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# OPTIMIZATION OF NODES OF COMPOSITE QUADRATURE FORMULAS IN THE PRESENCE OF A BOUNDARY LAYER

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Abstract. The problem of numerical integration of a function of one variable with large gradients in the region of the exponential boundary layer is studied. The problem is that the use of composite quadrature formulas on a uniform grid with decreasing of the small parameter value leads to significant errors, regardless of the number of nodes of the basic quadrature formula. In the paper it is proposed to choose nodes based on the composite quadrature formula error minimizing. Basic quadrature formula applied between grid nodes, takes into account the cases of the Newton-Cotes and Gauss formulas. It is proved that the minimum error is achieved on the Bakhvalov mesh, and the error of the quadrature formula becomes uniform in a small parameter.

**Keywords:** function of one variable, large gradients, numerical integration, Newton-Cotes formula, Gauss formula, optimization of nodes, error estimation.

## 1. Introduction

The problem of numerical integration of functions with singularities is of interest. For example, [1] presents some approaches to numerical integration functions with a singularity.

The solution of a singularly perturbed boundary value problem for an equation with a small parameter $\varepsilon$ at the highest derivative has large gradients in the boundary layer [2], [3]. The problem of numerical integrating of a function with large gradients in the exponential boundary layer considered in [4]. It is shown

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that when integrating such a function, the composite Newton-Cotes formulas on a uniform mesh with a step $h$ have an error of the order of $O(h)$, regardless of the number of nodes in the basic quadrature formula. For example, for $\varepsilon = 1$ the function does not have large gradients and the composite Simpson formula has an error of the order of $O(h^4)$, but for $\varepsilon = h$ the error becomes of the order of $O(h)$. Thus, the problem of constructing quadrature formulas in the presence of a boundary layer, the error of which is uniform in a parameter $\varepsilon \in (0,1]$, is important.

Consider the numerical integration problem

$$I(u) = \int_0^1 u(x)\,dx. \tag{1}$$

We assume that the function $u(x)$ has large gradients in the region of the exponential boundary layer. We use the decomposition

$$u(x) = p(x) + \Phi(x),\ x \in [0,1], \tag{2}$$

where

$$|p^{(j)}(x)| \le C_1,\quad |\Phi^{(j)}(x)| \le \frac{C_1}{\varepsilon^j}\, e^{-\alpha x/\varepsilon},\ 0 \le j \le J, \tag{3}$$

the functions $p(x)$ and $\Phi(x)$ are not explicitly specified, $\alpha > 0, \varepsilon \in (0,1]$. Coefficient $\alpha$ is separated from zero, parameter $\varepsilon$ can be close to zero, $J$ will be specified later.

Throughout this work, by $C$ and $C_j$ we mean positive constants independent of the parameter $\varepsilon$ and the number of mesh intervals $N$. In this case, one constant $C_j$ can be used to estimate different values if it is clear from the text.

According to (3), the regular component $p(x)$ has derivatives bounded to some order $J$, and the derivatives of the singular component $\Phi(x)$ are not uniformly bounded in the parameter $\varepsilon$.

In accordance with [2], [3], [5], decomposition (2) is valid for the solution of a singularly perturbed boundary value problem:

$$\varepsilon u''(x) + a_1(x)u'(x) - a_2(x)u(x) = f(x),\quad u(0) = A,\ u(1) = B, \tag{4}$$

where $a_1(x) \ge \alpha > 0,\ a_2(x) \ge 0,\ \varepsilon > 0$, functions $a_1(x), a_2(x), f(\mathrm{x})$ are sufficiently smooth. For small values of the parameter $\varepsilon$, the solution of the problem (4) has a boundary layer region of large gradients at the boundary $x = 0$, which corresponds to the representation (2).

In [4, 6] a modification of the Newton-Cotes formulas is proposed, based on making the formulas exact on the singular component $\Phi(x)$, which is assumed to be known up to a factor. This decomposition of the solution of the problem (4) was built in [7]. In [6] it is proved that the composite modified quadrature formula has an error of order $O(h^{k-1})$ uniformly in $\varepsilon$, where $k$ is the number of nodes in the mesh pattern of the basic formula.

It is possible to apply a mesh that is condensed in the boundary layer region to make the error of Newton-Cotes formulas uniform in a parameter $\varepsilon$. In [8] it is proposed to apply the Newton-Cotes formulas on the Shishkin mesh [2]. Then it is proved that the error of the composite Newton-Cotes formula with $k$ nodes becomes of order $O((\ln(N)/N)^k)$ uniformly with respect to the parameter $\varepsilon$. Here $N$ is the number of grid steps on the interval $[0,1]$. Note that in the regular case, when the grid is uniform and the derivatives of the integrable function are bounded, the composite Newton-Cotes formula has an error of order $O(1/N^k)$ [1].

It is known that Gauss quadrature formulas have a higher order of accuracy. The application of Gauss formulas for the numerical integration of functions of the form (2) was investigated in [9]. It is proposed to apply the Shishkin mesh and construct a Gauss formula with $k$ nodes on each interval $[x_{n-1}, x_n]$. For such a composite quadrature formula, an error estimate of order $O((\ln(N)/N)^{2k})$ is obtained uniformly with respect to the parameter $\varepsilon$.

The problem of constructing an optimal mesh for the numerical integration of functions with large gradients is considered in [1, Chapter 3]. The problem of minimizing the error of the composite quadrature formula of trapezoids by specifying the mesh nodes is considered. It is assumed that the estimate for the second derivative $|u''(x)| \le F(x)$ is given on the integration interval.

Let us apply this approach to numerically integrate the function $u(x)$ with large gradients in the exponential boundary layer, when the decomposition (2) with estimates of derivatives (3) is valid.

Let's set the mesh of the interval $[0, 1]$

$$\Omega^h = \{x_n : x_n = x_{n-1} + h_n,\ n = 1, 2, \dots, N,\ x_0 = 0, x_N = 1\},$$

whose nodes will be based on minimizing the error when calculating the integral (1).

## 2. Optimizing the nodes of the quadrature formula

Let's move on to constructing a composite quadrature formula. We represent $I(u)$ from (1) as

$$I(u) = \sum_{n=1}^{N} I_n(u),\ I_n(u) = \int_{x_{n-1}}^{x_n} u(x)\,dx. \tag{5}$$

Let the quadrature formula $S_{k,n}(u)$ with $k$ nodes be used to calculate the integral $I_n(u)$ from a sufficiently smooth function $u(x)$. We consider quadrature formulas for which, for some constant $C$, the error estimate holds

$$|S_{k,n}(u) - I_n(u)| \le C(x_n - x_{n-1})^{mk+1} \max_{s\in[x_{n-1},x_n]} |u^{(mk)}(s)|, \tag{6}$$

where $m$ is integer, $m \ge 1$. This parameter corresponds to the quadrature formula.

According to [1, p. 95], for $m = 1$ the estimate (6) is valid for the Newton-Cotes formula with $k$ equally spaced nodes.

According to [10], for $m = 2$ the estimate (6) is valid for the Gauss formula with $k$ nodes.

Let's define a composite quadrature formula

$$S_k(u) = \sum_{n=1}^{N} S_{k,n}(u). \tag{7}$$

We use approach [1] and assume that for a given $m$ for some constant $C$ the following estimate holds

$$|u^{(mk)}(x)| \le F_k(x). \tag{8}$$

Then according to (6)

$$|I(u) - S_k(u)| \le C \sum_{n=1}^{N} \max_{x\in[x_{n-1},x_n]} F_k(x)(x_n - x_{n-1})^{mk+1}. \tag{9}$$

We will define the nodes $x_n$ based on minimizing the error of the composite quadrature formula $S_k(u)$.

We define the nodes of the mesh $\Omega^h$ in the form

$$x_n = g(n/N),\ n = 0, 1, \dots, N,\ g(0) = 0,\ g(1) = 1, \tag{10}$$

where $g(t)$ is a strictly increasing function.

In accordance with the [1, Chapter 3] approach, the problem of minimizing the sum in (9) is reduced to minimizing the integral

$$\int_0^1 (g'(t))^{mk+1} F_k(g(t))\, dt = \int_0^1 (t'(g))^{-mk} F_k(g)\, dg = \int_0^1 G(g, t, t') dg. \tag{11}$$

The integral (11) is minimized, as in [1], based on the solution of the Euler equation

$$\frac{d}{dg}\Big(\frac{\partial G}{\partial t'}\Big) - \frac{\partial G}{\partial t} = 0. \tag{12}$$

Taking into account (11), from the equation (12) we obtain

$$F_k(g)(g'(t))^{mk+1} = M_1, \tag{13}$$

where $M_1$ is some constant. Taking into account the boundary conditions $g(0) = 0$, $g(1) = 1$, we find the function $g(t)$ and then we find the nodes $x_n = g(n/N),\ n = 0, 1, \dots, N$.

Now consider the case where the integrated function is of the form (2) with $J = mk$ in (3). When constructing the mesh, we will assume that in (3) $\varepsilon \le \varepsilon_0 < 1$, where some constant $\varepsilon_0$ is separated from 1. In the case of $\varepsilon \ge \varepsilon_0$ the derivatives of the function $u(x)$ are uniformly bounded, and to estimate the error, we can use well-known estimates, which, in order of accuracy, are not lower than those obtained estimates in the case $\varepsilon \le \varepsilon_0$.

In accordance with the estimates (3), (8) on the interval $[0, 1]$ we define

$$F_k(x) = 1 + \frac{1}{\varepsilon^{mk}} e^{-\alpha x/\varepsilon}.$$

We have taken into account that, according to (13) the function $F_k(x)$ can be specified up to a factor. Note that with this definition of $F_k(x)$ the equation (13) is not explicitly integrated. Taking into account the estimates (3), we set the interval $[0, \sigma]$, corresponding to the region of large gradients, on which we set $F_k(x) = \varepsilon^{-mk} e^{-\alpha x/\varepsilon}$. On the interval $[\sigma, 1]$ we set $F_k(x) = 1$.

Let us dwell on the case of the interval $[0, \sigma]$. Integrating (13) taking into account the boundary conditions $g(0) = 0, g(1/2) = \sigma$, we get

$$g(t) = -\frac{(mk+1)\varepsilon}{\alpha} \ln\Big[1 - 2(1 - e^{-\alpha\sigma/((mk+1)\varepsilon)})t\Big],\ t \in [0, 1/2]. \tag{14}$$

Similarly, on the interval $[\sigma, 1]$ we get

$$g(t) = 2\sigma - 1 + 2(1 - \sigma)t,\ t \in [1/2, 1]. \tag{15}$$

Thus, on the basis of minimizing the error of the composite quadrature formula, taking into account (14), (15), we obtain the mesh nodes:

$$x_n = -\frac{(mk+1)\varepsilon}{\alpha} \ln\Big[1 - 2(1 - e^{-\alpha\sigma/((mk+1)\varepsilon)})n/N\Big],\ n = 0, 1, 2, \dots, N/2, \tag{16}$$

$$x_n = 2\sigma - 1 + 2(1-\sigma)n/N,\ \ n = N/2, N/2+1, \dots, N. \tag{17}$$

**Theorem 1.** *Let the function $u(x)$ has the representation (2), $J = mk$ in (3), mesh nodes correspond to (16), (17),*

$$\sigma = -\frac{(mk+1)\varepsilon}{\alpha}\ln\varepsilon. \tag{18}$$

*Then for some constant $C$*

$$|I(u) - S_k(u)| \le \frac{C}{N^{mk}}. \tag{19}$$

*Proof.* According to (16), (18)

$$x_n = -\frac{(mk+1)\varepsilon}{\alpha}\ln\Big[1 - 2(1-\varepsilon)n/N\Big],\ \ n = 0, 1, \dots, N/2, \tag{20}$$

hence,

$$h_n = \frac{(mk+1)\varepsilon}{\alpha}\ln\Big[1 + \frac{2(1-\varepsilon)/N}{1 - 2(1-\varepsilon)n/N}\Big],\quad n = 1, 2, \dots, N/2. \tag{21}$$

It is easy to verify that the sequence of steps $\{h_n\},\ n = 1, 2, \dots, N/2$ is increasing, and for some constant $C_0$ the estimate is correct:

$$h_n \le \frac{C_0}{N},\ \ n = 1, 2, \dots, N. \tag{22}$$

According to estimates (3), (6), (7), (22) for some constant $C$

$$|I(p) - S_k(p)| \le \frac{C}{N^{mk}}. \tag{23}$$

Let's estimate $|I(\Phi) - S_k(\Phi)|$.

Let $r_n(u) = I_n(u) - S_{k,n}(u),\ n \le N/2$. According to (3), (6)

$$|r_n(\Phi)| \le f_n = \frac{C}{\varepsilon^{mk}} h_n^{mk+1} e^{-\alpha x_{n-1}/\varepsilon}. \tag{24}$$

Considering (20), we get

$$e^{-\alpha x_n/\varepsilon} = (1 - 2(1-\varepsilon)n/N)^{mk+1}. \tag{25}$$

Let's set $K = 2(1-\varepsilon)/N,\ 0 < K < 2/N$. Considering (21), (24), (25), we get

$$f_n \le C\varepsilon\Big(\ln\Big(1 + \frac{K}{1-Kn}\Big)(1 - K(n-1))\Big)^{mk+1}. \tag{26}$$

Let's define the function

$$R(x) = \ln\Big(1 + \frac{K}{1-Kx}\Big)(1 - K(x-1)),\ \ 1 \le x \le N/2.$$

Then

$$R'(x) = K\Big[\frac{K}{1-Kx} - \ln\Big(1 + \frac{K}{1-Kx}\Big)\Big].$$

It follows that $R'(x) > 0$. We see that the function $R(x)$ is positive and strictly increasing as $1 \le x \le N/2$. Therefore, $f_n \le f_{N/2},\ n \le N/2$.

Now we estimate $f_{N/2}$. From (26) it follows that

$$f_{N/2} \le C\varepsilon\Big(\ln\Big[1 + \frac{2(1-\varepsilon)}{N\varepsilon}\Big]\Big(\varepsilon + \frac{2(1-\varepsilon)}{N}\Big)\Big)^{mk+1}. \tag{27}$$

Let $\varepsilon = N^{-r},\ r > 0$. Considering the cases $r \geq 1$ and $r < 1$, we obtain that for some constant $C$ the estimate $f_{N/2} \leq C/N^{mk+1}$ is valid. Taking into account the inequality $f_n \leq f_{N/2},\ n \leq N/2$, we get

$$|I_n(\Phi) - S_{k,n}(\Phi)| \leq \frac{C}{N^{mk+1}}. \tag{28}$$

Consider the case $n > N/2$. Using (3), (18), we have $|\Phi^{(mk)}(x)| \leq C\varepsilon$ for $x \geq \sigma$. Therefore, in accordance with (6) an estimate (28) is also valid for $n > N/2$.

Considering (2), (23), (28), we get the estimate (19). □

**Remark 1.** In accordance with the estimate (19) the error of the composite formulas of Newton-Cotes (case $m = 1$) and Gauss (for $m = 2$) is of the same order as in the regular case when the derivatives of the function $u(x)$ are uniformly bounded [1].

**Remark 2.** The constructed mesh corresponds to the Bakhvalov mesh [11], [12]. In this case, the error of the considered quadrature formulas becomes uniform in the parameter $\varepsilon$.

## 3. Results of numerical experiments

Consider a function of the form (2)

$$u(x) = \cos\frac{\pi x}{2} + e^{-x/\varepsilon}, \tag{29}$$

where $\Phi(x) = e^{-x/\varepsilon}$.

Table 1. The error and the calculated order of accuracy of the trapezoid formula on the uniform grid

| $\varepsilon$ | $N$ | | | | | |
|---|---|---|---|---|---|---|
| | 16 | 32 | 64 | 128 | 256 | 512 |
| 1 | $3.1e-4$ | $7.6e-5$ | $1.9e-5$ | $4.8e-6$ | $1.2e-6$ | $3.0e-7$ |
| | 2.0 | 2.0 | 2.0 | 2.0 | 2.0 | |
| $10^{-1}$ | $2.7e-3$ | $6.9e-4$ | $1.7e-4$ | $4.2e-5$ | $1.1e-5$ | $2.7e-6$ |
| | 2.0 | 2.0 | 2.0 | 2.0 | 2.0 | |
| $10^{-2}$ | $2.1e-2$ | $6.9e-3$ | $1.9e-3$ | $5.8e-4$ | $1.3e-4$ | $3.1e-5$ |
| | 1.6 | 1.9 | 2.0 | 2.0 | 2.0 | |
| $10^{-3}$ | $3.0e-2$ | $1.5e-2$ | $6.8e-3$ | $2.9e-3$ | $1.0e-3$ | $3.1e-4$ |
| | 1.0 | 1.1 | 1.2 | 1.5 | 1.8 | |
| $10^{-4}$ | $3.1e-2$ | $1.5e-2$ | $7.7e-3$ | $3.8e-3$ | $1.9e-3$ | $8.8e-4$ |
| | 1.0 | 1.0 | 1.0 | 1.0 | 1.1 | |
| $10^{-5}$ | $3.1e-2$ | $1.6e-2$ | $7.8e-3$ | $3.9e-3$ | $1.9e-3$ | $9.7e-4$ |
| | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | |

In tables $e - m$ means $10^{-m}$.

In Table 1 for various values of $\varepsilon$ and $N$ is given the error $\Delta_{N,\varepsilon}$ of the trapezoid composite formula $S_2(u)$ from (7) in the case of the uniform grid $\Omega^h$,

$$\Delta_{N,\varepsilon} = |I(u) - S_2(u)|.$$

TABLE 2. The error and the calculated order of accuracy of the trapezoid formula on the optimal mesh

| $\varepsilon$ | $N$ | | | | | |
|---|---|---|---|---|---|---|
| | 16 | 32 | 64 | 128 | 256 | 512 |
| $10^{-1}$ | $5.0e-4$ | $9.4e-5$ | $2.1e-5$ | $5.2e-6$ | $1.3e-6$ | $3.2e-7$ |
| | 2.4 | 2.1 | 2.0 | 2.0 | 2.0 | |
| $10^{-2}$ | $9.2e-4$ | $2.5e-4$ | $7.0e-5$ | $1.9e-5$ | $4.9e-6$ | $1.2e-6$ |
| | 1.9 | 1.9 | 1.9 | 2.0 | 2.0 | |
| $10^{-3}$ | $1.9e-3$ | $4.7e-4$ | $1.2e-4$ | $2.9e-5$ | $7.3e-6$ | $1.8e-6$ |
| | 2.0 | 2.0 | 2.0 | 2.0 | 2.0 | |
| $10^{-4}$ | $2.0e-3$ | $5.1e-4$ | $1.3e-4$ | $3.2e-5$ | $7.9e-6$ | $2.0e-6$ |
| | 2.0 | 2.0 | 2.0 | 2.0 | 2.0 | |
| $10^{-5}$ | $2.0e-3$ | $5.1e-4$ | $1.3e-4$ | $3.2e-5$ | $8.0e-6$ | $2.0e-6$ |
| | 2.0 | 2.0 | 2.0 | 2.0 | 2.0 | |
| $10^{-6}$ | $2.1e-3$ | $5.1e-4$ | $1.3e-4$ | $3.2e-5$ | $8.0e-6$ | $2.0e-6$ |
| | 2.0 | 2.0 | 2.0 | 2.0 | 2.0 | |

TABLE 3. The error and the calculated order of accuracy of the Gauss formula with two nodes on the optimal mesh

| $\varepsilon$ | $N$ | | | | | |
|---|---|---|---|---|---|---|
| | 8 | 16 | 32 | 64 | 128 | 256 |
| 1 | $2.55e-7$ | $1.59e-8$ | $9.95e-10$ | $6.22e-11$ | $3.89e-12$ | $2.43e-13$ |
| | 4.00 | 4.00 | 4.00 | 4.00 | 4.00 | |
| $10^{-1}$ | $5.40e-5$ | $3.50e-6$ | $2.21e-7$ | $1.38e-8$ | $8.65e-10$ | $5.41e-11$ |
| | 3.95 | 3.99 | 4.00 | 4.00 | 4.00 | |
| $10^{-3}$ | $1.57e-5$ | $1.62e-6$ | $1.66e-7$ | $1.66e-8$ | $1.57e-9$ | $1.41e-10$ |
| | 3.28 | 3.28 | 3.33 | 3.40 | 3.48 | |
| $10^{-4}$ | $4.98e-6$ | $3.99e-7$ | $3.53e-8$ | $3.39e-9$ | $3.43e-10$ | $3.51e-11$ |
| | 3.64 | 3.50 | 3.38 | 3.31 | 3.29 | |
| $10^{-5}$ | $3.68e-6$ | $2.39e-7$ | $1.61e-8$ | $1.15e-9$ | $8.97e-11$ | $7.68e-12$ |
| | 3.94 | 3.89 | 3.80 | 3.69 | 3.55 | |
| $10^{-6}$ | $3.54e-6$ | $2.21e-7$ | $1.39e-8$ | $8.87e-10$ | $5.74e-11$ | $3.82e-12$ |
| | 4.00 | 3.99 | 3.97 | 3.95 | 3.91 | |

The calculated order of accuracy $M_{N,\varepsilon}$ is obtained using the error $\Delta_{N,\varepsilon}$,

$$M_{N,\varepsilon} = \log_2 \frac{\Delta_{N,\varepsilon}}{\Delta_{2N,\varepsilon}}.$$

As $\varepsilon$ decreases, the accuracy of the formula decreases and the order of accuracy decreases from 2 to 1.

Table 2 similarly shows the error and order of accuracy of the trapezoid formula $S_2(u)$ in the case of the optimal mesh $\Omega^h$. The obtained second order of accuracy corresponds to the estimate (19) for $m = 1$ and $k = 2$.

TABLE 4. The error and the calculated order of accuracy of the Gauss formula with three nodes on the uniform grid

| $\varepsilon$ | $N$ | | | | | |
|---|---|---|---|---|---|---|
| | 8 | 16 | 32 | 64 | 128 | 256 |
| $16^{-1}$ | $1.74e-6$ | $3.00e-8$ | $4.80e-10$ | $7.55e-12$ | $1.18e-13$ | $1.11e-15$ |
| | 5.86 | 5.96 | 5.99 | 6.00 | 6.73 | |
| $32^{-1}$ | $3.95e-5$ | $8.70e-7$ | $1.50e-8$ | $2.40e-10$ | $3.78e-12$ | $6.00e-14$ |
| | 5.51 | 5.86 | 5.96 | 5.99 | 5.99 | |
| $128^{-1}$ | $2.07e-3$ | $2.40e-4$ | $9.88e-6$ | $2.18e-7$ | $3.75e-9$ | $6.00e-11$ |
| | 3.11 | 4.60 | 5.51 | 5.86 | 5.96 | |
| $256^{-1}$ | $2.96e-3$ | $1.04e-3$ | $1.20e-4$ | $4.94e-6$ | $1.09e-7$ | $1.87e-9$ |
| | 1.52 | 3.11 | 4.60 | 5.51 | 5.86 | |
| $512^{-1}$ | $1.93e-3$ | $1.48e-3$ | $5.18e-4$ | $5.99e-5$ | $2.47e-6$ | $5.44e-8$ |
| | 0.38 | 1.52 | 3.11 | 4.60 | 5.51 | |

TABLE 5. The error and the calculated order of accuracy of the Gauss formula with three nodes on the optimal mesh

| $\varepsilon$ | $N$ | | | | | |
|---|---|---|---|---|---|---|
| | 8 | 16 | 32 | 64 | 128 | 256 |
| $16^{-1}$ | $1.74e-6$ | $3.00e-8$ | $4.80e-10$ | $7.55e-12$ | $1.18e-13$ | $1.11e-15$ |
| | 5.86 | 5.96 | 5.99 | 6.00 | 6.73 | |
| $32^{-1}$ | $3.95e-5$ | $8.70e-7$ | $1.50e-8$ | $2.40e-10$ | $3.78e-12$ | $6.00e-14$ |
| | 5.51 | 5.86 | 5.96 | 5.99 | 5.99 | |
| $128^{-1}$ | $8.82e-7$ | $1.24e-8$ | $1.82e-10$ | $2.56e-12$ | $2.22e-14$ | $2.22e-16$ |
| | 6.16 | 6.08 | 6.16 | 6.68 | 6.81 | |
| $256^{-1}$ | $4.70e-7$ | $6.50e-9$ | $9.58e-11$ | $1.44e-12$ | $2.07e-14$ | $2.22e-16$ |
| | 6.18 | 6.08 | 6.05 | 6.13 | 6.54 | |
| $512^{-1}$ | $2.45e-7$ | $3.36e-9$ | $4.91e-11$ | $7.44e-13$ | $1.11e-14$ | $1.73e-16$ |
| | 6.19 | 6.10 | 6.05 | 6.07 | 6.00 | |

Table 3 shows the error and order of accuracy of the composite Gauss formula with two nodes on the optimal mesh,

$$S_{2,n}(u) = \frac{h_n}{2}\Big[u\Big(\frac{x_{n-1}+x_n}{2} - \frac{h_n}{2\sqrt{3}}\Big) + u\Big(\frac{x_{n-1}+x_n}{2} + \frac{h_n}{2\sqrt{3}}\Big)\Big].$$

The obtained fourth order of accuracy corresponds to the estimate (19) for $m=2$ and $k=2$.

In Table 4 for various values of $\varepsilon$ and $N$ the error $\Delta_{N,\varepsilon}$ of the composite Gauss formula $S_3(u)$ on the uniform grid $\Omega^h$ is given,

$$S_{3,n}(u) = \frac{h_n}{18}\Big[5u\Big(\frac{x_{n-1}+x_n}{2} - \frac{h_n}{2}\sqrt{3/5}\Big) + 8u\Big(\frac{x_{n-1}+x_n}{2}\Big) +$$

$$+5u\Big(\frac{x_{n-1}+x_n}{2} + \frac{h_n}{2}\sqrt{3/5})\Big)\Big].$$

The order of accuracy on the uniform grid decreases if $\varepsilon \le 1/N$.

Table 5 shows the error and order of accuracy of the composite Gauss formula with three nodes on the optimal mesh. The obtained sixth order of accuracy corresponds to the estimate (19) for $m = 2$ and $k = 3$.

## 4. Conclusion

The problem of numerical integration of functions with large gradients is investigated. It is assumed that the decomposition in the form of the sum of the regular and singular components is valid for the integrable function. For the singular component, estimates for the derivatives corresponding to the presence of an exponential boundary layer are given. Cases when the basic Newton-Cotes or Gauss formulas are applied between the mesh nodes are considered. The mesh nodes are constructed on the basis of minimizing the error of the composite quadrature formula. It is proved that the minimum of the error is attained on a mesh that corresponds to the class of Bakhvalov meshes. The resulting error estimate for the composite quadrature formula is uniform in a small parameter. The order of error on the constructed mesh is the same as in the regular case when the function does not have large gradients. The results of computational experiments are consistent with the obtained error estimates.

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