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ITERATIVELY REGULARIZED GAUSS–NEWTON METHOD IN THE INVERSE PROBLEM OF IONOSPHERIC RADIOSONDING

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Abstract. The paper is concerned with the problem of reconstructing the vertical profile of the electron concentration of the ionosphere. The profile is reconstructed based on the results of measuring the incident phase of the probing signal from a moving satellite. The simplest measurement model with a single point of signal reception is adopted. The model under investigation takes into account the curvature of the probe beam when passing through the inhomogeneous ionosphere. The problem is reduced to a nonlinear integral equation. We prove that the resulting equation has a non–unique solution. To approximate the solution closest to the selected initial approximation, an iteratively regularized Gauss–Newton method is used with a projection on the set defined by a priori constraints on the solution. The results of numerical experiments are presented.

Keywords: nonlinear equation, irregular equation, iterative regularization, ionosphere, radiotomography.

1. Description of the model

The object of research in this work is the problem of ionospheric tomography using satellite radio signals. The ionosphere is the upper part of the atmosphere at altitudes of 60–2000 km, consisting of ionized gas. The causes of ionization are solar wind and solar radiation. The composition of the ionosphere varies in height, which causes the presence of areas of different ionization, called layers. The problem of ionospheric tomography has a variety of practical applications, for

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example, in assessing the total electronic content and taking into account plasma inhomogeneities for correcting the operation of short–wave radio navigation systems [1]. GPS and GLONASS satellite navigation systems are used for this purpose.

Free electrons are the main carriers of charges in the ionosphere, affecting the passage of electromagnetic radiation through the ionosphere. The concentration of free electrons $\mathcal{N}$ is one of the most important parameters of the ionosphere. In this paper we assume that the characteristics of the ionosphere depend only on the height z above the Earth's surface. The dependence of the electron concentration on the height is called the electron concentration profile. Figure 1 shows examples of typical electron concentration profiles $\mathcal{N} = \mathcal{N}(z)$ for day (solid line) and night (broken line). At night, only one maximum is usually observed on the profile, corresponding to the ionospheric layer F2. During the day, along with the increase in solar activity, the number of maxima also increases, but the maximum corresponding to the F2 layer remains the largest in magnitude. The paper studies the problem of reconstructing the profile $\mathcal{N}(z)$ based on results of measuring the phase incursion during the passage of the probing signal through the ionosphere. The main attention is paid to determining the maximum value of the electron concentration and its position in height.

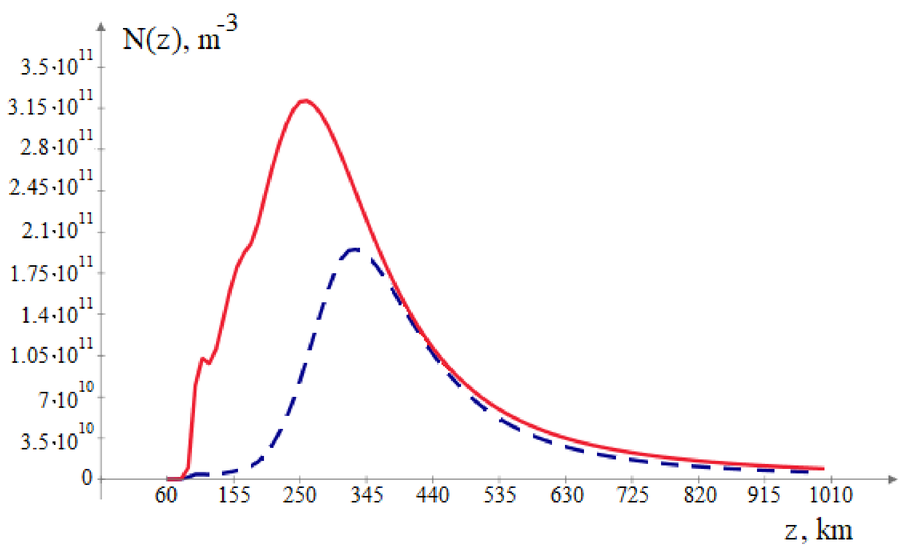

FIGURE 1 . Typical electron concentration profiles

The ionosphere is a randomly inhomogeneous medium and has significant dispersion properties, so that signals with different frequencies propagate in it at different speeds. Denote by c_0 and $c(z)$ the speeds of light in vacuum and in the medium at an altitude z, respectively. The ratio $n(z) = c(z)/c_0$ is called the refraction coefficient of the inhomogeneous medium. The friction force and electromagnetic force have a negligible effect on high–frequency signals. In this paper, for a frequency signal $\nu = 15.42$ MHz, we will consider the refraction coefficient as a value that depends only on the electrostatic force. This allows to write the representation [2], [3]

$$n(z) = \sqrt{1 - \varepsilon\mathcal{N}(z)}, \ \varepsilon = \frac{80.8}{\nu^2}. \tag{1.1}$$

Assume for simplicity that the moving satellite and the receiving point are in the same plane (x, z). The receiver is located at the point $(0, 0)$, the satellite has coordinates (χ, h), where $\chi \in [\chi_1, \chi_2]$ is the horizontal coordinate of the satellite, h is its height above the Earth's surface. The probing signal from the satellite propagates from the point (χ, h) along the geodesic curve $\gamma(\chi)$ in the Riemannian metric $d\sigma^2 = (dx^2 + dz^2)/n^2(z)$ connecting the points (χ, h) and $(0, 0)$. It is convenient to parametrize geodesics by the angle of inclination at the point $(0, 0)$ with respect to the positive direction of the axis x. Everywhere below we assume that $c(0) = c_0$. The geodesic coming out from $(0, 0)$ with the unit tangent vector $(\cos\alpha, \sin\alpha)$ is determined by the following system of differential equations with respect to $(x(t), y(t), z(t); p_x(t), p_y(t), p_z(t))$ [4]:

$$\dot{x} = c^2(z)p_x,\ \dot{y} = c^2(z)p_y,\ \dot{z} = c^2(z)p_z,$$

$$\dot{p}_x = -c^{-1}(z)\frac{\partial c(z)}{\partial x} \equiv 0,\ \dot{p}_y = -c^{-1}(z)\frac{\partial c(z)}{\partial y} \equiv 0,$$

$$\dot{p}_z = -c^{-1}(z)\frac{\partial c(z)}{\partial z} = -c^{-1}(z)c'(z).$$

Here t is the physical time of the signal movement, while along the geodesic $\gamma(\chi)$ we have $x = x(t)$, $y(t) \equiv 0$, $z = z(t)$. The system is solved together with the initial conditions

$$x(0) = 0,\ y(0) = 0,\ z(0) = 0;$$
$$p_x(0) = c_0^{-1}\cos\alpha,\ p_y(0) = 0,\ p_z(0) = c_0^{-1}\sin\alpha.$$

By virtue of these equalities, $p_x(t) \equiv c_0^{-1}\cos\alpha$, $p_y(t) \equiv 0$. Therefore the original system is simplified as follows:

$$\dot{x} = c^2(z)c_0^{-1}\cos\alpha,\ \dot{z} = c^2(z)p_z,\ \dot{p}_z = -c^{-1}(z)c'(z);$$
$$x(0) = 0,\ z(0) = 0,\ p_z(0) = c_0^{-1}\sin\alpha. \tag{1.2}$$

For all $t \geq 0$, the eikonal equation $p_x^2 + p_y^2 + p_z^2 = c^{-2}(z)$ holds. Thus,

$$(c_0^{-1}\cos\alpha)^2 + (p_z)^2 = c^{-2}(z).$$

It follows that

$$p_z = \sqrt{\frac{1}{c^2(z)} - \frac{\cos^2\alpha}{c_0^2}}.$$

The plus sign is taken before the root, since the function $z(t)$ increases along the ray, see equation $\dot{z} = c^2(z)p_z$ in (1.2). From (1.2) we obtain the following system for $(x(t), z(t))$:

$$\dot{x} = c^2(z)c_0^{-1}\cos\alpha,\ \dot{z} = c^2(z)\sqrt{\frac{1}{c^2(z)} - \frac{\cos^2\alpha}{c_0^2}}. \tag{1.3}$$

The initial conditions for (1.3) has the form

$$x(0) = 0,\ z(0) = 0. \tag{1.4}$$

From (1.3) with the use of equality $c(z) = c_0 n(z)$ we get

$$\frac{dx}{dz} = \frac{1}{\sqrt{\frac{1}{n^2(z)\cos^2\alpha} - 1}}. \tag{1.5}$$

Since according to (1.4), $x = 0$ for $z = 0$, this implies an explicit representation for the geodesic $\gamma(\chi)$:

$$x = x(z) = \int_0^z \frac{dz}{\sqrt{\frac{1}{n^2(z)\cos^2\alpha} - 1}}, \ z \in [0, h].$$

Letting $z = h$, for the horizontal coordinate of the satellite we get the representation

$$\chi = x(h) = \int_0^h \frac{dz}{\sqrt{\frac{1}{n^2(z)\cos^2\alpha} - 1}}. \tag{1.6}$$

As is known from [5], for a radio signal with a wavelength of Λ, the phase incursion is determined by the expression

$$\Phi(\chi) = \kappa \int_{\gamma(\chi)} \mathcal{N} d\sigma, \ \chi \in [\chi_1, \chi_2], \tag{1.7}$$

where $\kappa = \Lambda r_e$, r_e is the standard electron radius. By (1.1) and (1.5), with the use of equality

$$d\sigma = \frac{\sqrt{dx^2 + dz^2}}{n(z)} = \frac{1}{n(z)} \sqrt{1 + \Big(\frac{dx}{dz}\Big)^2} dz$$

we get

$$\mathcal{N}(z) = \frac{1 - n^2(z)}{\varepsilon}, \ d\sigma = \frac{dz}{n^2(z)\sqrt{\frac{1}{n^2(z)} - \cos^2\alpha}}. \tag{1.8}$$

From (1.7) and (1.8) we obtain

$$\Phi(\chi) = \frac{\kappa}{\varepsilon} \int_0^h \Big(\frac{1}{n^2(z)} - 1\Big) \frac{dz}{\sqrt{\frac{1}{n^2(z)} - \cos^2\alpha}}. \tag{1.9}$$

Denote

$$\mu(z) = \frac{1}{n^2(z)} - 1, \tag{1.10}$$

then in view of (1.8),

$$\mathcal{N}(z) = \frac{\mu(z)}{\varepsilon(1 + \mu(z))}. \tag{1.11}$$

According to (1.7), the measurement results give the function $\Psi(\chi) = \frac{\varepsilon}{\kappa}\Phi(\chi)$ for $\chi \in [\chi_1, \chi_2]$. From (1.9) and (1.10) we get

$$\Psi(\chi) = \int_0^h \frac{\mu(z) dz}{\sqrt{\mu(z) + \sin^2\alpha}}. \tag{1.12}$$

Suppose that the angle α changes on the segment $[a_1, a_2]$ when $\chi \in [\chi_1, \chi_2]$. Combining (1.6) and (1.12), we arrive at the following nonlinear integral equation for the function $\mu = \mu(z)$:

$$\int_0^h \frac{\mu(z)dz}{\sqrt{\mu(z)+\sin^2\alpha}} = \Psi\Big(\cos\alpha \int_0^h \frac{dz}{\sqrt{\mu(z)+\sin^2\alpha}}\Big), \ \alpha \in [a_1, a_2].$$

Let H_0 and H be, respectively, lower and upper bounds of heights such that below H_0 and above H the electron concentration $\mathcal{N}$ differs little from zero, so that the values $\mu(z)$ at these z give an insignificant contribution to the integral. Then the previous equality takes the form

$$\int_{H_0}^H \frac{\mu(z)dz}{\sqrt{\mu(z)+\sin^2\alpha}} = \Psi\Big(H_0\mathrm{ctg}\alpha + \cos\alpha \int_{H_0}^H \frac{dz}{\sqrt{\mu(z)+\sin^2\alpha}}\Big), \ \alpha \in [a_1, a_2]. \tag{1.13}$$

Equation (1.13) is the main object of theoretical and numerical analysis in this work.

2. Study of the mathematical model

Turning to the study of equation (1.13), we describe a class of functions μ, a priori containing the desired solutions. Following the discussion in §1, we assume that for some $0 < H_j < H_{j+1}$, $0 \le j \le 4$ with $H_5 = H$, the function $\mu \in C^l[H_0, H]$, $l \ge 2$, μ is convex on $[H_0, H_1]$, increases on $[H_1, H_2]$, concave on $[H_2, H_3]$, decreases on $[H_3, H_4]$ and convex on $[H_4, H_5]$. Denote this class of functions by $\mathfrak{M}$.

We now prove that the solution $\mu \in \mathfrak{M}$ of equation (1.13) is not unique for any $l \ge 2$. Let there be a solution $y = \mu(z)$ whose graph on some segment $[a_0, a] \subset (H_2, H_3)$, $\mu(a_0) = \mu(a)$, has a maximum at $z = z^* \in (a_0, a)$. Assume that

$$\overline{\mu} = \max_{z\in[a_0,a]} \mu''(z) < 0.$$

Below we will show how to obtain from this solution another smooth solution of equation (1.13) with a concave graph on $[a_0, a]$ and a different position of the extremum. The value of the extremum $M = \mu(z^*)$ will remain unchanged. Without changing the function μ outside of $[a_0, a]$, in this segment we will subject it to the variation described below. Denote $M_0 = \mu(a_0) = \mu(a)$. We choose an arbitrarily smooth monotonically increasing function $g : [M_0, M] \to [0, \tau]$ such that

$$g(M_0) = g'(M_0) = \cdots = g^{(l)}(M_0) = 0, \ g(M) = \tau, \ \tau \in (0, a - z^*), \tag{2.1}$$

and denote

$$\begin{aligned} \lambda_1 &= \max_{z\in[a_0,a]} |\mu'(z)|, \ \lambda_2 = \max_{z\in[a_0,a]} |\mu''(z)|, \\ \eta_1 &= \max_{y\in[M_0,M]} g'(y), \ \eta_2 = \max_{y\in[M_0,M]} |g''(y)|. \end{aligned} \tag{2.2}$$

Pick a function g subject to conditions

$$\eta_1\lambda_1 < 1, \ \overline{\mu} + \frac{\lambda_1(\eta_1\lambda_2 + \eta_2\lambda_1^2)}{1 - \eta_1\lambda_1} < 0. \tag{2.3}$$

It follows from (2.2) that the conditions (2.3) can be fulfilled by the choice of g having sufficiently small first and second derivatives on $[M_0, M]$. Below we need the auxiliary function

$$\varphi(z) = z + g(\mu(z)).$$

By virtue of the first condition in (2.3),

$$\varphi'(z) = 1 + g'(\mu(z))\mu'(z) \geq 1 - \eta_1\lambda_1 > 0,\ z \in [a_0, a]. \tag{2.4}$$

Therefore $\varphi(z)$ increases monotonically on $[a_0, a]$ and maps this segment into itself. Therefore, for any $w \in [a_0, a]$ there is a unique solution $z = z[w]$ of the equation $\varphi(z) = w$. Consider the function $\widetilde{\mu}$, which outside $[a_0, a]$ coincides with μ, and on $[a_0, a]$ is defined by the equality

$$\widetilde{\mu}(w) = \mu(z[w]),\ w \in [a_0, a]. \tag{2.5}$$

Symbolically, we write (2.5) as $\widetilde{\mu} = \mathcal{F}_g(\mu)$. Equality (2.5) means that the graph of $y = \widetilde{\mu}(z)$ is constructed as follows. Pick $y_0 \in [M_0, M)$ and consider the intersection points (z_1, y_0), (z_2, y_0) of the line $y = y_0$ with the graph $y = \mu(z)$. Then, on the specified line, points are marked that are spaced to the right of these points by the distance $g(y_0)$. The two points obtained are the intersection points of $y = y_0$ with the graph $y = \widetilde{\mu}(z)$. Due to condition (2.1), the smoothness of the joining of $y = \widetilde{\mu}(z)$ on $[a_0, a]$ to $y = \mu(z)$ outside $[a_0, a]$ is preserved.

It is easy to see that the operator $\mathcal{F}_g$ is also well defined for any convex or concave function $\theta \in C^2[a_0, a]$ satisfying the conditions

$$\theta(a_0) = \theta(a),\ \eta_1 \max_{z \in [a_0, a]} |\theta'(z)| < 1 \tag{2.6}$$

and having an extremum in (a_0, a).

According to (2.5), $\widetilde{\mu}(\varphi(z)) = \mu(z)$ for $z \in [a_0, a]$. Differentiating this equality twice, we get

$$\widetilde{\mu}''(\varphi(z))(\varphi'(z))^2 + \widetilde{\mu}'(\varphi(z))\varphi''(z) = \mu''(z). \tag{2.7}$$

Here,

$$\varphi''(z) = g''(\mu(z))(\mu'(z))^2 + g'(\mu(z))\mu''(z),\ \widetilde{\mu}'(\varphi(z)) = \frac{\mu'(z)}{\varphi'(z)}.$$

Therefore, taking into account (2.2) and (2.4),

$$|\widetilde{\mu}'(\varphi(z))| \leq \frac{\lambda_1}{1 - \eta_1\lambda_1},\ |\varphi''(z)| \leq \eta_1\lambda_2 + \eta_2\lambda_1^2. \tag{2.8}$$

From (2.7), (2.8) and the second condition in (2.3) it follows that

$$\widetilde{\mu}''(\varphi(z))(\varphi'(z))^2 < 0,\ z \in [a_0, a].$$

Thus $\widetilde{\mu}''(\varphi(z)) < 0$ for all $z \in [a_0, a]$. Since the function φ uniquely maps the segment $[a_0, a]$ to itself, for all $w \in [a_0, a]$ we have $\widetilde{\mu}''(w) < 0$. Thus, the constructed function $\widetilde{\mu}$ is concave on $[a_0, a]$, along with μ.

For functions

$$\theta_1(z) = \frac{1}{\sqrt{\mu(z) + \sin^2\alpha}},\ \theta_2(z) = \frac{\mu(z)}{\sqrt{\mu(z) + \sin^2\alpha}},$$

using (2.2) it is not difficult to get estimates

$$\max_{z \in [a_0, a]} |\theta_1'(z)| \leq \frac{\lambda_1}{2M_0^{3/2}},\ \max_{z \in [a_0, a]} |\theta_2'(z)| \leq \frac{\lambda_1}{\sqrt{M_0}}.$$

In addition, by virtue of the equality $\mu(a_0) = \mu(a)$ we have $\theta_1(a_0) = \theta_1(a)$, $\theta_2(a_0) = \theta_2(a)$. Further, the function θ_1 is convex, and θ_2 is concave on $[a_0, a]$. It follows from (2.6) that if the additional condition is fulfilled

$$\eta_1\lambda_1 \max\Big\{\frac{1}{2M_0^{3/2}}, \frac{1}{\sqrt{M_0}}\Big\} < 1, \tag{2.9}$$

then the operator $\mathcal{F}_g$ is well defined for functions θ_1, θ_2, and

$$\mathcal{F}_g(\theta_1)(z) = \frac{1}{\sqrt{\widetilde{\mu}(z) + \sin^2\alpha}}, \ \mathcal{F}_g(\theta_2)(z) = \frac{\widetilde{\mu}(z)}{\sqrt{\widetilde{\mu}(z) + \sin^2\alpha}}.$$

For any function $h = h(z)$, positive and continuous on $[a_0, a]$, we have the identity [6, Theorem 3.6.3]

$$\int_{a_0}^{a} h(z)dz = \int_0^\infty \mathrm{meas}\{z \in [a_0, a] : h(z) > t\}dt.$$

Here, meas$Q$ is the Lebesgue measure of the set $Q \subset \mathbb{R}$. By construction,

$$\mathrm{meas}\{z \in [a_0, a] : \mathcal{F}_g(\theta_j)(z) > t\} = \mathrm{meas}\{z \in [a_0, a] : \theta_j(z) > t\}.$$

Consequently,

$$\int_{a_0}^{a} \mathcal{F}_g(\theta_j)(z)dz = \int_{a_0}^{a} \theta_j(z)dz, \ j = 1, 2.$$

Thus, if g satisfies (2.1), (2.3), (2.4), then the function $\widetilde{\mu} = \mathcal{F}_g(\mu)$ is the solution of equation (1.13) in the class $\mathfrak{M}$. We see that equation (1.13) has a continuous family of smooth solutions.

3. Finite dimensional approximation and iteratively regularized Gauss–Newton method

In order to discretize equation (1.13), we introduce uniform grids $\{z_i\}_{i=0}^N \subset [H_0, H]$ and $\{\alpha_j\}_{j=0}^J \subset [a_1, a_2]$ with steps $h_z = (H - H_0)/N$ and $h_\alpha = (a_2 - a_1)/J$ respectively, where $J \geq N$. Using the rectangular scheme for approximating the integrals in (1.13), we obtain the following system of nonlinear equations

$$h_z \sum_{i=0}^{N} \frac{\mu_i}{\sqrt{\mu_i + \sin^2\alpha_j}} - \Psi\Big(H_0 \cot\alpha_j + h_z \sum_{i=0}^{N} \frac{\cos\alpha_j}{\sqrt{\mu_i + \sin^2\alpha_j}}\Big) = 0, \ 0 \leq j \leq J. \tag{3.1}$$

In (3.1), μ_i are the required approximations to $\mu(z_i)$, $0 \leq i \leq N$. The above–mentioned property of non–uniqueness of the solution to equation (1.13) is naturally transferred to its discrete approximation (3.1). It is not difficult to see that along with any of its solutions $(\mu_0, \dots, \mu_n)$, any of the $(N+1)! - 1$ permutations of the values of $(\mu_0, \dots, \mu_n)$ is also a solution of (3.1).

To narrow down the set of solutions taken for consideration, it is necessary to clarify a priori information about solutions of physical interest. Assume that we know the lower bound $b_0 > 0$ for $\min_{z\in[H_0,H_1]} \mu''(z)$, the lower bound $b_1 > 0$ for $\min_{z\in[H_1,H_2]} \mu'(z)$, the upper bound $b_2 < 0$ for $\max_{z\in[H_2,H_3]} \mu''(z)$, the upper bound $b_3 < 0$ for $\max_{z\in[H_3,H_4]} \mu'(z)$, and the lower bound $b_4 > 0$ for $\min_{z\in[H_4,H_5]} \mu''(z)$. Also suppose that functions $p(z) < q(z)$, $z \in (H_0, H)$ are given such that $p(z) \leq$

$\mu(z) \le q(z),\ z \in [H_0, H]$. In this case, the set of a priori constraints $\mathcal{D}$ in problem (3.1) can be described as follows:

(3.2)
$$\begin{aligned}
&\mathcal{D} = \{(\mu_0, \dots, \mu_N) : p(z_i) \le \mu_i \le q(z_i),\ 0 \le i \le N;\\
&\mu_{i-1} - 2\mu_i + \mu_{i+1} \ge b_0 h_z^2,\ 1 \le i \le K_1;\ \mu_{i+1} - \mu_i \ge b_1 h_z,\ K_1 \le i \le K_2;\\
&\mu_{i-1} - 2\mu_i + \mu_{i+1} \le b_2 h_z^2,\ K_2 \le i \le K_3;\\
&\mu_{i+1} - \mu_i \le b_3 h_z,\ K_3 \le i \le K_4,\ \mu_{i-1} - 2\mu_i + \mu_{i+1} \ge b_4 h_z^2,\ K_4 \le i \le K_5 - 1\}.
\end{aligned}$$

Here, $H_j = H_0 + K_j h_z,\ K_j \in \mathbb{N},\ 1 \le j \le 4$. Conditions (3.2) define a convex polyhedron in $\mathbb{R}^{N+1}$.

It is convenient to formalize the system (3.1) with a priori constraints (3.2) in the form of an operator equation

$$F(u) = f,\ u \in D. \tag{3.3}$$

Here, $F : \mathcal{H}_1 \to \mathcal{H}_2$ is a Frechet differentiable operator, $\mathcal{H}_1$, $\mathcal{H}_2$ are Hilbert spaces, and $D \subset \mathcal{H}_1$ is a convex closed set that a priori contains the solution of interest. In general, the element f in (3.3) is given with errors, so that an approximation $\widetilde{f}$ and the error level $\delta > 0$ are available instead, where $\|\widetilde{f} - f\|_{\mathcal{H}_2} \le \delta$. Additionally, we assume that the derivative F' satisfies the Lipschitz condition in a neighborhood of the desired solution $u^* \in D$, i.e.,

$$\|F'(u) - F'(v)\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} \le L\|u - v\|_{\mathcal{H}_1},\ u, v \in \Omega_R(u^*),$$

where $\Omega_R(u^*) = \{u \in \mathcal{H}_1 : \|u - u^*\|_{\mathcal{H}_1} \le R\}$. By $\|\cdot\|_z$ we denote the norm of the Hilbert or Banach space Z, $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is the space of linear continuous operators acting from $\mathcal{H}_1$ to $\mathcal{H}_2$. Under our conditions, equation (3.3) is irregular in the sense that the continuous invertibility of the derivative $F'(u)$ or the symmetrized derivative $F'^*(u)F'(u)$ for points u from the neighborhood of the solution is not guaranteed. This circumstance prevents the use of the standard iterative Newton–Kantorovich and Gauss–Newton processes [7],[8], since these processes assume a stable inversion of the mentioned operators at each iteration. Gradient type methods [7],[9] as applied to (3.3), are formally realizable, but the convergence of approximations they produce can be established only if restrictive nonlinearity conditions on F are fulfilled. In this paper, we turn to a group of iteratively regularized methods of the Gauss–Newton type [10, Ch.4] whose convergence can be justified without involving the regularity or nonlinearity conditions. To solve the system (3.1) with constraints (3.2), we use the iteratively regularized Gauss–Newton method, which, when applied to the problem (3.3), has the form [10, §4.2]

$$\begin{aligned}
&u^{n+1} = P_D(\overline{u}^{n+1}),\\
&\overline{u}^{n+1} = \xi - (F'^*(u^n)F'(u^n) + \alpha_n E_1)^{-1} F'^*(u^n)\cdot\\
&\cdot [F(u^n) - \widetilde{f} - F'(u^n)(u^n - \xi)].
\end{aligned} \tag{3.4}$$

By P_D we denote the operator of metric projection from $\mathcal{H}_1$ onto D,

$$P_D(u) \in D,\ \|P_D(u) - u\|_{\mathcal{H}_1} = \min_{v \in D} \|v - u\|_{\mathcal{H}_1};\ u \in \mathcal{H}_1,$$

E_1 is the identity operator in $\mathcal{H}_1$, $\xi \in \mathcal{H}_1$ is the parameter, the sequence of regularization parameters $\{\alpha_n\}$ is satisfies

$$0 < \alpha_{n+1} \le \alpha_n, \ \lim_{n\to\infty} \alpha_n = 0, \ \sup_{n=0,1,\dots} \frac{\alpha_n}{\alpha_{n+1}} < \infty.$$

The process (3.4) does not assume unique solvability of equation (3.3) and is intended to approximate the solution u^* satisfying, together with the control parameter ξ, the source condition

$$u^* - \xi = F'^*(u^*)v, \ v \in \mathcal{H}_2. \tag{3.5}$$

Denote by $\mathcal{R}(A)$ the image of a linear bounded operator A. If $\mathcal{R}(F'(u^*)) = \mathcal{H}_2$, then (3.5) is the necessary condition for u^* to be a solution of equation $F(u) = f$ closest to ξ, i.e., the solution of the problem

$$\min\{\|u-\xi\|^2_{\mathcal{H}_1} : F(u) = f, u \in \mathcal{H}_1\},$$

see, e.g., [11, §1.1].

The necessary approximation properties of the method (3.4) are provided by using suitable rules for stopping iterations. Of the greatest practical interest are a posteriori stopping schemes, according to which iterations continue for $n = 0, 1, \dots, N(\delta, \widetilde{f}) - 1$, while the stopping moment $N = N(\delta, \widetilde{f})$ is determined directly during iterative process. We fix the positive parameter m and assume that the initial guess u^0 in (3.4) satisfies

$$\|F(u^0) - \widetilde{f}\|^2_{\mathcal{H}_2} > m\delta.$$

As in [12, Ch.2,§4], we define the stopping moment by the rule

$$\begin{aligned} \|F(u^{N(\delta,\widetilde{f})}) - \widetilde{f}\|^2_{\mathcal{H}_2} \le m\delta < \|F(u^n) - \widetilde{f}\|^2_{\mathcal{H}_2}, \\ n = 0, 1, \dots, N(\delta, \widetilde{f}) - 1. \end{aligned} \tag{3.6}$$

Under a number of additional conditions on the elements of (3.3) and parameters of the process (3.4), it is established that the condition (3.6) determines the finite number $N(\delta, \widetilde{f})$. At the same time, for the resulting approximation $u^{N(\delta,\widetilde{f})}$ there is an accuracy estimate [12, Ch.2,§4,Theorem 3]

$$\|u^{N(\delta,\widetilde{f})} - u^*\|_{\mathcal{H}_1} \le l\sqrt{\alpha_{N(\delta,\widetilde{f})}}.$$

Here, u^* is the solution of (3.3) that satisfies condition (3.5).

We can write down the system (3.1) in the form (3.3), putting $u = \widehat{\mu} = (\mu_0, \dots, \mu_n)$, $f = 0$, $F(\widehat{\mu}) = (F_j(\widehat{\mu}))_{j=0}^J$, where $F_j(\widehat{\mu})$ is the left hand side of equation (3.1). We get

$$F'(\widehat{\mu}) = \Big(\frac{\partial F_j(\widehat{\mu})}{\partial \mu_k}\Big)_{j,k=0}^{J,N}, \ \frac{\partial F_j(\widehat{\mu})}{\partial \mu_k} =$$

$$= h_z(\mu_k + \sin^2\alpha_j)^{-3/2}\Big[(\mu_k/2 + \sin^2\alpha_j) + \frac{1}{2}\Psi'\Big(H_0 \cot\alpha_j + h_z \sum_{i=0}^{N} \frac{\cos\alpha_j}{\sqrt{\mu_i + \sin^2\alpha_j}}\Big)\cos\alpha_j\Big],$$

$$0 \le j \le J, \ 0 \le k \le N.$$

In numerical experiments we put

$$\widetilde{f} = (\widetilde{f}_j)_{j=0}^J, \ \widetilde{f}_j = (2\zeta_j - 1)\Delta,$$

where ζ_j are independent random variables uniformly distributed on $[0,1]$, Δ is the level of noise in data. Thus,

$$\delta = \Big(\frac{1}{J}\sum_{j=0}^{J}(\widetilde{f}_j - f_j)^2\Big)^{1/2} = \Delta\Big(\frac{1}{J}\sum_{j=0}^{J}(2\zeta_j - 1)^2\Big)^{1/2}. \tag{3.7}$$

Numerical simulation of measuring the signal received at $(0,0)$ from a satellite moving at the altitude h consists in that we fix the electron concentration density to be reconstructed $\mathcal{N}(z) = \mathcal{N}^*(z)$. Then using $\mathcal{N}^*(z)$, by (1.11) we determine $\mu^*(z) = \varepsilon\mathcal{N}^*(z)/(1-\varepsilon\mathcal{N}^*(z))$, the desired solution of equations (1.13). To stimulate the observation function $\Psi = \Psi(\chi)$, on a suitable grid

$$\chi_j = H_0\cot\alpha_j + h_z\sum_{i=0}^{N}\frac{\cos\alpha_j}{\sqrt{\mu^*(z_i)+\sin^2\alpha_j}},\ j = j_0,\dots,j_s$$

by (3.1) we find corresponding values

$$\eta_j = h_z\sum_{i=0}^{N}\frac{\mu^*(z_i)}{\sqrt{\mu^*(z_i)+\sin^2\alpha_j}},$$

where

$$\Psi(\chi_j) = \eta_j,\ j = j_0,\dots,j_s. \tag{3.8}$$

In the calculations, we let $\Psi(\chi) = P_s(\chi)$, where P_s is the interpolation polynomial of degree s defined by conditions (3.8). We now turn to results of the numerical experiment.

4. Numerical experiment

The discretized problem (3.1), (3.2) was solved with $J = N = 50$. When implementing iterations (3.4), we put $\widehat{\mu}^0(z_i) = \xi(z_i) = p(z_i)$, $0 \le i \le N$, iterations were stopped according to criterion (3.6) with $m = 160$. The sequence of regularization parameters $\{\alpha_n\}$ was $\alpha_n = 0.9^n\alpha_0$ with $\alpha_0 = 100$. The noise level δ, determined according to (3.7), was $\delta \approx 1$.

To generate the model observation function $\Psi(\chi)$, we used the real profile of the electron concentration $\mathcal{N}(z)$, obtained from the International Reference Model of the Ionosphere IRI–2016, which is available at
https://ccmc.gsfc.nasa.gov /modelweb /models/iri2016_vitmo.php.

In (3.8) we put $s = 14$, the array of interpolation points $\{\chi_{j_k}\}_{k=0}^{14}$ was chosen as $\{5, 23, 40, 58, 76, 93, 111, 129, 146,$
$164, 181, 198, 215, 232, 248\}$ (kms). The resulting observation function Ψ is shown on Figure 2.

When constructing the set of constraints (3.2), the functions p and q a priori limiting the desired profile, were determined by $p(z) = 2\cdot 10^{-5}(z - H_0)(H - z) + 0.25\mu^*(z_1)$ and $q(z) = 10^{-4}(z - H_0)(H - z) + 0.25\mu^*(z_1)$ respectively. We put $H = 700$, $\{H_j\}_{j=0}^4 = \{50, 200, 250, 400, 510\}$, where the values are given in kms. The constants b_j defining a priori constraints of the derivatives and second derivatives of the desired solution were chosen as $b_j = \epsilon b_j^*$, where b_j^* is the value of the minimum or maximum of the related derivative for the exact function μ^* on the corresponding interval, $0 \le j \le 4$. The parameter $\epsilon \in (0,1)$ characterizes the degree of availability of the specified information when processing the observation results.

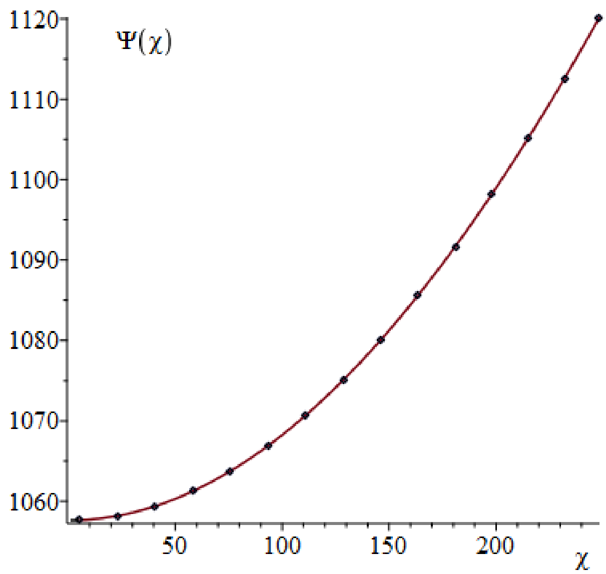

FIGURE 2 . Model observation function Ψ

In the calculations, we used the values $\epsilon = 0.32$, $\epsilon = 0.45$, $\epsilon = 0.5$. The number of iterations determined by criterion (3.6) in all cases turned out to be equal to 3. The results for different ϵ are given on Figure 3.

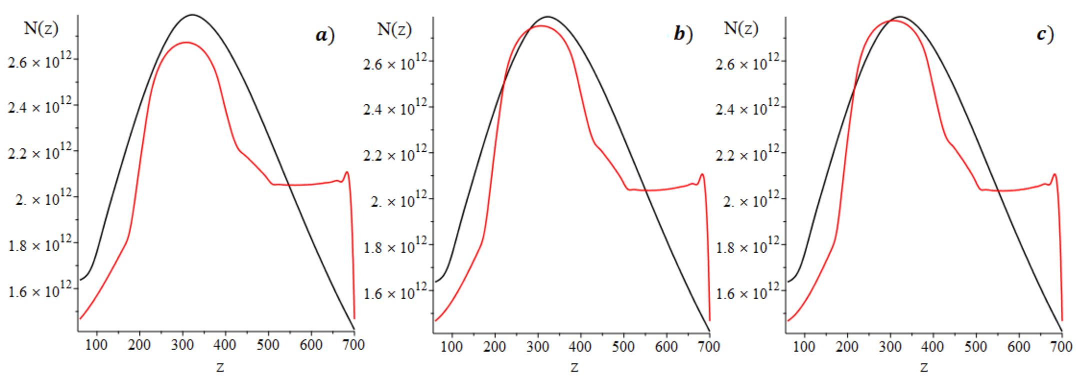

FIGURE 3 . Reconstructed electron concentration profiles (smooth line) against the true ones (polyline): a) ϵ=0.32, b) ϵ=0.45, c) ϵ=0.5

When modeling the processes of propagation of radio waves in the ionosphere, the localization and the value of the maximum electron concentration are of considerable interest. In the example, the error in determining the localization of the maximum point of the function $\mathcal{N}$ is about 9% and weakly depends on the parameter ϵ. On the other hand, the accuracy of the approximation of the maximum value increases with the increase of ϵ and reaches about 2% at $\epsilon = 0.5$, see Figure 3b).

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