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# THE MODERATE DEVIATIONS PRINCIPLE FOR THE TRAJECTORIES OF COMPOUND RENEWAL PROCESSES ON THE HALF – LINE

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Abstract. The moderate deviations principle is obtained for the trajectories of compound renewal processes on the half – line under the Cramèr moment condition.

**Keywords:** large deviations principle, moderate deviations principle, compound renewal process, Cramer's condition, rate function.

## 1. Introduction. The main theorem

Let's define the objects under study. We consider an initial random vector

$$\boldsymbol{\xi}_1 = (\tau_1, \zeta_1)$$

and a sequence of identically distributed random vectors

$$\boldsymbol{\xi} = (\tau, \zeta),\ \boldsymbol{\xi}_2 = (\tau_2, \zeta_2),\ \boldsymbol{\xi}_3 = (\tau_3, \zeta_3), \ldots$$

We will assume that the vectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3, \ldots$ are independent and $\tau > 0$ a.s., $\tau_1 \geq 0$ a.s.

Let's put $T_0 = Z_0 = 0$ and denote

$$T_n := \sum_{j=1}^{n} \tau_j, \quad Z_n := \sum_{j=1}^{n} \zeta_j, \quad S_n := \sum_{j=1}^{n} \boldsymbol{\xi}_j = (T_n, Z_n) \text{ for } n \geq 1.$$

We define random processes

$$\eta(t) := \min\{k \geq 0 : \ T_k \geq t\}, \quad \nu(t) := \max\{k \geq 0 : \ T_k < t\}, \quad \eta(0) = \nu(0) = 0$$

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for $t \geq 0$. It is clear that for $t > 0$

$$\nu(t) = \eta(t) - 1.$$

The *first compound renewal process* (CRP) $Z(t),\ t \geq 0$ is defined as

$$Z(t) := Z_{\nu(t)} \quad \text{for} \quad t > 0, \quad Z(0) = 0.$$

Along with the first CRP $Z(t)$ we study the process

$$Y(t) := Z_{\eta(t)} = Z(t) + \zeta_{\eta(t)} \quad \text{for} \quad t > 0, \quad Y(0) = 0, \quad Y(0+) = \zeta_1,$$

which we call the *second CRP*.

CRP $Z(t)$ and $Y(t)$ occur in both applied and theoretical problems (see, for example, [1, §0.1]). It is shown in [1, Chapters 1, 2] and [3] that if the corresponding conditions are met, then the limit laws are the same for CRP $Z(t)$ and $Y(t)$ in the domain of normal and moderate deviations. In the domain of large deviations (see [1, 3–6]) this isn't true always.

Assume henceforth that Cramér's condition holds in the following form:

$[\mathbf{C}_0]$. $\mathbf{E}e^{\lambda(\tau+|\zeta|)} < \infty$, $\mathbf{E}e^{\lambda(\tau_1+|\zeta_1|)} < \infty$ *for some* $\lambda > 0$.

Moreover, we will assume that the random vector $\boldsymbol{\xi} = (\tau, \zeta)$ is *nondegenerate*, i.e. for any $b, c, d \in \mathbb{R}$, $|b| + |c| \neq 0$ the inequality $\mathbf{P}(b\tau + c\zeta = d) < 1$ is true. To avoid repetition, we omit these two conditions in our main statements.

If the random vectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}$ have the same distribution, then we will call this case *homogeneous*, if the distributions are different, then *inhomogeneous*.

The standard, generally accepted CRP model assumes that the time of occurrence of the first jump $\tau_1$ and its value $\zeta_1$ have a joint distribution different, in general, from the joint distribution $(\tau, \zeta)$ (see, for example, [2]). This is implemented, for example, for an CRP with stationary increments.

Denote for $t \geq 0$

$$Z_1(t) := Z(t) - at, \quad Y_1(t) := Y(t) - at,$$

$$Z_2(t) := Z(t) - a_\zeta \nu(t), \quad Y_2(t) := Y(t) - a_\zeta \eta(t),$$

where $a := \frac{a_\zeta}{a_\tau}$, $a_\zeta := \mathbf{E}\zeta$, $a_\tau := \mathbf{E}\tau$.

Let's fix the function $x = x(T)$, such that

$$\lim_{T\to\infty} \frac{x(T)}{\sqrt{T}} = \infty, \quad \lim_{T\to\infty} \frac{x(T)}{T} = 0. \tag{1.1}$$

Wherever it does not interfere with the explanation, the argument $T$ will omit in the function $x(T)$.

The main object of study is four families of processes

$$z_{1,T}(t) := \frac{1}{x} Z_1(tT), \quad z_{2,T}(t) := \frac{1}{x} Z_2(tT), \quad 0 \leq t < \infty;$$

$$y_{1,T}(t) := \frac{1}{x} Y_1(tT), \quad y_{2,T}(t) := \frac{1}{x} Y_2(tT), \quad 0 \leq t < \infty.$$

Usually, when normal and moderate deviations are studying (see, for example, [1, Chapters 1, 2]), we deal with the families $z_{1,T}$, $y_{1,T}$, i.e. we choose $f(t) = at$ as the "centering function". We extend the class of situations in which the limit theorems can be applied, by involving the families $z_{2,T}$, $y_{2,T}$ in [3] and this paper.

This work is a continuation of the series of authors' works devoted to the principles of large deviations (LDP) on the half–line $[0, \infty)$. Thus, the authors have previously obtained: LDP on the half–line for the trajectories of random walks, processes with independent increments and solutions of stochastic differential equations [4]; extended LDP on the half–line for trajectories of random walks and processes with independent increments [5], [6]. We will be interested in the moderate deviations principle (MDP) for the families of processes $z_{1,T}$, $z_{2,T}$, $y_{1,T}$, $y_{2,T}$ on the half–line $t \in [0,\infty)$. The MDP was obtained by us earlier in [3] for these processes on the finite segment. In particular, the results of [3] imply convergence

$$z_{i,T}(t) \to 0,\ y_{i,T}(t) \to 0 \text{ for any fixed } t \geq 0,\ i = 1,2$$

in probability when $T \to \infty$.

We denote an arbitrary metric space (MS) by $\mathbf{X}_\rho$, the Borel $\sigma$–algebra of its subsets by $\mathfrak{B}_{\mathbf{X}_\rho}$, and the complement, closure, and interior of the set $B$ by $\overline{B}$, $[B]$, and $(B)$, respectively.

Recall the necessary definitions (see, for example, [7]).

**Definition 1.** *A family of random processes $s_T$ satisfies LDP in MS $\mathbf{X}_\rho$ with a rate function (RF) $I = I(f) : \mathbf{X} \to [0,\infty]$ and the normalizing function (NF) $\psi(T) : \lim_{T\to\infty} \psi(T) = \infty$, if for any $c \geq 0$ the set $\{f \in \mathbf{X} : I(f) \leq c\}$ is a compact set in MS $\mathbf{X}_\rho$ and for any set $B \in \mathfrak{B}_{\mathbf{X}_\rho}$ the following inequalities hold:*

$$\limsup_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbf{P}(\, s_T \in B\,) \leq -I([B]),$$

$$\liminf_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbf{P}(\, s_T \in B\,) \geq -I((B)),$$

*where $I(B) = \inf_{y\in B} I(y)$ for $B \in \mathfrak{B}_{\mathbf{X}_\rho}$, $I(\varnothing) = \infty$.*

In what follows, the words "the family of random processes $s_T$ satisfies $(I, \psi, \mathbf{X}_\rho)$ – LDP" means that the family of random processes $s_T$ satisfies LDP in MS $\mathbf{X}_\rho$ with RF $I = I(f)$ and NF $\psi = \psi(T)$.

**Definition 2.** *The families of random processes $v_T(t)$ and $s_T(t)$, whose trajectories belong to MS $\mathbf{X}_\rho$, are equivalent from the viewpoint of LDP ($v_T \overset{L.D.}{\sim} s_T$), if for any $\varepsilon > 0$*

$$\limsup_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbf{P}\left(\rho(v_T, s_T) > \varepsilon\right) = -\infty.$$

It is easy to prove that if $v_T \overset{L.D.}{\sim} s_T$, MS $\mathbf{X}_\rho$ is complete, and one of the families of processes satisfies LDP, then the second family satisfies the same LDP (see, for example, [7, theorem 4.2.13]).

We will use the following notations:
$\mathbb{C}[0,u]$ is the space of continuous functions on the segment $[0,u]$ with the metric

$$\rho_u(f,g) = \sup_{t\in[0,u]} \frac{|f(t) - g(t)|}{1+t};$$

$\mathbb{C}$ is the space of continuous functions on the half–line $[0,\infty)$ with the metric

$$\rho(f,g) = \sup_{t\in[0,\infty)} \frac{|f(t) - g(t)|}{1+t};$$

$\mathbb{C}_0 \subset \mathbb{C}$ is the class of functions $f \in \mathbb{C}$, such that $f(0) = 0$ and $\lim_{t\to\infty} \frac{f(t)}{1+t} = 0$;
$\mathbb{D}[0,u]$ is the space of functions, which are continuous on the left and have limits on the right on the segment $[0,u]$, with uniform metric $\rho_u$;

$\mathbb{D}$ is the space of functions, which are right continuous with left limits on the half–line $[0,\infty)$ with a metric $\rho$;
$\mathbb{AC}_0[0,u]$ is a set of functions, which are absolutely continuous on the segment $[0,u]$ and start from zero;
$\mathbb{AC}_0$ is a set of functions, which are absolutely continuous on half–line $[0,\infty)$ and start from zero.

We will be interested in LDP on the space $\mathbb{D}$, but due to the inseparability, the Borel $\sigma$–algebra constructed by sets that are open relative to the metric $\rho$ will contain sets nonmeasurable for the probabilistic measure $\mathbf{P}$, see [9, §18]. Therefore, in what follows, we will consider the measure $\mathbf{P}$ on the sets that belong the $\sigma$–algebra constructed by open cylindrical subsets of the space $\mathbb{D}$, this approach is used, for example, in [10], [11]. It can be shown that the closure and interior of any such set in the space $\mathbb{D}$ will also belong to this $\sigma$–algebra. We will also need the LDP in the space $\mathbb{C}$, for this space the $\sigma$–algebra constructed by its open cylindrical subsets coincides with the $\sigma$–algebra constructed by the sets that open with respect to the metric $\rho$.

We denote by $\tilde{z}_{1,T}(t)$, $\tilde{z}_{2,T}(t)$, $\tilde{y}_{1,T}(t)$, $\tilde{y}_{2,T}(t)$ a continuous random polygons with nodes at the points of jumps of processes $z_{1,T}(t)$, $z_{2,T}(t)$, $y_{1,T}(t)$, $y_{2,T}(t)$, i.e. for $i\in\{1,2\}$

$$\tilde{z}_{i,T}(t) := z_{i,T}(T_{k-1}+)+\frac{t-T_{k-1}}{T_k-T_{k-1}}(z_{i,T}(T_k+)-z_{i,T}(T_{k-1}+)),\ \text{if}\ t\in[T_{k-1},T_k),\ k\in\mathbb{N};$$

$$\tilde{y}_{i,T}(t) := y_{i,T}(T_{k-1}+)+\frac{t-T_{k-1}}{T_k-T_{k-1}}(y_{i,T}(T_k+)-y_{i,T}(T_{k-1}+)),\ \text{if}\ t\in[T_{k-1},T_k),\ k\in\mathbb{N}.$$

Let's formulate the main result.

**Theorem 1.1.** *1) The families of stochastic processes $z_{1,T}$, $y_{1,T}$, $\tilde{z}_{1,T}(t)$, $\tilde{y}_{1,T}(t)$ satisfy $(I_1, \frac{x^2}{T}, \mathbb{D})$ – LDP, where*

$$I_1(f) := \begin{cases} \frac{a_\tau}{2\sigma_1^2}\int_0^\infty (f'(t))^2dt, & \text{for} \quad f\in\mathbb{AC}_0, \\ \infty, & \text{otherwise}, \end{cases}$$

$\sigma_1^2=\mathbf{D}(\zeta-a\tau)$.

*2) The families of stochastic processes $z_{2,T}$, $y_{2,T}$, $\tilde{z}_{2,T}(t)$, $\tilde{y}_{2,T}(t)$ satisfy $(I_2, \frac{x^2}{T}, \mathbb{D})$ – LDP, where*

$$I_2(f) := \begin{cases} \frac{a_\tau}{2\sigma_2^2}\int_0^\infty (f'(t))^2dt, & \text{for} \quad f\in\mathbb{AC}_0, \\ \infty, & \text{otherwise}, \end{cases}$$

$\sigma_2^2=\mathbf{D}\zeta$.

Traditionally, the MDP is a statement of the LDP type in which the RF depends only on the first two moments of the control vector $(\tau,\zeta)$, and which is an extension of the invariance principle in the domain of large deviations. It is easy to see, that this statement is Theorem 1.1.

Theorem 1.1 will be proved in section 3. We will have obtained the MDP for continuous versions of $\widetilde{z}_{1,T}$, $\widetilde{y}_{1,T}$, $\widetilde{z}_{2,T}$, $\widetilde{y}_{2,T}$ before we obtain the corresponding MDP for the families $z_{1,T}$, $y_{1,T}$, $z_{2,T}$, $y_{2,T}$. In the second section, we prove the auxiliary statements.

## 2. Auxiliary results

We denote the projection of the function $f \in \mathbb{C}$ on the space $\mathbb{C}[0,u]$ by

$$f^{(u)} = f^{(u)}(t) := f(t), \quad t \in [0,u].$$

Let the family of the processes $s_T(t)$, $T \geq 0$, $t \in [0,\infty)$ is defined on the space $\mathbb{C}$. We need the following theorem [4].

**Theorem 2.1.** *Let the family of processes $s_T$ satisfies the conditions:*

**I**. *The family of processes $s_T^{(u)}$ satisfies $(I_u, \psi, \mathbb{C}[0,u])$ – LDP for all $u \in (0,\infty)$. And for every function $f \in \mathbb{C}[0,u]$ there exists a function $g = g_f \in \mathbb{C}_0$ such that $g^{(u)} = f$ and equality*

$$I_v(g^{(v)}) = I_u(f); \tag{2.1}$$

*holds for all $v \geq u$.*

**II**. *The equality*

$$\lim_{u\to\infty} \sup_{f \in B_r^+} \sup_{t \geq u} \frac{|f(t)|}{1+t} = 0,$$

*holds for all $r \geq 0$, where*

$$B_r^+ := \{f \in \mathbb{C} : \varlimsup_{u\to\infty} I_u(f^{(u)}) \leq r\}.$$

**III**. *For every $N < \infty$ and $\varepsilon > 0$ there exists $M = M_{N,\varepsilon} < \infty$ such that*

$$\varlimsup_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbf{P}\left( \sup_{t \geq M} \frac{|s_T(t)|}{1+t} > \varepsilon \right) \leq -N.$$

*Then the family of processes $s_T$ satisfies the $(I, \psi, \mathbb{C})$ – LDP, where*

$$I(f) := \lim_{u\to\infty} I_u(f^{(u)}).$$

Let's explain condition (2.1), it means that any function $f \in \mathbb{C}[0,u]$ can be continued for $t > u$ so that the RF does not increase. It is natural to call the function $g = g_f$ *the most probable extension of the function $f$ outside the segment* $[0,u]$.

We note that in [4] Theorem 2.1 was proved for the integer parameter $T = n \in \mathbb{N}$. However, this proof is fully preserved for the case when parameter $T > 0$ is real.

We need the following lemma.

**Lemma 2.1.** *$z_{1,T} \overset{L.D.}{\sim} \tilde{z}_{1,T}$, $z_{2,T} \overset{L.D.}{\sim} \tilde{z}_{2,T}$, $y_{1,T} \overset{L.D.}{\sim} \tilde{y}_{1,T}$, $y_{2,T} \overset{L.D.}{\sim} \tilde{y}_{2,T}$ in MS $\mathbb{D}$ with NF $\psi(T) = \frac{x^2}{T}$.*

P r o o f. Let's prove that $z_{1,T} \overset{L.D.}{\sim} \tilde{z}_{1,T}$. The jumps of the process $z_{1,T}(t)$ occur at time points $T_k$, $k \in \mathbb{N}$. Since $\sup\limits_{t \in (T_{k-1},T_k]} |z_{1,T}(t) - \tilde{z}_{1,T}(t)|$ is equal to $|\zeta_k|/x$ (the absolute value of the jump at time $T_k$), then for any $C > 0$, $\varepsilon > 0$ we have

$$\mathbf{P}\left(\rho(z_{1,T}, \tilde{z}_{1,T}) > \varepsilon\right) = \mathbf{P}\left( \sup_{t \geq 0} \frac{|z_{1,T}(t) - \tilde{z}_{1,T}(t)|}{1+t} > \varepsilon \right)$$

$$\leq \mathbf{P}\left( \sup_{k \in \mathbb{N}} \frac{|\zeta_k|}{1+T_{k-1}} > x\varepsilon \right) \leq \mathbf{P}\left( \sup_{k \leq x} |\zeta_k| > x\varepsilon \right)$$

$$+\mathbf{P}\left(\sup_{k>x}\frac{|\zeta_k|}{1+T_{k-1}}>x\varepsilon\right)=:\mathbf{P}_1+\mathbf{P}_2. \tag{2.2}$$

Let us estimate $\mathbf{P}_1$ from above. Using conditions $[\mathbf{C}_0]$, (1.1) and Chebyshev inequality, we obtain for sufficiently large $T$

$$\mathbf{P}_1\le\mathbf{P}\left(e^{\lambda|\zeta_1|}>e^{\lambda x\varepsilon}\right)+(x-1)\,\mathbf{P}\left(e^{\lambda|\zeta|}>e^{\lambda x\varepsilon}\right)$$

$$\le\frac{\mathbf{E}e^{\lambda|\zeta_1|}}{e^{\lambda x\varepsilon}}+(x-1)\,\frac{\mathbf{E}e^{\lambda|\zeta|}}{e^{\lambda x\varepsilon}}\le cxe^{-\lambda x\varepsilon}\le e^{-\frac{\lambda x\varepsilon}{2}}, \tag{2.3}$$

where $c:=\max\left(\mathbf{E}e^{\lambda|\zeta_1|},\mathbf{E}e^{\lambda|\zeta|}\right)$, the parameter $\lambda>0$ from condition $[\mathbf{C}_0]$ [1].

Let us find the upper estimate for $\mathbf{P}_2$. For any $C>0$ we have

$$\mathbf{P}_2\le\sum_{k>x}\mathbf{P}\left(\frac{|\zeta_k|}{1+T_{k-1}}>x\varepsilon\right)=:\sum_{k>x}\mathbf{P}_{2,k}. \tag{2.4}$$

Let us estimate $\mathbf{P}_{2,k}$ from above.

$$\mathbf{P}_{2,k}\le\mathbf{P}\left(\frac{|\zeta_k|}{C(k-1)}>x\varepsilon,T_{k-1}>C(k-1)\right)+\mathbf{P}\left(T_{k-1}\le C(k-1)\right)$$

$$\le\mathbf{P}\left(\frac{|\zeta_k|}{C(k-1)}>x\varepsilon\right)+\mathbf{P}\left(T_{k-1}\le C(k-1)\right)=:\mathbf{P}_{2,1,k}+\mathbf{P}_{2,2,k}. \tag{2.5}$$

Using condition $[\mathbf{C}_0]$ and Chebyshev inequality, we obtain

$$\mathbf{P}_{2,1,k}\le\mathbf{P}\left(e^{\lambda|\zeta_k|}>e^{\lambda\varepsilon C(k-1)x}\right)\le\frac{\mathbf{E}e^{\lambda|\zeta_k|}}{e^{\lambda\varepsilon C(k-1)x}}\le ce^{-\lambda\varepsilon C(k-1)x}. \tag{2.6}$$

Now we find the upper estimate for $\mathbf{P}_{2,2,k}$. Applying the Chebyshev inequality, we have

$$\mathbf{P}_{2,2,k}=\mathbf{P}\left(T_{k-1}\le C(k-1)\right)\le\mathbf{P}\left(e^{-\sum\limits_{r=2}^{k-1}\tau_r}\ge e^{-C(k-1)}\right)$$

$$\le\frac{\mathbf{E}e^{-\sum\limits_{r=2}^{k-1}\tau_r}}{e^{-C(k-1)}}\le\frac{b^{k-2}}{e^{-C(k-1)}}, \tag{2.7}$$

where $b:=\mathbf{E}e^{-\tau}$. Since $\mathbf{P}(\tau>0)=1$ we obtain $b<1$. Therefore, choosing the constant $C:=C(b)=-\frac{\ln b}{2}$, using inequality (2.7), we have

$$\mathbf{P}_{2,2,k}\le\frac{1}{b}b^{\frac{k-1}{2}}. \tag{2.8}$$

It follows from inequalities (2.5), (2.6) and (2.8) that

$$\mathbf{P}_{2,k}\le ce^{-\lambda\varepsilon C(k-1)x}+\frac{1}{b}b^{\frac{k-1}{2}}\le\frac{2}{b}b^{\frac{k-1}{2}} \tag{2.9}$$

for sufficiently large $T$.

Applying inequalities (2.4) and (2.9), we obtain

$$\mathbf{P}_2\le\frac{2}{b}\sum_{k>x}b^{\frac{k-1}{2}}\le\frac{2}{b^{3/2}}b^{\frac{x}{2}}\sum_{k=0}^{\infty}b^{\frac{k}{2}}\le\frac{2\tilde{C}}{b^{3/2}}e^{-\frac{x}{2}\ln\left(\frac{1}{b}\right)}, \tag{2.10}$$

[1] everywhere below, the parameter $\lambda$ is used only in this sense, i.e. as a constant from condition $[\mathbf{C}_0]$

where $\tilde{C} := \sum\limits_{k=0}^{\infty} b^{\frac{k}{2}}$.

It follows from inequalities (2.2), (2.3), (2.10) and condition (1.1) that for sufficiently large $T$

$$\mathbf{P}\left(\rho(z_{1,T}, \tilde{z}_{1,T}) > \varepsilon\right) \le e^{-\frac{\lambda x \varepsilon}{2}} + \frac{2\tilde{C}}{b^{3/2}} e^{-\frac{x}{2}\ln(\frac{1}{b})} \le h e^{-xl},$$

where $h := 1 + \frac{2\tilde{C}}{b^{3/2}}$, $l := \min(\frac{\lambda\varepsilon}{2}, \frac{1}{2}\ln(\frac{1}{b}))$.

Therefore, for any $\varepsilon > 0$ we have

$$\limsup_{T\to\infty} \frac{T}{x^2} \ln \mathbf{P}\left(\rho(z_{1,T}, \tilde{z}_{1,T}) > \varepsilon\right) \le -\lim_{T\to\infty} \frac{T}{x} l = -\infty.$$

The other equivalences are proved in a similar way.□

**Remark 1.** *Obviously, Lemma* 2.1 *implies for any* $u \in [0, \infty)$ *the equivalences* $z_{1,T} \overset{L.D.}{\sim} \tilde{z}_{1,T}$, $z_{2,T} \overset{L.D.}{\sim} \tilde{z}_{2,T}$, $y_{1,T} \overset{L.D.}{\sim} \tilde{y}_{1,T}$, $y_{2,T} \overset{L.D.}{\sim} \tilde{y}_{2,T}$ *on MS* $\mathbb{D}[0, u]$ *with NF* $\psi(T) = \frac{x^2}{T}$.

The following assertion follows from Corollary 3.2 and Lemma 4.1 in [3].

**Theorem 2.2.** *1) For any* $u \in [0, \infty)$ *the families of random processes* $z_{1,T}$, $y_{1,T}$ *satisfy* $(I_{1,u}, \frac{x^2}{T}, \mathbb{D}[0, u])$ *– LDP, where*

$$I_{1,u}(f) := \begin{cases} \frac{a_\tau}{2\sigma_1^2} \int_0^u (f'(t))^2 dt, & \textit{for} \quad f \in \mathbb{AC}_0[0, u], \\ \infty, & \textit{otherwise}, \end{cases}$$

$\sigma_1^2 = \mathbf{D}(\zeta - a\tau)$.

*2) For any* $u \in [0, \infty)$ *the families of random processes* $z_{2,T}$, $y_{2,T}$ *satisfy* $(I_{2,u}, \frac{x^2}{T}, \mathbb{D}[0, u])$ *– LDP, where*

$$I_{2,u}(f) := \begin{cases} \frac{a_\tau}{2\sigma_2^2} \int_0^u (f'(t))^2 dt, & \textit{for} \quad f \in \mathbb{AC}_0[0, u], \\ \infty, & \textit{otherwise}, \end{cases}$$

$\sigma_2^2 = \mathbf{D}\zeta$.

**Remark 2.** *Since the trajectories of the processes* $\tilde{z}_{1,T}$, $\tilde{z}_{2,T}$, $\tilde{y}_{1,T}$, $\tilde{y}_{2,T}$ *are almost surely continuous, it follows from Remark 1 and Theorem 2.2 that for any* $u \in [0, \infty)$ *the families of random processes* $\tilde{z}_{1,T}$, $\tilde{y}_{1,T}$ *satisfy* $(I_{1,u}, \frac{x^2}{T}, \mathbb{C}[0, u])$ *– LDP, and the families of random processes* $\tilde{z}_{2,T}$, $\tilde{y}_{2,T}$ *satisfy* $(I_{2,u}, \frac{x^2}{T}, \mathbb{C}[0, u])$ *– LDP.*

## 3. Proof of Theorem 1.1

P r o o f. It follows from Lemma 2.1 (see section 2) that it is sufficient to prove that the families of random processes $\tilde{z}_{1,T}$, $\tilde{y}_{1,T}$ satisfy $(I_1, \frac{x^2}{T}, \mathbb{C})$ – LDP, and the families of random processes $\tilde{z}_{2,T}$, $\tilde{y}_{2,T}$ satisfy $(I_2, \frac{x^2}{T}, \mathbb{C})$ – LDP.

It follows from Remark 2 (see section 2) that for any $u \in [0, \infty)$ random processes $\tilde{z}_{1,T}$, $\tilde{y}_{1,T}$ satisfy $(I_{1,u}, \frac{x^2}{T}, \mathbb{C}[0, u])$ – LDP, and the families of random processes $\tilde{z}_{2,T}$, $\tilde{y}_{2,T}$ satisfy $(I_{2,u}, \frac{x^2}{T}, \mathbb{C}[0, u])$ – LDP. Therefore, it is sufficient to verify conditions I–III of Theorem 2.1 (see section 2).

It is easy to see that condition I is met, as a function $g_f$ one can choose

$$g(t) = \begin{cases} f(t), & \text{for} \quad t \in [0, u], \\ f(u), & \text{for} \quad t \in (u, \infty). \end{cases}$$

Let's check condition II for the functional $I_{1,u}$ (it is defined in Theorem 2.2 of Section 2), i.e. let us show that for any $r \geq 0$

$$\lim_{u\to\infty} \sup_{f\in B_r^+} \sup_{t\geq u} \frac{|f(t)|}{1+t} = 0,$$

where

$$B_r^+ := \{f \in \mathbb{C} : \ \overline{\lim_{u\to\infty}} I_{1,u}(f^{(u)}) \leq r\}.$$

Using the Cauchy – Schwarz inequality, we obtain

$$\lim_{u\to\infty} \sup_{f\in B_r^+} \sup_{t\geq u} \frac{|f(t)|}{1+t} = \lim_{u\to\infty} \sup_{f\in B_r^+} \sup_{t\geq u} \frac{1}{1+t}\left|\int_0^t f'(s)ds\right|$$

$$\leq \lim_{u\to\infty} \sup_{f\in B_r^+} \sup_{t\geq u} \frac{t^{1/2}}{1+t}\left(\int_0^t (f'(s))^2 ds\right)^{1/2} \leq \lim_{u\to\infty} \sup_{f\in B_r^+} \frac{1}{u^{1/2}}\left(\sup_{t\geq u} \frac{2\sigma_1^2}{a} I_{1,t}(f^{(t)})\right)^{1/2}$$

$$= \lim_{u\to\infty} \sup_{f\in B_r^+} \frac{\sqrt{2}\sigma_1}{\sqrt{a}u^{1/2}}(\overline{\lim_{u\to\infty}} I_{1,u}(f^{(u)}))^{1/2} \leq \lim_{u\to\infty} \frac{\sqrt{2r}\sigma_1}{\sqrt{a}u^{1/2}} = 0.$$

Condition II for the functional $I_{2,u}$ (it is defined in Theorem 2.2 of Section 2) can be verified in a completely similar way.

Let's check condition III for the family of processes $\tilde{z}_{1,T}$, i.e. show that for any $N < \infty$ and $\varepsilon > 0$ there exists $M = M_{N,\varepsilon} < \infty$ such that

$$\overline{\lim_{n\to\infty}} \frac{T}{x^2} \ln \mathbf{P}\left(\sup_{t\geq M} \frac{|\tilde{z}_{1,T}(t)|}{1+t} > \varepsilon\right) \leq -N.$$

We will consider the case when the CRP is homogeneous. We will have

$$\mathbf{P}\left(\sup_{t\geq M} \frac{|\tilde{z}_{1,T}(t)|}{1+t} > \varepsilon\right) \leq \sum_{k=[M]+1}^{\infty} \mathbf{P}\left(\sup_{t\in[k-1,k]} \frac{|\tilde{z}_{1,T}(t)|}{1+t} > \varepsilon\right)$$

$$\leq \sum_{k=[M]+1}^{\infty} \mathbf{P}\left(\sup_{t\in[0,k]} \frac{|\tilde{z}_{1,T}(t)|}{k} > \varepsilon\right) =: \sum_{k=[M]+1}^{\infty} \mathbf{P}_k. \tag{3.1}$$

Let us estimate $\mathbf{P}_k$ from above. Since a polygon can reach its maximum on the segment only at nodes, we will have

$$\mathbf{P}_k \leq \mathbf{P}\left(\sup_{t\in[0,k]} |Z(tT) - atT| > \varepsilon kx\right) = \mathbf{P}\left(\sup_{t\in[0,k]} |Z(tT) \pm a_\zeta \nu(tT) - atT| > \varepsilon kx\right)$$

$$\leq \mathbf{P}\left(\sup_{t\in[0,k]} |Z(tT) - a_\zeta \nu(tT)| > \frac{\varepsilon kx}{2}\right)$$

$$+\mathbf{P}\left(\sup_{t\in[0,k]} \left|\nu(tT) - \frac{1}{a_\tau} tT\right| > \frac{\varepsilon kx}{2|a_\zeta|}\right) =: \mathbf{P}_{1,k} + \mathbf{P}_{2,k}. \tag{3.2}$$

Now we find the upper estimate for $\mathbf{P}_{1,k}$. For any $C > 0$ we have

$$\mathbf{P}_{1,k} \leq \mathbf{P}\left(\max_{1\leq r\leq [CT]} \left|\sum_{j=1}^{r} (\zeta_j - a_\zeta)\right| > \frac{\varepsilon kx}{2}, \ \nu(kT) \leq CT\right)$$

$$+\mathbf{P}(\nu(kT) > CT) =: \mathbf{P}_{1,1,k} + \mathbf{P}_{1,2,k}. \tag{3.3}$$

First, let's find the upper estimate for $\mathbf{P}_{1,2,k}$. Applying the Chebyshev inequality, we obtain

$$\mathbf{P}_{1,2,k} = \mathbf{P}\left(\nu(kT) > CT\right) \leq \mathbf{P}\left(T_{[CT]} < kT\right)$$

$$= \mathbf{P}\left(e^{\sum_{r=1}^{[CT]} \tau_r} < e^{kT}\right) = \mathbf{P}\left(e^{-\sum_{r=1}^{[CT]} \tau_r} > e^{-kT}\right) \leq \frac{(\mathbf{E}e^{-\tau})^{[CT]}}{e^{-kT}}. \tag{3.4}$$

Since $\mathbf{P}(\tau > 0) = 1$, we will have $c := \mathbf{E}e^{-\tau} < 1$. Therefore, by choosing a constant

$$C := C(c,k) = -\frac{3k}{\ln c}, \tag{3.5}$$

using inequality (3.4), we obtain

$$\mathbf{P}_{1,2,k} \leq \frac{c^{-\frac{2kT}{\ln c}}}{e^{-kT}} = e^{-kT}. \tag{3.6}$$

Now let us estimate $\mathbf{P}_{1,1,k}$ from above. We will have

$$\mathbf{P}_{1,1,k} \leq \mathbf{P}\Bigg(\max_{1\leq r\leq [CT]} \exp\left\{\left|\sum_{j=1}^{r}(\zeta_j - a_\zeta)\right|\right\}$$

$$> \exp\left\{\frac{\varepsilon k x}{2}\right\}\Bigg) \leq \mathbf{P}\left(\max_{1\leq r\leq [CT]} e^{\sum_{j=1}^{r}(\zeta_j - a_\zeta)} > e^{\frac{\varepsilon k x}{2}}\right)$$

$$+\mathbf{P}\left(\max_{1\leq r\leq [CT]} e^{-\sum_{j=1}^{r}(\zeta_j - a_\zeta)} > e^{\frac{\varepsilon k x}{2}}\right) =: \mathbf{P}_{1,1,1,k} + \mathbf{P}_{1,1,2,k}. \tag{3.7}$$

Let us find an upper estimate for $\mathbf{P}_{1,1,1,k}$. Since the random variables $\zeta_1, \ldots, \zeta_{[CT]}$ are independent and satisfy condition $[\mathbf{C}_0]$, it is easy to see that for any $l \in [0, \lambda]$ the sequence

$$M(r) := \frac{e^{l\sum_{j=1}^{r}(\zeta_j - a_\zeta)}}{\mathbf{E}e^{l\sum_{j=1}^{r}(\zeta_j - a_\zeta)}}, \quad 1 \leq r \leq [CT]$$

is a positive martingale and $\mathbf{E}M(r) = 1$.

Let's denote $b(l) := \mathbf{E}e^{l(\zeta - a_\zeta)}$. Using Jensen's inequality, we obtain

$$b(l) = \mathbf{E}e^{l(\zeta - a_\zeta)} \geq e^{\mathbf{E}l(\zeta - a_\zeta)} = 1. \tag{3.8}$$

Applying inequality (3.8) and Dub's inequality (see [8] Theorem 3.2 on p. 317), we will have

(3.9)

$$\mathbf{P}_{1,1,1,k} \leq \mathbf{P}\left(\max_{1\leq r\leq [CT]} M(r) > \frac{e^{\frac{l\varepsilon k x}{2}}}{(b(l))^{[CT]}}\right) \leq \frac{(b(l))^{[CT]}\mathbf{E}M([CT])}{e^{\frac{l\varepsilon k x}{2}}} = \frac{(b(l))^{[CT]}}{e^{\frac{l\varepsilon k x}{2}}}.$$

Let's choose $l := \frac{x}{T\sqrt{k}}$ and find an upper estimate for $b(l)$. We will have

$$b(l) = 1 + \mathbf{E}\sum_{r=2}^{\infty}\frac{(\zeta - a_\zeta)^r l^r}{r!} \leq 1 + \frac{l^2}{2}\mathbf{E}(\zeta - a_\zeta)^2 e^{|\zeta - a_\zeta| l}.$$

It follows from conditions $[\mathbf{C}_0]$ and (1.1), that there exists a constant $h > 0$ such that for all sufficiently large $T$ the inequality

$$b(l) \leq 1 + \frac{hl^2}{2} = 1 + \frac{hx^2}{2T^2k}. \tag{3.10}$$

will be satisfied. It follows from (3.5) and inequality (3.10) that

$$(b(l))^{[CT]} \le \left(1+\frac{hx^2}{2T^2k}\right)^{[CT]} \sim e^{-\frac{3hx^2}{2T\ln c}} \quad \text{when} \quad T\to\infty. \tag{3.11}$$

By choosing $M > \left(\frac{6h}{\varepsilon \ln c}\right)^2$, using inequalities (3.9) and (3.11), we have for $k \ge [M]+1$ and large enough $T$

$$\mathbf{P}_{1,1,1,k} \le 2e^{-\frac{\sqrt{k}\varepsilon x^2}{4T}}. \tag{3.12}$$

The upper estimate for $\mathbf{P}_{1,1,2,k}$ is obtained in a completely similar way and has the same form. Therefore, it follows from inequalities (3.7), (3.12) that

$$\mathbf{P}_{1,1,k} \le 4e^{-\frac{\sqrt{k}\varepsilon x^2}{4T}}. \tag{3.13}$$

It follows from condition (1.1), inequalities (3.3), (3.6), (3.13) that for sufficiently large $T$

$$\mathbf{P}_{1,k} \le e^{-kT} + 4e^{-\frac{\sqrt{k}\varepsilon x^2}{4T}} \le 5e^{-\frac{\sqrt{k}\varepsilon x^2}{4T}}. \tag{3.14}$$

Let's find the upper estimate for $\mathbf{P}_{2,k}$.

$$\mathbf{P}_{2,k} \le \mathbf{P}\biggl(\sup_{t\in[0,k]}\left|\nu(tT)-\frac{1}{a_\tau}tT\right| > \frac{\varepsilon kx}{2|a_\zeta|},\ \nu(kT)\le CT\biggr)$$

$$+\mathbf{P}(\nu(kT) > CT) =: \mathbf{P}_{2,1,k} + \mathbf{P}_{1,2,k}. \tag{3.15}$$

Let's estimate $\mathbf{P}_{2,1,k}$ from above. Let us denote

$$A := \bigcap_{r=1}^{CT}\left\{|T_r - a_\tau r| \le \frac{\varepsilon kxa_\tau}{3|a_\zeta|}\right\}.$$

It is easy to see that for sufficiently large $T$

$$A\cap\{\nu(kT)\le CT\}$$

$$\subseteq \left\{\forall\ t\in[0,k]\ \nu(tT) \le \frac{tT}{a_\tau}+\frac{\varepsilon kx}{3|a_\zeta|},\ \nu(tT) \ge \frac{tT}{a_\tau}-\frac{\varepsilon kx}{3|a_\zeta|}-1\right\}\cap\{\nu(kT)\le CT\}$$

$$\subseteq \left\{\sup_{t\in[0,k]}\left|\nu(tT)-\frac{1}{a_\tau}tT\right| \le \frac{\varepsilon kx}{2|a_\zeta|}\right\}\cap\{\nu(kT)\le CT\} =: B\cap\{\nu(kT)\le CT\}.$$

Hence $\overline{A}\cap\{\nu(kT)\le CT\} \supseteq \overline{B}\cap\{\nu(kT)\le CT\}$, and therefore

$$\mathbf{P}_{2,1,k} \le \mathbf{P}(\overline{A},\nu(kT)\le CT) \le \mathbf{P}\biggl(\max_{1\le r\le[CT]}\left|\sum_{j=1}^{r}(\tau_j-a_\tau)\right| > \frac{\varepsilon kxa_\tau}{3|a_\zeta|}\biggr).$$

By applying arguments completely similar to those that we used to estimate the term $\mathbf{P}_{1,1,k}$, we obtain

$$\mathbf{P}_{2,1,k} \le 4e^{-\frac{\sqrt{k}\varepsilon a_\tau x^2}{6T|a_\zeta|}}. \tag{3.16}$$

It follows from condition (1.1), inequalities (3.6), (3.15) and (3.16) that for sufficiently large $T$

$$\mathbf{P}_{2,k} \le e^{-kT} + 4e^{-\frac{\sqrt{k}\varepsilon a_\tau x^2}{6T|a_\zeta|}} \le 5e^{-\frac{\sqrt{k}\varepsilon a_\tau x^2}{6T|a_\zeta|}}. \tag{3.17}$$

Using condition (1.1), inequalities (3.1), (3.2), (3.14) and (3.17), assuming $v := \min(\frac{1}{4}, \frac{a_\tau}{6|a_\zeta|})$, we obtain for sufficiently large $T$ and $M$

$$\mathbf{P}\bigg(\sup_{t\geq M}\frac{|\tilde{z}_{1,T}(t)|}{1+t} > \varepsilon\bigg) \leq 5\sum_{k=[M]+1}^{\infty}\Big(e^{-\frac{\sqrt{k}\varepsilon x^2}{4T}} + e^{-\frac{\sqrt{k}\varepsilon a_\tau x^2}{6T|a_\zeta|}}\Big) \leq 10\sum_{k=[M]+1}^{\infty} e^{-\frac{\sqrt{k}\varepsilon v x^2}{T}}$$

$$\leq 10 e^{-\frac{\sqrt{M}\varepsilon v x^2}{2T}}\sum_{k=1}^{\infty} e^{-\frac{\sqrt{k}\varepsilon v x^2}{2T}} \leq 10 e^{-\frac{\sqrt{M}\varepsilon v x^2}{T}}\sum_{k=1}^{\infty} e^{-\sqrt{k}} =: 10\tilde{C} e^{-\frac{\sqrt{M}\varepsilon v x^2}{T}}, \tag{3.18}$$

where $\tilde{C} := \sum\limits_{k=1}^{\infty} e^{-\sqrt{k}}$. It follows from inequality(3.18) that for any

$$M \geq \max\big(\big(\frac{6h}{\varepsilon \ln c}\big)^2, \big(\frac{N}{\varepsilon v}\big)^2\big)$$

the inequality

$$\overline{\lim_{n\to\infty}}\frac{T}{x^2}\ln\mathbf{P}\bigg(\sup_{t\geq M}\frac{|\tilde{z}_{1,T}(t)|}{1+t} > \varepsilon\bigg) \leq -\sqrt{M}\varepsilon v \leq -N$$

is satisfied. Thus, condition III is met for the homogeneous process $\tilde{z}_{1,T}(t)$.

The inhomogeneous case is considered completely similar, we should put

$$b(l) := \max\big(\mathbf{E}e^{l(\zeta_1 - a_{\zeta_1})}, \mathbf{E}e^{l(\zeta - a_\zeta)}\big),$$

and replace the right-hand side of inequality (3.4) by $\frac{(\mathbf{E}e^{-\tau})^{[CT]-1}}{e^{-kT}}$.

Condition III is verified similarly for the process $\tilde{z}_{2,T}(t)$, but the proof is simplified since we do not need to estimate $\mathbf{P}_{2,k}$ from above.

Condition III is also verified in a similar way for the processes $\tilde{y}_{1,T}(t)$, $\tilde{y}_{2,T}(t)$, but some estimated sums will have one more term.□

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