, \tag{2.2}$$

is called a generalized solution of the boundary value problem (2.1), if for every function $v(t,x) \in W^{1,2}_{2,add}(\Pi)$ satisfying

$$v|_{t=T} = 0, \quad v|_{x=0} = 0, \quad D_x v|_{x=0} = 0, \quad v|_{x=1} = 0, \quad D_x v|_{x=1} = 0, \tag{2.3}$$

the following equality holds:

$$\int_0^T\int_0^1 \Big[D_t u D_t v + D^2_{tx}u D^2_{tx}v - a^2 D_x u D_x v - D^2_x u D^2_x v\Big]dx\,dt = -\int_0^T\int_0^1 f(t,x)v(t,x)dx\,dt$$

$$-\int_0^1 \Big[\varphi_2(x)v(t,x)\big|_{t=0} + D_x\varphi_2(x)D_x v(t,x)\big|_{t=0}\Big]dx. \tag{2.4}$$

Theorem 1. *The boundary value problem* (2.1) *cannot have more than one generalized solution.*

Theorem 2. *Suppose that $f(t,x) \in L_2(\Pi)$ and $\varphi_1(x) = 0$, $\varphi_2(x) = 0$. Then the boundary value problem (2.1) has a unique generalized solution $u(t,x) \in W^{1.2}_2(\Pi)$, moreover,*

$$\|u(t,x), W^{1.2}_2(\Pi)\| \le c\|f(t,x), L_2(\Pi)\|, \tag{2.5}$$

where the constant $c > 0$ does not depend on f.

Corollary 1. *Let $f(t,x) = 0$ and $\varphi_1(x) \in \overset{\circ}{W}{}^2_2(0,1)$, $\varphi_2(x) \in \overset{\circ}{W}{}^1_2(0,1)$. Then the boundary value problem (2.1) has a unique generalized solution $u(t,x) \in W^{1,2}_2(\Pi)$, moreover,*

$$\|u(t,x), W^{1,2}_2(\Pi)\| \le c\Big[\|\varphi_1(x), W^2_2(0,1)\| + \|\varphi_2(x), W^1_2(0,1)\|\Big],$$

where the constant $c > 0$ does not depend on φ_1 and φ_2.

3. Proof of uniqueness

We will carry out the proof of uniqueness using the standard scheme (see, for example, [12]). We will need the following formula:

$$\int_0^t \varphi(\tau)\Big(\int_\tau^t \varphi(s)\,ds\Big)d\tau \equiv \frac{1}{2}\Big(\int_0^t \varphi(\tau)\,d\tau\Big)^2, \quad t \in [0,T], \tag{3.1}$$

which holds for every integrable function φ.

We will prove the uniqueness of a generalized solution by contradiction. Assume that there exist two distinct generalized solutions of the boundary value problem (2.1): $u_1(t,x)$, $u_2(t,x)$. Then the function

$$u(t,x) = u_1(t,x) - u_2(t,x) \neq 0,\ u(t,x) \in W^{1,2}_{2,add}(\Pi),$$

satisfies (2.2) for $\varphi_1(x) = 0$ and

$$\int_0^T \int_0^1 \Big[D_t u D_t v + D^2_{tx} u D^2_{tx} v - a^2 D_x u D_x v - D^2_x u D^2_x v \Big] dx\, dt = 0 \tag{3.2}$$

holds for the derivative $v(t,x) \in W^{1,2}_{2,add}(\Pi)$, satisfying (2.3). We fix an arbitrary number $\tau \in (0,T)$ and take the following function for $v(t,x)$:

$$v(t,x) = \begin{cases} \int_t^\tau u(s,x) ds & \text{for} \quad t \in (0,\tau), \\ 0 & \text{for} \quad t \in (\tau, T). \end{cases} \tag{3.3}$$

It is easy to verify that $v(t,x) \in W^{1,2}_2(\Pi)$, $D^2_{tx} v(t,x) \in L_2(\Pi)$, (2.3) are fulfilled, moreover,

$$D_t v(t,x) = \begin{cases} -u(t,x) & \text{for} \quad t \in (0,\tau), \\ 0 & \text{for} \quad t \in (\tau,T), \end{cases}$$

$$D_x v(t,x) = \begin{cases} \int_t^\tau D_x u(s,x) ds & \text{for} \quad t \in (0,\tau), \\ 0 & \text{for} \quad t \in (\tau,T), \end{cases}$$

$$D^2_{tx} v(t,x) = \begin{cases} -D_x u(t,x) & \text{for} \quad t \in (0,\tau), \\ 0 & \text{for} \quad t \in (\tau,T), \end{cases}$$

$$D^2_x v(t,x) = \begin{cases} \int_t^\tau D^2_x u(s,x) ds, & \text{for} \quad t \in (0,\tau), \\ 0 & \text{for} \quad t \in (\tau,T). \end{cases}$$

We substitute the mentioned function $v(t,x)$ from (3.3) into (3.2), now we have

$$\int_0^1 \int_0^\tau \Big[D_t u(t,x) u(t,x) + D^2_{tx} u(t,x) D_x u(t,x) + a^2 D_x u(t,x) \Big(\int_t^\tau D_x u(s,x)\, ds \Big)$$

$$+ D^2_x u(t,x) \Big(\int_t^\tau D^2_x u(s,x)\, ds \Big) \Big] dt\, dx = 0.$$

Noting that

$$D_t u(t,x) u(t,x) + D^2_{tx} u(t,x) D_x u(t,x) = \frac{1}{2} D_t (u(t,x))^2 + \frac{1}{2} D_t (D_x u(t,x))^2,$$

and taking into account (3.1), we have that

$$\frac{1}{2} \int_0^1 \int_0^\tau D_t u^2(t,x)\, dt\, dx + \frac{1}{2} \int_0^1 \int_0^\tau D_t (D_x u(t,x))^2\, dt\, dx + \frac{a^2}{2} \int_0^1 \Big(\int_0^\tau D_x u(t,x)\, dt \Big)^2 dx$$

$$+\frac{1}{2}\int_0^1\Big(\int_0^\tau D_x^2u(t,x)\,dt\Big)^2dx=0.$$

Integrating, we obtain

$$\int_0^1 u^2(\tau,x)\,dx+\int_0^1(D_xu(\tau,x))^2\,dx+a^2\int_0^1\Big(\int_0^\tau D_xu(t,x)\,dt\Big)^2dx$$
$$+\int_0^1\Big(\int_0^\tau D_x^2u(t,x)\,dt\Big)^2dx=0.$$

From here, in particular, we have

$$\int_0^1 u^2(\tau,x)\,dx=0.$$

Due to the fact that $\tau\in(0,T)$ is arbitrary, we obtain $u(\tau,x)=0$ almost everywhere in Π, which is a contradiction.
The theorem is proved.

4. Proof of existence of a generalized solution

We will perform the proof of existence of a generalized solution of the boundary value problem (2.1) using a well-known scheme, constructing a sequence of approximate solutions by the Galerkin method (see, for example, [12, 13]).

We will describe the proof of Theorem 2 in detail.

Let $\{v_p(x)\}$ be an orthonormal basis in $\mathring{W}_2^2(0,1)$. We can assume that $v_p(x)\in C_0^\infty(0,1)$. We will seek for a sequence of approximate solutions in the form

$$u^m(t,x)=\sum_{p=1}^m c_p(t)v_p(x), \tag{4.1}$$

where

$$c_p(t)\in W_2^2(0,T),\quad c(0)=0,\quad D_tc(0)=0,\quad p=1,\dots,m,$$

moreover, for almost every $t\in(0,T)$ we assume that the following relations are fulfilled:

$$\int_0^1\Big[(I-D_x^2)D_t^2u^m(t,x)+D_x^4u^m(t,x)-a^2D_x^2u^m(t,x)\Big]v_k(x)\,dx$$
$$=\int_0^1 f(t,x)v_k(x)\,dx,\quad k=1,\dots,m. \tag{4.2}$$

Taking into account definition (4.1) of the functions $u^m(t,x)$ and the equalities

$$-\int_0^1 v_p''(x)v_k(x)\,dx=\int_0^1 v_p'(x)v_k'(x)\,dx,$$

relations (4.2) can be rewritten in the following way:

$$\sum_{p=1}^{m} D_t^2 c_p(t)\Big(\int_0^1 v_p(x)v_k(x)\,dx + \int_0^1 v_p'(x)v_k'(x)\,dx\Big)$$
$$+\sum_{p=1}^{m} c_p(t)\Big(\int_0^1 v_p''(x)v_k''(x)\,dx + a^2\int_0^1 v_p'(x)v_k'(x)\,dx\Big)$$
$$= \int_0^1 f(t,x)v_k(x)\,dx, \quad k=1,\dots,m. \tag{4.3}$$

With the notations

$$a_{pk} = \int_0^1 v_p(x)v_k(x)\,dx + \int_0^1 v_p'(x)v_k'(x)\,dx, \quad A=(a_{pk}),$$
$$b_{pk} = \int_0^1 v_p''(x)v_k''(x)\,dx + a^2\int_0^1 v_p'(x)v_k'(x)\,dx, \quad B=(b_{pk}),$$
$$F(t) = \begin{pmatrix} F_1(t) \\ \cdots \\ \cdots \\ F_m(t)\end{pmatrix}, \text{ where } F_k(t) = \int_0^1 f(t,x)v_k(x)\,dx,$$
$$c(t) = \begin{pmatrix} c_1(t) \\ \cdots \\ \cdots \\ c_m(t)\end{pmatrix},$$

relation (4.3) can be rewritten in the form

$$AD_t^2 c + Bc = F(t), \tag{4.4}$$

moreover, as it follows from the definition of $u^m(t,x)$, we have that

$$c(0)=0, \quad D_t c(0) = 0. \tag{4.5}$$

Since the vector function $F(t)$ has components from $L_2(0,T)$, it is easy to show that there exists a unique solution of Cauchy problem (4.4), (4.5) is the vector function of $c(t)\in W_2^2(0,T)$, and the sequence of Galerkin approximations u^m is well-defined (see, for example, [12, 13]).

Lemma 4.1. *For every $m\ge 1$, the following estimate holds*

$$\|u^m(t,x), W_2^{1,2}(\Pi)\| \le c\|f(t,x), L_2(\Pi)\|,$$

where the constant $c>0$ does not depend on m and f.

Proof. Multiplying the k-th relation in (4.2) by $D_t c_k$ and taking a sum over k from 1 to m, given the definition of the function u^m, we obtain

$$\int_0^1 \Big[(I-D_x^2)D_t^2 u^m(t,x) + D_x^4 u^m(t,x) - a^2 D_x^2 u^m(t,x)\Big] D_t u^m(t,x)\,dx$$

$$= \int_0^1 f(t,x) D_t u^m(t,x)\,dx.$$

Integrating from 0 to t, using the formula for integrating by parts, and taking into account

$$u^m(t,x)\big|_{x=0} = u^m(t,x)\big|_{x=1} = 0, \quad D_x u^m(t,x)\big|_{x=0} = D_x u^m(t,x)\big|_{x=1} = 0,$$

we obtain that

$$\int_0^t \int_0^1 \Big[D_\tau\big((D_\tau u^m(\tau,x))^2\big) + D_\tau\big(\big(D^2_{\tau x} u^m(\tau,x)\big)^2\big) + D_\tau\big((D^2_x u^m(\tau,x))^2\big)$$
$$+ a^2 D_\tau\big(\big(D_x u^m(\tau,x)\big)^2\big)\Big]\,dx\,d\tau = 2\int_0^t \int_0^1 f(\tau,x) D_\tau u^m(\tau,x)\,dx\,d\tau.$$

Since $u^m(t,x)\big|_{t=0} = 0$, $D_t u^m(t,x)\big|_{t=0} = 0$, we will have

$$\int_0^1 \Big[|D_t u^m(t,x)|^2 + |D^2_{tx} u^m(t,x)|^2 + a^2 |D_x u^m(t,x)|^2 + |D^2_x u^m(t,x)|^2 \Big] dx$$
$$= 2\int_0^t \int_0^1 f(\tau,x) D_\tau u^m(\tau,x)\,dx\,d\tau.$$

We integrate this relation with respect to t from 0 to T. Then, taking into account the formula

$$\int_0^T \int_0^t F(\tau) d\tau dt \equiv \int_0^T (T-t) F(t) dt,$$

we obtain

$$\int_0^T \int_0^1 \Big[|D_t u^m(t,x)|^2 + |D^2_{tx} u^m(t,x)|^2 + a^2 |D_x u^m(t,x)|^2 + |D^2_x u^m(t,x)|^2 \Big] dx\,dt$$
$$\le 2T \int_0^T \int_0^1 \big| f(t,x) D_t u^m(t,x) \big| dx\,dt.$$

From here due to Hölder's inequality, it follows that

$$\int_0^T \int_0^1 \Big[|D_t u^m(t,x)|^2 + |D^2_{tx} u^m(t,x)|^2 + a^2 |D_x u^m(t,x)|^2 + |D^2_x u^m(t,x)|^2 \Big] dx\,dt$$
$$\le 2T \| f(t,x), L_2(\Pi) \| \| D_t u^m(t,x), L_2(\Pi) \|.$$

We have

$$\Big(\int_0^T \int_0^1 \Big[|D_t u^m(t,x)|^2 + |D^2_{tx} u^m(t,x)|^2 + a^2 |D_x u^m(t,x)|^2 + |D^2_x u^m(t,x)|^2 \Big] dx\,dt \Big)^{\frac{1}{2}}$$
$$\le 2T \| f(t,x), L_2(\Pi) \|.$$

Since for almost every $t \in (0,T)$, we have that $u^m(t,x) \in \overset{\circ}{W}{}_2^2(0,1)$, then the Steklov's inequality holds:

$$\|u^m, L_2(0,1)\| \le c\|D_x^2 u^m, L_2(0,1)\|.$$

Therefore, we will have

$$\left(\int_0^T\int_0^1 \left[|u^m(t,x)|^2 + |D_t u^m(t,x)|^2 + |D_{tx}^2 u^m(t,x)|^2 + a^2|D_x u^m(t,x)|^2 + |D_x^2 u^m(t,x)|^2\right] dx\,dt\right)^{\frac{1}{2}} \le c\|f(t,x), L_2(\Pi)\|.$$

Since from every sequence bounded in $L_2(\Pi)$ we can extract a subsequence weakly converging to a function from $L_2(\Pi)$ (see, for example, [14, 15]), then, taking into account the theorem on weak closedness of a generalized differentiation operator (see, for example, [12]), we obtain that from $\{u^m\}$ we can extract a subsequence, weakly converging in $W_{2,add}^{1,2}(\Pi)$ to some function $u \in W_{2,add}^{1,2}(\Pi)$. For brevity, we will denote this subsequence by the same symbol $\{u^m\}$.

We will study some properties of a limit function u. To do that, we will use Mazur's theorem [16].

Theorem 3. *Suppose that the sequence of elements $\{u^m\}$ of the Hilbert space H weakly converges to $u \in H$. Then there exists a sequence of convex linear combinations*

$$\left\{\sum_{i=1}^N \lambda_{i,N} u^i\right\}, \quad \lambda_{i,N} \ge 0, \quad \sum_{i=1}^N \lambda_{i,N} = 1,$$

strongly converging to u.

Using this theorem and the weakly converging Galerkin sequence $\{u^m\}$, we construct a strongly converging sequence $\{\tilde{u}^N\}$ of convex combinations

$$\{\tilde{u}^N\} = \sum_{i=1}^N \lambda_{i,N} u^i(t,x), \quad \lambda_{i,N} \ge 0, \quad \sum_{i=1}^N \lambda_{i,N} = 1, \tag{4.6}$$

$$\|\tilde{u}^N(t,x) - u(t,x), W_2^{1,2}(\Pi)\| \to 0, N \to \infty. \tag{4.7}$$

$$\|D_{tx}^2\tilde{u}^N(t,x) - D_{tx}^2 u(t,x), L_2(\Pi)\| \to 0, N \to \infty. \tag{4.8}$$

From definition (4.6) and the properties of the functions u^i, we get

$$\begin{gathered}\tilde{u}^N(t,x)\big|_{t=0} = 0, \quad \tilde{u}^N(t,x)\big|_{x=0} = 0, \quad \tilde{u}^N(t,x)\big|_{x=1} = 0,\\ D_t\tilde{u}^N(t,x)\big|_{t=0} = 0, \quad D_x\tilde{u}^N(t,x)\big|_{x=0} = 0, \quad D_x\tilde{u}^N(t,x)\big|_{x=1} = 0,\end{gathered} \tag{4.9}$$

and also

$$\begin{aligned}&\|\tilde{u}^N(t,x), W_2^{1,2}(\Pi)\| \le \sum_{i=1}^N \lambda_{i,N}\|u^i(t,x), W_2^{1,2}(\Pi)\| \le c\|f(t,x), L_2(\Pi)\|,\\ &\|D_{tx}^2\tilde{u}^N(t,x), L_2(\Pi)\| \le c\|f(t,x), L_2(\Pi)\|.\end{aligned} \tag{4.10}$$

Taking into account the theorem on embedding for anisotropic Sobolev spaces (see, for example, [17]), we have that

$$\max_{(t,x)\in\overline{\Pi}}|\tilde{u}^N(t,x) - u(t,x)| \le c\|\tilde{u}^N(t,x) - u(t,x), W_2^{1,2}(\Pi)\|.$$

Due to (4.7), (4.9), we obtain

$$|u(t,x)\big|_{x=0}| = |\tilde{u}^N(t,x)\big|_{x=0} - u(t,x)\big|_{x=0}| \le \max_{(t,x)\in\overline{\Pi}} |\tilde{u}^N(t,x) - u(t,x)|$$

$$\le c\|\tilde{u}^N(t,x) - u(t,x), W_2^{1,2}(\Pi)\| \to 0, \quad N \to \infty,$$

therefore, $u(t,x)\big|_{x=0} = 0$.

By the same reasoning, it can be proved that

$$u(t,x)\big|_{t=0} = 0, \quad u(t,x)\big|_{x=1} = 0.$$

Taking into account the theorem on traces for the functions from Sobolev spaces $W_2^{1,2}(\Pi)$ (see, for example, [17]), and also $D_x\tilde{u}^N(t,x)\big|_{x=0} = 0$, we have

$$\|D_x u(t,x)\big|_{x=0}, L_2(0,T)\| = \|D_x u(t,x)\big|_{x=0} - D_x\tilde{u}^N(t,x)\big|_{x=0}, L_2(0,T)\|$$

$$\le c\|\tilde{u}^N(t,x) - u(t,x), W_2^{1,2}(\Pi)\| \to 0, \quad N \to \infty.$$

Therefore,

$$D_x u(t,x)\big|_{x=0} = 0.$$

Similarly we get that $D_x u(t,x)\big|_{x=1} = 0$.

Taking into account relation (4.10), we get

$$\|u(t,x), W_2^{1,2}(\Pi)\| \le c\|f(t,x), L_2(\Pi)\| + \|\tilde{u}^N(t,x) - u(t,x), W_2^{1,2}(\Pi)\|,$$

$$\|D_{tx}^2 u(t,x), L_2(\Pi)\| \le c\|f(t,x), L_2(\Pi)\| + \|D_{tx}^2 u(t,x) - D_{tx}^2\tilde{u}^N(t,x), L_2(\Pi)\|.$$

From here due to the convergence (4.7), (4.8), for the function $u(t,x) \in W_2^{1,2}(\Pi)$ follows the enequality (2.5) and

$$\|D_{tx}^2 u(t,x), L_2(\Pi)\| \le c\|f(t,x), L_2(\Pi)\|.$$

We will show that the limit function $u \in W_2^{1,2}(\Pi)$ is a generalized solution of the first boundary value problem (2.1) given $\varphi_1(x) = 0$, $\varphi_2(x) = 0$.

Multiplying the k-th relation in (4.2) by $\tilde{c}_k(t)$ and taking a sum over k from 1 to l, $l \in \mathbb{N}$, given the notations

$$v^l(t,x) = \sum_{k=1}^{l} \tilde{c}_k(t)v_k(x), \quad \tilde{c}_k(t)\big|_{t=T} = 0, \quad \tilde{c}_k(t) \in W_2^2(0,T), \tag{4.11}$$

we obtain

$$\int_0^1 \Big[(I - D_x^2)D_t^2 u^m(t,x) + D_x^4 u^m(t,x) - a^2 D_x^2 u^m(t,x)\Big] v^l(t,x)\,dx$$

$$= \int_0^1 f(t,x)v^l(t,x)\,dx, \quad m \ge l.$$

We integrate from 0 to t and use the formula for integrating by parts, taking into account that $D_t u^m(t,x)\big|_{t=0} = 0$ and $v^l(t,x)$ by construction satisfies (2.3), we will have

$$\int_0^T\int_0^1 \Big[D_t u^m(t,x)D_t v^l(t,x) + D_{tx}^2 u^m(t,x)D_{tx}^2 v^l(t,x) - a^2 D_x u^m(t,x)D_x v^l(t,x)$$

$$-D_x^2 u^m(t,x) D_x^2 v^l(t,x)\Big] dx\,dt = -\int_0^T\int_0^1 f(t,x)v^l(t,x)dx\,dt, \quad m \geq l,$$

or, taking into account (4.6),

$$\int_0^T\int_0^1 \Big[D_t\tilde{u}^N(t,x)D_tv^l(t,x) + D_{tx}^2\tilde{u}^N(t,x)D_{tx}^2v^l(t,x) - a^2 D_x\tilde{u}^N(t,x)D_xv^l(t,x)$$

$$-D_x^2\tilde{u}^N(t,x)D_x^2v^l(t,x)\Big] dx\,dt = -\int_0^T\int_0^1 f(t,x)v^l(t,x)dx\,dt, \quad N \geq l.$$

Due to the fact that $\tilde{u}^N$ strongly converges to u in the norm $W_2^{1,2}(\Pi)$ and $D_{tx}^2\tilde{u}^N$ strongly converges to $D_{tx}^2 u$ in the norm $L_2(\Pi)$ as $N \to \infty$, taking into account Hölder's inequality, we obtain

$$\int_0^T\int_0^1 \Big|\big(D_t\tilde{u}^N(t,x) - D_tu(t,x)\big)D_tv^l(t,x) + \big(D_{tx}^2\tilde{u}^N(t,x) - D_{tx}^2u(t,x)\big)D_{tx}^2v^l(t,x)$$

$$-a^2\big(D_x\tilde{u}^N - D_xu\big)D_xv^l - \big(D_x^2\tilde{u}^N - D_x^2u\big)D_x^2v^l\Big| dx\,dt$$

$$\leq c\bigg(\|\tilde{u}^N(t,x) - u(t,x), W_2^{1,2}(\Pi)\| + \|D_{tx}^2\tilde{u}^N(t,x) - D_{tx}^2u(t,x), L_2(\Pi)\|\bigg) \to 0$$

as $N \to \infty$. Therefore, relation (2.4) holds for the functions $v(t,x) = v^l(t,x)$ from (4.11):

$$\int_0^T\int_0^1 \Big[D_tu(t,x)D_tv^l(t,x) + D_{tx}^2u(t,x)D_{tx}^2v^l(t,x) - a^2 D_xu(t,x)D_xv^l(t,x)$$

$$-D_x^2u(t,x)D_x^2v^l(t,x)\Big] dx\,dt = -\int_0^T\int_0^1 f(t,x)v^l(t,x)dx\,dt. \tag{4.12}$$

Since $C^\infty(\overline{\Pi})$ is everywhere dense in $W_2^{1,2}(\Pi)$ (see, for example, [17]), then it suffices to justify relation (2.4) for an arbitrary function $v(t,x) \in C^\infty(\overline{\Pi})$, satisfying (2.3).

Recall that $v_k(x)$ is an orthonormal basis in $\mathring{W}_2^2(0,1)$, therefore, taking into account that for almost every $t \in (0,T)$ $v(t,x) \in \mathring{W}_2^2(0,1)$, $D_tv(t,x) \in \mathring{W}_2^2(0,1)$, we obtain the following representations:

$$v(t,x) = \sum_{k=1}^\infty \tilde{c}_k(t)v_k(x), \tag{4.13}$$

$$D_tv(t,x) = \sum_{k=1}^\infty D_t\tilde{c}_k(t)v_k(x), \tag{4.14}$$

where $\tilde{c}_k(t)$ are the Fourier coefficients of the function v, that is,

$$\tilde{c}_k(t) = \langle v(t,x), v_k(x)\rangle_{\mathring{W}_2^2(0,1)},$$

$\langle\,,\,\rangle_{\mathring{W}_2^2(0,1)}$ is a scalar product in $\mathring{W}_2^2(0,1)$. We denote by

$$v_l(t,x) = \sum_{k=1}^{l} \tilde{c_k}(t) v_k(x),$$

$$D_t v_l(t,x) = \sum_{k=1}^{l} D_t \tilde{c_k}(t) v_k(x)$$

the partial sums of the series (4.13) and (4.14) respectively. The following Parseval–Steklov identity holds:

$$\sum_{k=1}^{\infty} |\tilde{c_k}(t)|^2 + |D_t \tilde{c_k}(t)|^2 = \|v, \mathring{W}_2^2(0,1)\|^2 + \|D_t v, \mathring{W}_2^2(0,1)\|^2. \qquad (4.15)$$

Due to Steklov's inequality,

$$\|D_t v(t,x) - D_t v_l(t,x), L_2(0,1)\| \le c\|D_t v(t,x) - D_t v_l(t,x), \mathring{W}_2^2(0,1)\|,$$

$$\|D_{tx}^2 v(t,x) - D_{tx}^2 v_l(t,x), L_2(0,1)\| \le c\|D_t v(t,x) - D_t v_l(t,x), \mathring{W}_2^2(0,1)\|.$$

Then

$$\|v(t,x) - v_l(t,x), \mathring{W}_2^2(0,1)\|^2 + \|D_t v(t,x) - D_t v_l(t,x), L_2(0,1)\|^2$$

$$+\|D_{tx}^2 v(t,x) - D_{tx}^2 v_l(t,x), L_2(0,1)\|^2 \le \|v(t,x) - v_l(t,x), \mathring{W}_2^2(0,1)\|^2$$

$$+\tilde{C}\|D_t v(t,x) - D_t v_l(t,x), \mathring{W}_2^2(0,1)\|^2 = \sum_{k=l+1}^{\infty} \left(|\tilde{c_k}(t)|^2 + \tilde{C}|D_t \tilde{c_k}(t)|^2 \right).$$

Due to (4.15), for every $t \in (0,T)$ the series converges. We integrate with respect to t from 0 to T

$$\int_0^T \Bigg[\|v(t,x) - v_l(,t,x), \mathring{W}_2^2(0,1)\|^2 + \|D_t v(t,x) - D_t v_l(t,x), L_2(0,1)\|^2$$

$$+\|D_{tx}^2 v(t,x) - D_{tx}^2 v_l(t,x), L_2(0,1)\|^2 \Bigg] dt \le \tilde{C} \int_0^T \sum_{k=l+1}^{\infty} \left(|\tilde{c_k}(t)|^2 + |D_t \tilde{c_k}(t)|^2 \right) dt.$$

Therefore,

$$\|v(t,x) - v_l(t,x), W_2^{1,2}(\Pi)\| \to 0, \quad \|D_{tx}^2 v(t,x) - D_{tx}^2 v_l(t,x), L_2(\Pi)\| \to 0 \qquad (4.16)$$

as $l \to \infty$. Substituting into (4.11), (4.12) as $v^l(t,x)$ a partial sum (4.13), and taking into account the convergence (4.16), we obtain the required equality (2.4) for every $v(t,x) \in C^\infty(\overline{\Pi})$, satisfying (2.3).

Theorem 2 is proved.

The proof of existence of a generalized solution of the boundary value problem (2.1) in the case when $f(t,x) = 0$, $\varphi_j(x) \ne 0$, $j = 1, 2$, replicates the reasoning used in Theorem 2.

The authors would like to thank G. V. Demidenko for the statement of the problem and for the attention to their work.

References

[1] S.L. Sobolev, *Selected works of S.L. Sobolev. Volume I: Equations of mathematical physics, computational mathematics, and cubature formulas*, Springer, New York, 2006. Zbl 1117.01022

[2] G.V. Demidenko, S.V. Uspenskii, *Partial differential equations and systems not solvable with respect to the highest-order derivative*, Marcel Dekker, New York, 2003. Zbl 1061.35001

[3] G.V. Demidenko, *Solvability conditions of the Cauchy problem for pseudohyperbolic equations*, Sib. Math. J., **56**:6 (2015), 1028–1041. Zbl 1336.35250

[4] G.V. Demidenko, *On solvability of the Cauchy problem for pseudohyperbolic equations*, Sib. Adv. Math., **11**:4 (2001), 25–40. Zbl 1049.35141

[5] Kh.G. Umarov, *Cauchy problem for the torsional vibration equation of a nonlinear-elastic rod of infinite length*, Mech. Solids, **54**:5 (2019), 726–740. Zbl 1459.74080

[6] I. Fedotov, L.R. Volevich, *The Cauchy problem for hyperbolic equations not resolved with respect to the highest time derivative*, Russ. J. Math. Phys., **13**:3 (2006), 278–292. Zbl 1113.35134

[7] I. Fedotov, M. Shatalov, J. Marais, *Hyperbolic and pseudo-hyperbolic equations in the theory of vibration*, Acta Mech., **227**:11 (2016), 3315–3324. Zbl 1388.74061

[8] S.I. Gerasimov, V.I. Erofeev, Problems of wave dynamics for structural elements, Ross. Fed. Yad. Tsentr-Vseross. Nauchn.-Issled. Inst. Eksp. Fiz., Sarov, 2014.

[9] V.Z. Vlasov, *Thin-walled elastic beams*, National Science Foundation, Washington, D.C., 1961.

[10] R.E.D. Bishop, *Longitudinal waves in beams*, Aero. Quart., **3**:4 (1952), 280–293. MR0066902

[11] J.S. Rao, *Advanced theory of vibration*, Wiley Eastern, New Delhi, 1992.

[12] G.V. Demidenko, *Sobolev spaces and generalized solutions*, NSU, Novosibirsk, 2015.

[13] V.P. Mikhailov, *Partial differential equations*, Mir, Moscow, 1978. MR0601389

[14] S.L. Sobolev, *Introduction to the theory of cubature formulas*, Nauka, Moscow, 1974. Zbl 0294.65013

[15] V.A. Trenogin, *Functional analysis*, Nauka, Moscow, 1980. Zbl 0517.46001

[16] K.Yosida, *Functional analysis*, Springer-Verlag, Berlin, 1994. Zbl 0830.46001

[17] S.V. Uspenskii, G.V. Demidenko, V.G. Perepelkin, *Imbedding theorems and applications to differential equations*, Nauka, Novosibirsk, 1984. Zbl 0575.46030

Lina Nikolaevna Bondar
Sobolev Institute of Mathematics,
4, Koptyuga ave.,
Novosibirsk, 630090, Russia
Email address: b_lina@ngs.ru

Vakhobbiddin Nurmakhmatov
Novosibirsk State University,
1, Pirogova str.,
Novosibirsk, 630090, Russia
Email address: v.nurmakhmatov@g.nsu.ru