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ON FACETS OF THE NEWTON POLYTOPE FOR THE DISCRIMINANT OF THE POLYNOMIAL SYSTEM

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Abstract. We study normal directions to facets of the Newton polytope of the discriminant of the Laurent polynomial system via the tropical approach. We use the combinatorial construction proposed by Dickenstein, Feichtner and Sturmfels for the tropicalization of algebraic varieties admitting a parametrization by a linear map followed by a monomial map.

Keywords: discriminant, Newton polytope, tropical variety, Bergman fan, matroid.

1. Introduction

Consider a system of n polynomial equations of the form

$$P_i := \sum_{\lambda \in A^{(i)}} a_\lambda^{(i)} y^\lambda = 0,\ i = 1, \dots, n \tag{1}$$

with unknowns $y = (y_1, \dots, y_n) \in (\mathbb{C} \setminus 0)^n$, variable complex coefficients $a = (a_\lambda^{(i)})$, where the sets $A^{(i)} \subset \mathbb{Z}^n$ are fixed, finite and contain the zero element $\overline{0}$, $\lambda = (\lambda_1, \dots, \lambda_n)$, $y^\lambda = y_1^{\lambda_1} \cdot \dots \cdot y_n^{\lambda_n}$. Solution $y(a) = (y_1(a), \dots, y_n(a))$ of (1) has a polyhomogeneity property, and thus the system usually can be reduced by means of monomial transformations $x = x(a)$ of coefficients (see [1]). To do this, it is necessary to distinguish a collection of n exponents $\omega^{(i)} \in A^{(i)}$ such that the matrix

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$\omega = \left(\omega^{(1)}, \dots, \omega^{(n)}\right)$ is non-degenerated. As a result, we obtain a reduced system of the form

$$Q_i := y^{\omega^{(i)}} + \sum_{\lambda \in \Lambda^{(i)}} x_\lambda^{(i)} y^\lambda - 1 = 0, \ i = 1, \dots, n, \tag{2}$$

where $\Lambda^{(i)} := A^{(i)} \setminus \{\omega^{(i)}, \bar{0}\}$ and $x = (x_\lambda^{(i)})$ are variable complex coefficients. Denote by Λ the disjoint union of the sets $\Lambda^{(i)}$ and by N the cardinality of this union, that is the number of variable coefficients in the system (2). The coefficients of the system vary in the vector space $\mathbb{C}_x^N$ in which points $x = (x_\lambda)$ are indexed by elements $\lambda \in \Lambda$.

Denote by ∇° the set of all coefficients $x = \left(x_\lambda^{(i)}\right)$ such that the polynomial mapping $Q = (Q_1, \dots, Q_n)$ has multiple zeros in the complex algebraic torus $(\mathbb{C} \setminus 0)^n$, that is zeros for which the determinant of the Jacobi matrix $\frac{\partial Q}{\partial y}$ of the mapping Q equals zero. *The discriminant set* ∇ of the system (2) is defined to be the closure of the set ∇° in the space of coefficients. If ∇ is a hypersurface, then its defining polynomial $\Delta(x)$ is said to be *the discriminant* of the system (2). The set ∇ is also called *the reduced discriminant set* of the system (1). Following the concept of the A–discriminant developed in the book [8], the discriminant of a system of equations is appropriate to call the $(A^{(1)}, \dots, A^{(n)})$–discriminant.

The goal of our research is to construct the tropicalization of the discriminant set ∇ of the system (2) and to find normal vectors to facets of the Newton polytope of the discriminant $\Delta(x)$. Recall that *the Newton polytope* $\mathcal{N}_\Delta$ of the polynomial $\Delta(x)$ is defined to be the convex hull of its support in $\mathbb{R}^n$.

The idea of computing the tropical discriminant is adopted from the paper [5], where the general combinatorial construction of the tropicalization is given for algebraic varieties that admit a parametrization in the form of the product of linear forms. We use the parametrization of the discriminant set for the system of n Laurent polynomials (2) proposed and comprehensively studied in [1].

We write the set Λ in the form of the block matrix $\Lambda = (\Lambda^{(1)}| \dots |\Lambda^{(n)})$ whose columns are vectors of exponents of monomials of the system. Introduce $(n \times N)$–matrices

$$\Psi := \omega^* \Lambda, \ \ \tilde{\Psi} := \Psi - |\omega| \chi,$$

where ω^* is the adjoint matrix to the matrix ω, χ is the matrix whose i-th row represents the characteristic function of the subset $\Lambda^{(i)} \subset \Lambda$ and $|\omega|$ is the determinant of the matrix ω. In what follows, we denote the rows of the matrices Ψ and $\tilde{\Psi}$ by $\psi_1, \dots, \psi_n$ and $\tilde{\psi}_1, \dots, \tilde{\psi}_n$ correspondingly, and we denote the rows of the identity matrix E_N by $e_1, \dots, e_N$. One of the main results of this paper is

Theorem 1. *The vectors*

$$e_1, \dots, e_N, -\psi_1, \dots, -\psi_n, \tilde{\psi}_1, \dots, \tilde{\psi}_n \in \mathbb{Z}^N \tag{3}$$

define the normal directions of facets of the Newton polytope of the discriminant for the system (2).

The discriminant we study is the generalization of the notion of A–discriminant [8]. In [9], Kapranov proved that A–discriminants characterize hypersurfaces with birational logarithmic Gauss mappings, and also that any reduced discriminant hypersurface admit the parametrization by monomials depending on linear forms, which is called the *Horn-Kapranov uniformization*. The tropical analogue of the Horn-Kapranov uniformization was obtained in [5].

In the present research, we use the parametrization of the discriminant set that occurs to be the inverse of the logarithmic Gauss mapping in the case when the set ∇ is an irreducible hypersurface that depends on all groups of the variable coefficients of the system (see [1]). However, the logarithmic Gauss mapping for the hypersurface ∇ is not always birational, thus, in general, $(A^{(1)},\dots,A^{(n)})$–discriminants are not reducible to A–discriminants, and consequently, require careful research.

It is important to note that Theorem 1 does not bring all normals of the Newton polytope to light, but only those that are represented in the parametrization of the set ∇ explicitly, as vectors of coefficients in linear forms. Section 2 of the paper discusses the properties of parametrization. Section 3 presents the construction of the tropical variety and the proof of Theorem 1. Finally, in Section 4, we explain how the constructed tropical variety brings out the so-called «hidden» facets of the Newton polytope for the discriminant of the system.

2. Parametrization of the discriminant set ∇

We introduce two copies of the space $\mathbb{C}^N$: the space $\mathbb{C}^N_x$ of variables $x=(x_\lambda)$, and the space $\mathbb{C}^N_s$ of variables $s=(s_\lambda)$. In both cases, coordinates of points are indexed by elements $\lambda\in\Lambda$. The space $\mathbb{C}^N_s$ is considered as the space of homogeneous coordinates for $\mathbb{CP}^{N-1}_s$. It is proved in [1] that the parametrization of the discriminant set ∇ of the system (2) is determined by the multivalued algebraic mapping from the projective space $\mathbb{CP}^{N-1}_s$ into the space $\mathbb{C}^N_x$ of the coefficients of the system with components

$$x^{(i)}_\lambda=-\frac{s^{(i)}_\lambda}{\langle\tilde{\varphi}_i,s\rangle}\prod_{k=1}^{n}\left(\frac{\langle\tilde{\varphi}_k,s\rangle}{\langle\varphi_k,s\rangle}\right)^{\varphi_{k\lambda}},\ \lambda\in\Lambda^{(i)},\ i=1,\dots,n, \tag{4}$$

where φ_k and $\tilde{\varphi}_k$ are rows of matrices $\Phi:=\omega^{-1}\Lambda$ and $\tilde{\Phi}:=\Phi-\chi$ correspondingly, $\varphi_{k\lambda}$ is a coordinate of the row φ_k indexed by $\lambda\in\Lambda^{(i)}\subset\Lambda$, the angle brackets denote the inner product. The number of branches in (4) equals to the absolute value of the determinant $|\omega|$ but some branches may coincide. If the discriminant set ∇ of the system (2) is an irreducible hypersurface that depends on all groups of variables, then the mapping (4) parametrizes it with the multiplicity that is equal to the index $|\mathbb{Z}^n:H|$ of the sublattice $H\subset\mathbb{Z}^n$ generated by columns of the matrix $(\omega|\Lambda)$, i.e., by all exponents of the monomials from the system (2). The image of the mapping (4) is a hypersurface if all coordinates of vectors φ_k, $\tilde{\varphi}_k$ are non-zero, $k=1,\dots,n$.

We consider the rational mapping $\mathbb{CP}^{N-1}_s\to\mathbb{C}^N_w$ that is obtained from the mapping (4) as a result of raising of all its coordinates to the power $|\omega|$. It has components

$$w^{(i)}_\lambda=\left(-\frac{|\omega|s^{(i)}_\lambda}{\langle\tilde{\psi}_i,s\rangle}\right)^{|\omega|}\prod_{k=1}^{n}\left(\frac{\langle\tilde{\psi}_k,s\rangle}{\langle\psi_k,s\rangle}\right)^{\psi_{k\lambda}}, \tag{5}$$

where $\psi_{k\lambda}$ is a coordinate of the row ψ_k indexed by $\lambda\in\Lambda^{(i)}\subset\Lambda$. The mapping (5) defines the hypersurface $\tilde{\nabla}\subset\mathbb{C}^N_w$ whose amoeba has the same asymptotic directions as the amoeba of the discriminant hypersurface ∇. Recall that the *amoeba* $\mathcal{A}_\nabla$ of the algebraic hypersurface

$$\nabla = \{x \in (\mathbb{C} \setminus 0)^N : \Delta(x) = 0\}$$

is determined to be the image of ∇ under the mapping

$$\mathrm{Log} : (x_1, \dots, x_N) \to (\log|x_1|, \dots, \log|x_N|).$$

Each component of the mapping (5) is a monomial with integer exponents composed with linear functions. It is convenient to associate the mapping (5) with two block matrices

$$U = \left(-|\omega|E_N \left|\Psi^T\right| \tilde{\Psi}^T\right)^T, \quad V = \left(|\omega|E_N \left|-\Psi^T\right| \tilde{\Psi}^T\right), \tag{6}$$

where E_N is the identity matrix. The rows of the matrix U determine linear functions, and the rows of V determine exponents of monomials in the parametrization (5).

3. Tropicalization of the rational variety

Further, we pay attention to the algebraic variety $\tilde{\nabla} \subset \mathbb{C}^N$. Let us study its tropicalization $\tau(\tilde{\nabla})$. At the very beginning of the section, we recall some notions and facts from the tropical geometry. Basic concepts of this theory and numerous references to fundamental research can be found in the book [10]. Since the variety $\tilde{\nabla}$ admits the parametrization in the form of the product of linear forms, we implement the general combinatorial construction proposed in [5] in order to compute the tropical variety $\tau(\tilde{\nabla})$.

A tropical variety has the structure of the polyhedral fan. In particular, the tropicalization of a linear subspace $X \subset \mathbb{C}^k$ is the Bergman fan of the matroid $M(X)$ associated with this subspace (see [2], [7]). The construction of the Bergman fan of a variety X is related to the notion of the logarithmic limit-set of the variety X introduced in [3]. The Bergman fan of an irreducible subvariety $X \subset \mathbb{C}^k$ is the finite union of convex polyhedral cones with the vertex at the origin of the dimension that coincides with the dimension of the variety [4].

Let us consider in more detail the definition of the matroid, which covers the general combinatorial essence of the concepts of independence in linear algebra, graph theory, etc. There exist several different equivalent systems of axioms that define the matroid (see, for instance, [2], [10], [13]). Let us determine it as follows.

Definition 1. *The matroid M is defined to be a pair $(\mathcal{E}, \mathcal{I})$ that consists of a finite set $\mathcal{E}$ and a collection $\mathcal{I}$ of subsets of $\mathcal{E}$ such that*

(I-1) $\emptyset \in \mathcal{I}$.
(I-2) *If $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$.*
(I-3) *If $I, J \in \mathcal{I}$ and $|I| < |J|$, then there exists $j \in J - I$ such that $I \cup j \in \mathcal{I}$.*

Elements of $\mathcal{I}$ are called independent sets.

It is supposed that all one-element subsets of $\mathcal{E}$ are independent. Maximal independent sets are said to be *bases* of a matroid. From the axiom (I-3), it follows that all bases of a matroid M have the same cardinality $r = r(M)$ that is called the *rank* of the matroid. If A is a subset of the set $\mathcal{E}$, then the cardinality of maximal independent set in A is called the rank of A, and it is denoted by $r(A)$. The minimal by inclusion dependent set $C \subset \mathcal{E}$ is said to be the *cycle* of the matroid M. A subset

$F \subseteq \mathcal{E}$ is said to be the *flat* if $r(F \cup e) > r(F)$ for all $e \notin F$. We say F is *proper* if its rank equals neither 0 nor r. Every flat F of the matroid M is represented by the incidence vector

$$e_F := \sum_{i \in F} e_i.$$

The vector e_F is considered to be an element of the *tropical projective space* $\mathbb{R}^{|\mathcal{E}|}/\mathbb{R}\mathbf{1}$, where $\mathbf{1} = (1, 1, \ldots, 1)$. The partially ordered set of all flats forms the lattice of flats of the matroid.

The Bergman fan is one of geometric models of matroids. We use the first statement of Theorem 2 as the definition of the Bergman fan of the matroid M.

Theorem 2 (Ardila–Klivans[2])**.**

(1) *The Bergman fan* $\mathfrak{B}_M$ *of a matroid* M *on* $\mathcal{E}$ *is the polyhedral complex in* $\mathbb{R}^{|\mathcal{E}|}/\mathbb{R}\mathbf{1}$ *that consists of the cones*

$$\sigma_{\mathcal{F}} = cone\{e_F : F \in \mathcal{F}\}$$

for each flag $\mathcal{F} = \{F_1 \subsetneq \ldots \subsetneq F_l\}$ *of proper flats of* M*. Here,* $e_F := e_{f_1} + \ldots + e_{f_k}$ *for* $F = \{f_1, \ldots, f_k\}$*.*

(2) *The tropicalization of a linear subspace* X *is the Bergman fan* $\mathfrak{B}_{M(X)}$ *of the matroid* $M(X)$ *related to the subspace* X*.*

Proof of Theorem 1. We first find the tropicalization $\tau(\tilde{\nabla})$ of the hypersurface $\tilde{\nabla}$. Note that in the case of an irreducible hypersurface the tropicalization is the union of all cones of codimension one of the normal fan for the Newton polytope of the polynomial defining the hypersurface. Moreover, the collection of one-dimensional generators of the normal fan for the Newton polytope of the polynomial defining $\tilde{\nabla}$ coincides with the same collection for the polynomial defining the discriminant hypersurface ∇.

Let X be a linear subspace in $\mathbb{C}^{N+2n}$, and let it be the image of the linear mapping $s \to Us$ given by the matrix U, where $s \in \mathbb{CP}_s^{N-1}$. We consider the matroid $M(X)$ on the set of rows of the matrix U (on the set $\mathcal{E} = \{1, 2, \ldots, N + 2n\}$). By Theorem 2, the Bergman fan $\mathfrak{B}_{M(X)}$ of the matroid $M(X)$ is the tropicalization of the subspace X. The following lemma holds.

Lemma 1. *The tropical variety* $\tau(\tilde{\nabla})$ *is the image of the Bergman fan* $\mathfrak{B}_{M(X)}$ *under the mapping* $\mathbb{R}^{N+2n} \to \mathbb{R}^N$ *given by the matrix* V*.*

Proof of Lemma 1. Coordinates of the parametrization of the variety $\tilde{\nabla}$ given by (5) are monomials composed with linear forms. The linear forms are defined by the rows of the matrix U that is associated with the matroid $M(X)$ of the rank N on the set $\mathcal{E}$. Exponents of the monomials in parametrization are rows of the matrix V. According to [5, Theorem 3.1], the tropicalization of the variety $\tilde{\nabla}$ can be defined in terms of the matrices U and V. More precisely, the tropicalization is the image of the Bergman fan $\mathfrak{B}_{M(X)}$ under the linear mapping V. $\square$

It follows from Lemma 1 that one-dimensional cones (rays) of the tropical variety $\tau(\tilde{\nabla})$ are generated by the columns of the matrix V. They do determine the normal directions for the facets of the Newton polytope for the discriminant of the system. Thus Theorem 1 is proved. $\square$

It is important to note that the acquired information about the structure of the normal fan of the Newton polytope for the discriminant, as well as the parametrization (4), can be used for the study of truncations of the discriminant $\Delta(x)$ of the system (2). The *truncation* of the polynomial $\Delta(x)$ with respect to a face h of the Newton polytope $\mathcal{N}_\Delta$ is defined to be the sum of all monomials of $\Delta(x)$ whose exponents lie in the face h. The geometric proofs of factorization identities for truncations of the classical discriminant with respect to faces of its Newton polytope are given in two recent papers [11], [12]. The proof is based on the blow-up property of the logarithmic Gauss map for the zero set of the discriminant. These identities were proved earlier in the book [8, Chapter 10] using the complicated machinery of the theory of A–determinants.

4. «Hidden» normal directions

Consider a system of two equations with two unknowns y_1, y_2 and three variable coefficients x_1, x_2, x_3:

$$\begin{cases} {y_1}^2 + x_1 y_1 y_2 + x_2 y_1 {y_2}^2 - 1 = 0, \\ {y_2}^2 + x_3 {y_1}^2 y_2 - 1 = 0. \end{cases} \tag{7}$$

The matrix of exponents of the system (7) has the form

$$(\omega|\Lambda) = \left(\begin{array}{cc|ccc} 2 & 0 & 1 & 1 & 2 \\ 0 & 2 & 1 & 2 & 1 \end{array} \right).$$

The index of the sublattice in $\mathbb{Z}^2$ generated by its columns equals one. It is the greatest common divisor of all second order minors of the matrix. Moreover, matrices

$$\Phi = \begin{pmatrix} 1/2 & 1/2 & 1 \\ 1/2 & 1 & 1/2 \end{pmatrix}, \tilde{\Phi} = \begin{pmatrix} -1/2 & -1/2 & 1 \\ 1/2 & 1 & -1/2 \end{pmatrix}$$

do not contain zero elements, so the discriminant set ∇ of the system (7) is the hypersurface, which can be parametrized with the multiplicity one by the mapping $\mathbb{CP}^2_s \to \mathbb{C}^3_x$ of the form

$$\begin{aligned} x_1 &= -\frac{2s_1}{-s_1 - s_2 + 2s_3} \left(\frac{-s_1 - s_2 + 2s_3}{s_1 + s_2 + 2s_3} \right)^{\frac{1}{2}} \left(\frac{s_1 + 2s_2 - s_3}{s_1 + 2s_2 + s_3} \right)^{\frac{1}{2}}, \\ x_2 &= -\frac{2s_2}{-s_1 - s_2 + 2s_3} \left(\frac{-s_1 - s_2 + 2s_3}{s_1 + s_2 + 2s_3} \right)^{\frac{1}{2}} \left(\frac{s_1 + 2s_2 - s_3}{s_1 + 2s_2 + s_3} \right), \\ x_3 &= -\frac{2s_3}{s_1 + 2s_2 - s_3} \left(\frac{-s_1 - s_2 + 2s_3}{s_1 + s_2 + 2s_3} \right) \left(\frac{s_1 + 2s_2 - s_3}{s_1 + 2s_2 + s_3} \right)^{\frac{1}{2}}. \end{aligned} \tag{8}$$

Here $s = (s_1, s_2, s_3) \in \mathbb{C}^3$ are homogeneous coordinates in $\mathbb{CP}^2_s$.

Let us study the tropicalization $\tau(\tilde{\nabla})$ of the rational variety $\tilde{\nabla} \subset \mathbb{C}^3_w$ given as follows

(9)

$$\begin{aligned} w_1 &= (-4s_1)^4 (2s_1 + 2s_2 + 4s_3)^{-2} (2s_1 + 4s_2 + 2s_3)^{-2} (-2s_1 - 2s_2 + 4s_3)^{-2} (2s_1 + 4s_2 - 2s_3)^2, \\ w_2 &= (-4s_2)^4 (2s_1 + 2s_2 + 4s_3)^{-2} (2s_1 + 4s_2 + 2s_3)^{-4} (-2s_1 - 2s_2 + 4s_3)^{-2} (2s_1 + 4s_2 - 2s_3)^4, \\ w_3 &= (-4s_3)^4 (2s_1 + 2s_2 + 4s_3)^{-4} (2s_1 + 4s_2 + 2s_3)^{-2} (-2s_1 - 2s_2 + 4s_3)^4 (2s_1 + 4s_2 - 2s_3)^{-2}. \end{aligned}$$

As we noted above, the mapping (9) is associated with matrices

$$U = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \\ 2 & 2 & 4 \\ 2 & 4 & 2 \\ -2 & -2 & 4 \\ 2 & 4 & -2 \end{pmatrix} \quad \text{and } V = \begin{pmatrix} 4 & 0 & 0 & -2 & -2 & -2 & 2 \\ 0 & 4 & 0 & -2 & -4 & -2 & 4 \\ 0 & 0 & 4 & -4 & -2 & 4 & -2 \end{pmatrix},$$

which play a crucial role in constructing of the tropical variety $\tau(\tilde{\nabla}) \subset \mathbb{R}^3$. Consider a matroid M on the set $\mathcal{E} = \{1, 2, 3, 4, 5, 6, 7\}$ of rows of the matrix U. The tropical linear space associated with the matroid M is the Bergman fan $\mathfrak{B}_M$. It is a two-dimensional fan in $\mathbb{R}^7/\mathbb{R}\mathbf{1}$ or a graph depicted in Fig. 1.

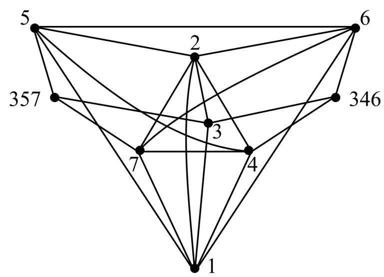

FIG. 1. The Bergman complex $\mathfrak{B}_M$

The graph has nine vertices. Seven vertices correspond to one-element flats 1,2, 3, 4, 5, 6, 7, and two ones correspond to cycles 346 and 357 of the matroid M.

The image of the fan $\mathfrak{B}_M$ under the mapping V is the two-dimensional fan in $\mathbb{R}^3$ (see Fig. 2). It consists of sixteen cones. Fifteen cones are spanned by pairs 12, 13, 14, 15, 16, 17, 23, 24, 25, 26, 27, 45, 47, 56, 67 of columns of the matrix V, and one cone is generated by the triple 346.

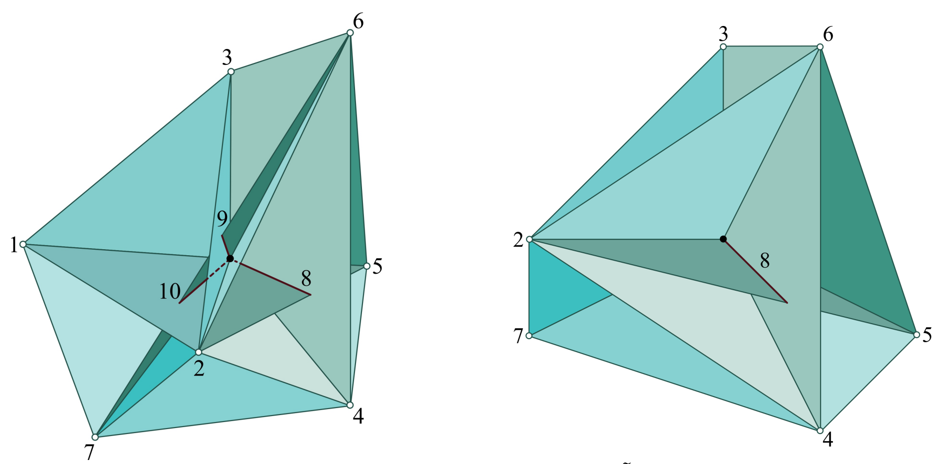

FIG. 2. The tropical variety $\tau(\tilde{\nabla})$

The tropical variety $\tau(\tilde{\nabla})$ constructed in this way yields the normal fan for the Newton polytope $\mathcal{N}_\Delta$ of the discriminant of the system (7). Seven rays of the fan $\tau(\tilde{\nabla})$ spanned by columns of the matrix V define normal directions of facets of the Newton polytope $\mathcal{N}_\Delta$. In Fig. 2, they are denoted by 1, 2, 3, 4, 5, 6, 7 . These are exactly the normal directions obtained in Theorem 1. Besides these normals, there are three more rays arising as a result of the intersection of cones of the fan $\tau(\tilde{\nabla})$. In particular, cones 346 and 25 intersect at the ray $\mathbb{R}_{\geqslant 0}(-1,-1,-1)^T$ (number 8 in Fig. 2), cones 67 and 23 intersect at $\mathbb{R}_{\geqslant 0}(0,1,1)^T$ (number 9 in Fig. 2), while cones 67 and 12 intersect at $\mathbb{R}_{\geqslant 0}(1,3,0)^T$ (number 10 in Fig. 2). All in all, ten inward normals of the Newton polytope $\mathcal{N}_\Delta$ are found:

$$
\begin{aligned}
n_1 &= (4,0,0), & n_6 &= (-2,-2,4),\\
n_2 &= (0,4,0), & n_7 &= (2,4,-2),\\
n_3 &= (0,0,4), & n_8 &= (-1,-1,-1),\\
n_4 &= (-2,-2,-4), & n_9 &= (0,1,1),\\
n_5 &= (-2,-4,-2), & n_{10} &= (1,3,0).
\end{aligned}
$$

The discriminant of the system (7) looks as follows:

$$
\begin{aligned}
\Delta(x) = {} & 432x_1^{11}x_3^3 - 1296x_1^9x_2^2x_3^3 - 64x_1^9x_3^5 + 1296x_1^7x_2^4x_3^3 + 192x_1^7x_2^2x_3^5 - 432x_1^5x_2^6x_3^3 - 192x_1^5x_2^4x_3^5\\
& + 64x_1^3x_2^6x_3^5 + 2976x_1^8x_2x_3^4 - 20916x_1^6x_2^3x_3^4 - 384x_1^6x_2x_3^6 + 9576x_1^4x_2^5x_3^4 + 2928x_1^4x_2^3x_3^6\\
& - 3300x_1^2x_2^7x_3^4 - 1248x_1^2x_2^5x_3^6 + 432x_2^7x_3^6 + 4032x_1^9x_3^3 + 13272x_1^7x_2^2x_3^3 - 3684x_1^7x_3^5\\
& - 45864x_1^5x_2^4x_3^3 + 8580x_1^5x_2^2x_3^5 + 432x_1^5x_3^7 + 43560x_1^3x_2^6x_3^3 - 91176x_1^3x_2^4x_3^5 - 1200x_1^3x_2^2x_3^7\\
& - 15000x_1x_2^8x_3^3 + 96x_1x_2^6x_3^5 + 12000x_1x_2^4x_3^7 - 4068x_1^8x_2x_3^2 + 25136x_1^6x_2^3x_3^2 - 10192x_1^6x_2x_3^4\\
& - 50568x_1^4x_2^5x_3^2 + 3584x_1^4x_2^3x_3^4 - 1056x_1^4x_2x_3^6 + 42000x_1^2x_2^7x_3^2 - 467344x_1^2x_2^5x_3^4 + 97335x_1^2x_2^3x_3^6\\
& - 12500x_2^9x_3^2 + 4800x_2^7x_3^4 + 4800x_2^5x_3^6 - 12500x_2^3x_3^8 + 11360x_1^7x_3^3 + 102792x_1^5x_2^2x_3^3 + 3552x_1^5x_3^5\\
& + 172032x_1^3x_2^4x_3^3 - 132850x_1^3x_2^2x_3^5 - 509960x_1x_2^6x_3^3 + 567418x_1x_2^4x_3^5 + 15000x_1x_2^2x_3^7\\
& - 36944x_1^6x_2x_3^2 + 108528x_1^4x_2^3x_3^2 + 20003x_1^4x_2x_3^4 + 53328x_1^2x_2^5x_3^2 - 856566x_1^2x_2^3x_3^4 - 3300x_1^2x_2x_3^6\\
& - 118000x_2^7x_3^2 + 417147x_2^5x_3^4 - 118000x_2^3x_3^6 - 64x_1^7x_3 + 7328x_1^5x_2^2x_3 + 8192x_1^5x_3^3 - 14464x_1^3x_2^4x_3\\
& + 477776x_1^3x_2^2x_3^3 - 64x_1^3x_3^5 + 7200x_1x_2^6x_3 - 621232x_1x_2^4x_3^3 + 130440x_1x_2^2x_3^5 - 99720x_1^4x_2x_3^2\\
& + 264032x_1^2x_2^3x_3^2 - 27048x_1^2x_2x_3^4 - 340328x_2^5x_3^2 - 340328x_2^3x_3^4 - 512x_1^5x_3 + 57952x_1^3x_2^2x_3\\
& - 512x_1^3x_3^3 + 56736x_1x_2^4x_3 + 320064x_1x_2^2x_3^3 + 432x_1^4x_2 - 864x_1^2x_2^3 - 61440x_1^2x_2x_3^2 + 432x_2^5\\
& - 265536x_2^3x_3^2 + 432x_2x_3^4 - 1024x_1^3x_3 + 125568x_1x_2^2x_3 + 3456x_1^2x_2 + 3456x_2^3 + 3456x_2x_3^2 + 6912x_2.
\end{aligned}
$$

It is computed using the computer algebra system for polynomial computations Singular [6].

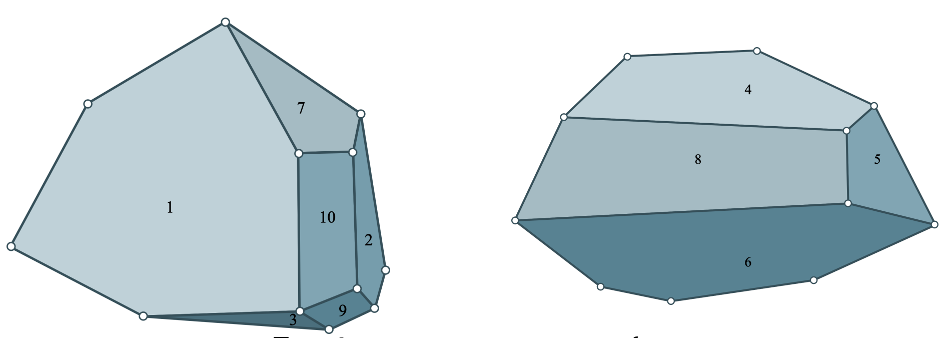

FIG. 3. The Newton polytope $\mathcal{N}_\Delta$

The Newton polytope $\mathcal{N}_\Delta$ is depicted in Fig. 3 from two angles. It has ten two-dimensional faces. They are enumerated in accordance with the sequence of rays of the polyhedral fan in Fig. 2.

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